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Pre-Godelian, Post-Godelian and non-Godelian Philosophy of Mathematics

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The features of the Pre-Godelian (mostly Hilbert style), Post-Godelian and Non-Godelian (mostly paraconsistent style) philosophy of mathematics described. The core of Pre-Godelian philosophy of mathematics - the belief that every mathematically true statement could be proved. Godel's theorems clarified that formally proved disjoint with informally true in mathematics and the latter plays pivotal role in any formal discourse. Recently has been realized that the necessary precondition of the Godel's results - the consistency - actually (in general case) failes in the paraconsistent systems, where the standart methods and even status of the Godel's theorems should be reassessed. Consistency is not a sine qua non condition for such systems within the frame of which the relationship between provability and truth radically weakens, the condition of non-triviality along with the principle of non-inference from external premises has to replace that of consistency. The study of external premises and conditions for non-triviality of the systems lies in the center of the non-Godelian philosophy of mathematics.

1. Mathematics/1/ and philosophy are tightly bounded. This bond manifested, for instance, in the fact that precisely philosophical orientations and preferences of mathematicians largely determine the choice of basic assumptions, abstractions and methods of discourse, thus, the fundamental mathematics and its foundations conceptions. Such a bond may be viewed as a two-way street: the development of mathematics influence the philosophy of mathematics and vice versa (perhaps in more delicate manner) - shifts in the philosophy of mathematics has impact upon architecture and, so to speak, every day life of mathematicians (especially those who deals with pure, fundamental mathematics).

Being a study of the nature of mathematics and methods of mathematical knowledge and its regularities development, philosophy of mathematics has been greatly influenced by almost any prominent discovery, which threw new light on foundational problems. One of those discoveries - probably, the most significant one in the foundations of mathematics in the XX century - was the discovery by K.Godel in 1931 of his (famous now) theorems.

2. Godel's theorems abruptly changed approach to foundations of mathematics; essentually they served as a scaffold of new philosophy of mathematics, replaced the philosophy of mathematics the final accord and transit link to which was Hilbert's program. J. von Neumann was not exaggerating when pointed out: "Godel's achievement in modern logic is singular and monumental - indeed it is more than a monument, it is a landmark" /Tribute to Dr.Godel. - In: Foundations of mathematics. Ed. J.Bulloff et al. 1969, p. X/.

The significance of Godel's theorems is not limited to logic and mathematics, but valid for all systems with certain kind of formalization. In mathematics and formalized systems Godel's theorems (and the rest restrictive theorems) could manifest themselfs in various situations, worthy only of their reliability degree discussion.

2.1 The situation mentioned can arise already in cases when mechanical problems formulated for periodical are generalized on quasiperiodical movement. This implies that classical dynamics is not a secluded theory for it language enables us to formulate true statements, which could't be derived within the theory itself.

It may be assumed that any physical theory with a measurement subtheory is incomplete in Godel's sense for the existence of such subtheory presupposes an arithmetics (W.Yourgrau). If it is really admittable to discuss the applicability of Godel's results to physics, then, more likely, we should deal with semantic version of these theorems (Bazhanov V.A. The completeness problem in quantum physics. Kazan Univ.press, 1983, p.66-69).

It may be argued that Godel's theorems are also the theorems of theoretical programming (V.Glushkov), the attempts are made to implement these theorems to linguistics, biology, social doctrine and even to literature. Notwithstanding the disputability of such attempts one may judge that Godel's theorems could claim for common validity. We may argue that shift in the philosophy of mathematics overstep the limits of mathematics and has impact upon the rationality as a whole.

3. Godel's theorems are watershed between two epochs in the foundations of mathematics and its philosophy studies. What is the main features of Pre-Godelian philosophy of mathematics? Going to describe them in general terms I must stress that it'll be, so to speak, some average characteristics, smooth over and leveled various rival directions in mathematics and mathematical schools, being valid for mathematics in the fall of XIX - turn of XX century, but not for the particular trend of mathematical thought.

3.1 At the period mentioned in the philosophy of mathematics have been a belief that mathematical assumptions and abstractions are derived from the experience and at least to many of them the definite physical sense could be prescribed. This was the main reason why, for example, the point according to which reasoning about the inconsistensies of some mathematical systems "reflecting the world" seemed senseless. In the same way the thesis that some sets of formulas can be true under one interpretation and false under another seemed to be awkward. Whereas the intensive development of algebraic methods implied the bewitching of the if-then, symbolic character of mathematics: symbols were trusted much more then constraint nature of discipline. Meanwhile the empirical ideology caused doubtfulness in the objective meaning of certain far off abstractions. The introduction by G.Cantor of transfinite numbers added the feelings of diffedence in acceptability of such strong abstractions. The discovery later of the theory of sets paradoxes only whiped the process, making him the real crisis of mathematics foundations and its philosophy.

