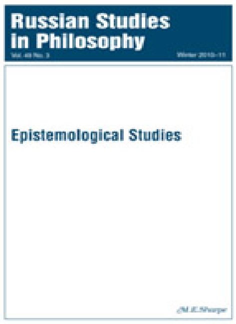

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Mathematical Proof as a Form of Appeal to a Scientific Community

The author analyzes proof and argumentation (mainly in mathematics and logic) as a form of appeal to a scientific community with deep ethical meaning. He presents proof primarily as an effort to persuade a scientific (sub) community rather than a search for true knowledge, as an instrument by which responsibility is taken for the correctness of the thesis being proved, which usually originates in a sudden flash of insight.

Proof as the most important component of mathematical argumentation

At the very core of rationally understood science lies the powerful system of argumentation that is brought to bear in the process of substantiating various propositions. This system is multifaceted, dynamic, and multidimensional; its influence affects any "cell" of scientific knowledge to the same degree to which the whole "body" of science is permeated by rational procedures for the derivation and substantiation of new knowledge. The most important element here is the procedure of proof.

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Translated by Stephen D. Shenfield.

It is no coincidence that G.A. Brutian emphasized that "in any kind of argumentation we are dealing above all with proof."[1] He has in mind here the logical aspect of argumentation and logical proof, although argumentation is not identical to logical proof insofar as it is precisely "the choice of social position [that] conditions the strategy of argumentation."[2] However, the social aspect may also pertain in a certain manner to logical proof as such, which it would be very blinkered, not to say detrimental, to understand only as an ordered chain of symbols. Let us try to substantiate this proposition.

What is a proof? The matter would seem to be clear enough. In logic it is customary to regard as a proof some finite sequence of formulas in which any formula is either an axiom or a consequence drawn from preceding formulas in accordance with one of the rules of inference. Thirty years ago, Morris Kline observed that "the most disagreements arise out of the question: *what is a mathematical proof*?"[3] There is still no unequivocal solution to this question. The nature and meaning of mathematical proof is a topic of lively discussion within the mathematical community and outside it.[4] This is no cause for surprise: mathematics remains "the Queen of the Sciences"—energetically renewing itself and expanding into fields where mathematics was not typically used in the past—for example, the decoding of the human genome, the modeling of apoptosis—or of Down's syndrome, and handwriting recognition.[5] However, we also encounter pessimistic assessments of the discipline's prospects, based on set-theoretical considerations: it is argued that mathematicians can prove only a small proportion of true propositions even in the field of whole numbers, so there approaches "the twilight of the glorious epoch opened by the [ancient—V.B.] Greeks," although the hope is expressed that "mathematics lives for a very long time, and does not reach a dead end for many generations to come."[6] A very significant interest in mathematical proof is shown even by psychologists, who find fundamental differences between the deductive reasoning that constitutes the basis of mathematical (logical) proof and the process of natural human reasoning.[7]

The deep essence of mathematics is often seen in the place that proof occupies in mathematics. However, the concept of proof goes far beyond the bounds of mathematical knowledge properly speaking, although it is, of course, precisely in mathematics that the most important properties of proof most clearly manifest themselves. Argumentation and proof are universal procedures that encompass a great variety of disciplines, of which mathematics is only one. And in mathematics itself there is a constant rethinking of the status and nature of proof as a purely formal operation, an awareness of the significance of the substantive and intuitive components of reasoning, and a growing conviction that informal proof is still of extraordinary importance and does not cede its place to formal methods in the era of their triumph.[8] The discovery of the incompleteness theorems of metamathematics has seriously undermined the philosophical norm that was the pivot of the paradigm of rationality that was characteristic of nineteenth- and early twentieth-century thought and had been accepted for two millennia. While according to this norm "it was assumed that the very nature of mathematical truth consists in its provability"[9] and (to quote Richard Dedekind) "nothing that is provable in science should be accepted without proof,"[10] the development of mathematics in the twentieth century made it necessary to rethink this norm in light of the more complex and contradictory character of proof when considered in its social and psychological context—a context in which mathematical truth is sought after and attained as a result of nontrivial cognitive procedures and a special type of creativity. As V.A. Uspenskii writes, "the concept of proof does not belong to mathematics (only its formal model—formal proof—belongs to mathematics). It belongs to logic, linguistics, and above all psychology."[11]