Indeed, in Hilbert's program were expressed not only those features of mathematical thought and philosophical standards of mathematics, directly bounded with the crisis, but the pivotal idea of Pre-Godelian philosophy of mathematics fixed in the Hilbert's program: the idea that all true sentenses are provable (the epistemological optimism "burst out of banks"). As Hilbert put it, in mathematics there is no ignorabimus: vice versa, we can answer every meaningful question. We just should find the means of proof; there is no doubt sooner or later they'll be found.

The conviction in provability of every true statement inspried the spread and estimation of (pure) proofs of existence. Indeed, the crisis in the foundations of mathematics launched the examination of what should be considered the existence in mathematics.

3.2 Hilberts confidence in indisputability of classical logic and uniqueness of its laws forced him (to use the modern expression) to make an attempt to prove the conservativeness of classical mathematics abstract methods. If by chance the attempt has to be success the problem of ideal elements simultaneously has to be resolve. The requirement of conservativeness implicitly was presented at Hilbert's speech on the mathematical Congress at Heidelberg, where he expressed the intention to prove the consistency of the Cantorian theory of sets, although the program of the demonstration of consistency emerged from just mentioned one. To fulfil the task Hilbert put forward the finitary method, presupposed tools quite acceptable even by those who were engaged in "revision" of mathematics - by intuitionists.

Hilbert's program was not totally consequent: thus, understanding the ideal nature of mathematical abstractions, Hilbert hasn't even a spark of doubt in unique character of "Cantor's Paradise", i.e. in uniqueness and absoluteness of certain mathematical system. When Hilbert with collaborators seem to be nearly succeded Godel proved the completeness of the first order predicate calculus, - the theorem suggested the goal is close and its attainment depends only upon the application of adequate technique. The completeness theorem strengthened the core of Pre-Godelian philosophy of mathematics - the conviction of provability of all true statements. However as early as next year Godel proved his famous incompleteness theorems/2/ crossed out the strategic aspiration for fulfilment of the program (though some tactic points of the program were not affected by the theorems and preserve their value so far).

Godel's theorems virtually overturned the basis of the traditional philosophy of mathematics and become the cornerstone of the new approach to the philosophy of mathematics. If the Pre-Godelian epoch in the philosophy of mathematics implied the identity of logically proved and informally true, then Post-Godelian epoch marked by the difference of the features alleged. Moreover, the notions of logical inference and truth turned to belong to the principally different types of arithmetically expressible sets. This enables to call the Godel's theorems - "theorems of the century", manifested the most common traits of the knowledge acquisition process of XX century (B.G.Kuznetsov).

The comprehension of Godel's theorems revealed the scope and limits of formalization and following axiomatisation, the point that mathematics deals not only with absolute truths and, hence, is much more close to natural sciences then it was accepted before. Post-Godelian philosophy of mathematics based upon the realization of the significance of informal, intuitive, "personal knowledge" in mathematical creativity and the fact that their authority should be not less than that of formal components. To cut the long story short, the restrictive theorems (of Godel, Church, Tarsky etc.) may be conceived as an indicators of the primacy of informal upon formal components of mathematical knowledge acquisition and as a mark of non-existence of universal algorithmic procedures.

4.1 Godel's theorems reveal how to construct the statements expressing their own undecidability. These statements, strictly speaking, of metamathematical, not mathematical character. Thats why it was vital to construct an example of independent from arithmetics though of the same evidence undecidable sentence. The reinforcement of Godel's theorems required the proof of undecidability and independance of certain sentence A with the help of method distinguished from the Godel numeration, for instance, via construction of two models of arithmetics in first of which A would be valid and in the second would be valid non-A. Such reinforcement actually was gained in the late 1970-th - early 1980-th by Harrington, Paris, Kirby. Alongside with purely mathematical significance this result has haughty philosophical impact on the philosophy of mathematics, for it has become absolutely clear that reasoning which involve Godel's theorems has force really upon mathematics, not metamathematics alone.

4.2 What the main features of the Post-Godelian philosophy of mathematics (we again try to mark those features which are common to the most mathematical schools)? To begin with the idea of the plurality of formal systems and realization that various paradigmas in the foundations of mathematics are relative to definite basic assumptions, abstractions and principles, chosen according to philosophical sympathies should be named. The standpoint of relativism admitted by almost everyone has been combined with points of absolutism. The most important among the latter is consistency requirement, inhereted from the previous stage of the philosophy of mathematics. Godel's theorems shed the light on the conditions and circumstances of the provability of the particular kind of formal systems consistency.

Godel's theorems restored the authority of intuition and informal considerations, promoting understanding of the futility of the reduction of mathematics to the "game with symbols", to the pure syntax. They gave a powerful impedus to the metatheoretical studies and analysis of mathematics self-control tools (in the sense of using particular assumptions, principles, methods, axioms, say, axiom of choice or abstraction of actual infinity).

However the Post-Godelian philosophy of mathematics evolved largely from the rational kernel of Hilbert's program and restrictive theorems (first of all Godel's theorems) in certain aspect is imperfect. The imperfection can be, probably, considered as a starting point of the non-Godelian philosophy of mathematics.

5. Before we'll discuss the alleged imperfection it should be pointed out that the catalyst of non-Godelian philosophy of mathematics birth is the springing up and development of paraconsistent logic and mathematics. The study of paraconsistent systems are, so to speak, still in the cradle, but to my mind, the impact in the future on mathematics and its philosophy may be very profound. In these systems the traditional interpretation of Godel's theorems is subject to reevaluation. As a matter of fact the necessary conditions of Godel's theorems proof is that of consistency, - the requirement not valid for paraconsistent systems. The chance of contradictory statements, the situation when theorems A and non-A are both true indicate that not only standard methods of Godel's theorems proving is to be reconsidered, but their status as well.

5.1 Inadmissibility of two judgements one of which is negation of another is not an ideal, but a standard of any classical formal system (and, strictly speaking, of some non-classical systems, say, intuitionistic). If the system is inconsistent within its frame "everything is provable" (ex contractione quodlibet), i.e. the system is trivial. For the classical systems the notions of inconsistency is coincide with the notion of triviality.

5.2 Classical logic and mathematics are forced to restrict their own linguistic tools due to the implicit threat of inconsistencies and paradoxes. The basic reason of the latter is seen - ponder over! - in excessive richness of alleged linguistic tools. According to the classical standpoint the existence of a paradox in the system indicate of its serious troubles (if not a catastrophe). One may compare with the healthy man accused of his health. It is not accidently the foundation of mathematics crisis was provoked due to dislosure of the set-theoretical paradoxes and the search for the way out of crisis has had far reaching consequenses.

Restrictive tendencies are strong and natural in the classical mathematics. Various considerations justifying these tendencies were proposed. The most radical forms of such tendencies lead to the exclusion of whole fragments from traditional mathematical theories. We may judge that pursuit of an unconditionally consistent system is a sort of restriction, the converse of which is the phenomenon of the unprovability of a systems consistency by its own means. Meanwhile the very aspiration for consistent systems is the main stimulus of restrictive tendencies and of Hilbert's program, - the aspiration occured to be unattainable.

Hence, consistency is not a sine qua non condition for the formal system. The only reasonable condition of that kind is a non-triviality of formal system, i.e. not all judgements, expressed in a certain language, are equally provable. Paraconsistent logic is par excellence aimed at studying inconsistent, but non-trivial systems.

5.3 Paraconsistent mathematics and logic development entails important consequences for the philosophy of mathematics. On the foreground of non-Godelian philosophy of mathematics are the concepts of triviality and paracompleteness (the system K is paracomplete if in addition of the formula A, undecidable within K, makes the system paraconsistent). The principle of non-contradiction yield the place for the principle of non-inference from the external (outside) premises. The latter seems to be more fundamental in the scope of paraconsistent systems then the former. Namely, the sentences making the system trivial, will be the external premises. Failure to comply with this principle in the classical (consistent) mathematics is due to accomplishment of the Scott principle (ex contradictione quodlibet). As a result we are obliged to rethink the relationship between validity and provability. So the paraconsistency of definite formal systems means that formulas A and non-A are both may be theorems. To ascribe the formula A the meaning "true" one must establish that non-A is not provable in this system. Otherwise one may speak only about the "non-falsity" of A. Thus, in non-Godelian philosophy of mathematics we should take into account the weakening of the relationship between "true" and "false". The chief thing, above all is that, in the focus we have now the study of external (outside) premises but not the consistency.

Allow for paraconsistent systems being in their infancy one can see that even the concise description of the related (non-Godelian) philosophy show its non-conjugation with the Post-Godelian version of the philosophy of mathematics.

/1/ Speaking of mathematics I mean logic as well, though I fully realize their non-identity. But for the purpose of this paper such insert is permissible (see also item 4.1).

/2/ In addition to deductive, semantic and descriptive completeness (see: Hintikka J. Is there completeness in mathematics after Godel? - In: Philosophical Topics,1989,vol.XVII, N2) we can speak of sintactical completeness as well.

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