The new philosophical norm that replaced the boundless epistemological optimism of David Hilbert and now lies at the foundation of mathematics entailed substantial changes in the standards of logical—and, more saliently, mathematical—proof. New norms of proof started to emerge as "counterweights" to the standards of traditional mathematical proof, in which rigor in reasoning had predominated over value in application. Thus, in catastrophe theory a "threshold" for proof was accepted that, as V.I. Arnold remarked, was "implausibly (from the traditional point of view) low."[12] René Thom observes that when he became a colleague of A. Grotendik, his example "inspired me [Thom—V.B.] to regard rigor as a somewhat unnecessary detail of mathematical thinking. . . . I still believe that rigor is a relative and not an absolute concept. Since the collapse of Hilbert's program we have known that rigor may be no more than a local and sociological criterion" (translated from Russian).[13]

Such lowering of the "threshold" of proof is usually associated with a period of rapid development of a discipline, marked by the neglect of rigor; from time to time it is observed in various branches of mathematics. For example, "by the mid-nineteenth century the importance of proof had fallen so far that some mathematicians did not consider it necessary to adduce complete proofs even in those cases where this was possible." What mathematicians operated with at that time "would be more correctly called mere snatches of reasoning" (translated from Russian).[14] However—and this may, of course, evoke astonishment—even in eighteenth-century mathematics the number of erroneous arguments was small: unerring intuition was at work.[15]

It is precisely in the informal components of mathematical reasoning and proof that Imre Lakatos saw the guarantee of their effectiveness. He counterposed the *deductivist* and the *heuristic approach*. The deductivist approach cultivates an "authoritarian" atmosphere in mathematics, turning it into an agglomerate of "sacred" and unquestionable truths, depriving the concepts generated by proofs of their preceding history, and treating them as artificial concepts born out of a "vacuum." This approach conceals the clash of opinions in the history of mathematics. The heuristic approach, by contrast, emphasizes precisely the aforementioned factors. It places the stress on the problem situation and boldly presents the "logic" thanks to which a new concept was born.[16] The deductivist approach, according to Lakatos, leads to the "degeneration" of mathematics: when the style of mathematics acquires features of the baroque, this should be seen as a danger signal.[17]

Indeed, even now many prominent mathematicians take a rather cool attitude toward considerations of rigor in proof and emphasize the wealth of informal proof procedures: "much of mathematical reasoning presents genuine meaning-dependent mathematical characteristics that cannot be captured by formal calculi" (Georg Kreisel[18]); "now mathematical practice strongly supports the view that important notion of proof in mathematics is not derivation, but informal proof" (B. Löwe, T. Müller[19]); "complete formalization and complete proof, even if possible in principle, are impossible in practice" (Reuben Hersh[20]). Many mathematicians draw attention to the extreme importance of informal proofs in contemporary mathematics, while Don Fallis points out the inevitable—conscious or unconscious—presence of important intentional gaps in proofs.[21] In connection with the rapid development of paraconsistent systems, it has become possible to assert that a certain proposition is not simply proved or unproved but, for instance, is "60 percent proved" in Lakatos' sense that the appearance of counterexamples to the given proposition makes its proof not absolute but relative, corresponding to the conception of fallibilism and recognizing the method of trial and error as also immanent to the development of mathematics.[22] Thus, the proof of the celebrated Kepler conjecture by Thomas Hales in 1998 is assessed as 99 percent valid because 250 pages of analytical proof (reduced to 100 pages in 2005) are supplemented by complex programs that go through numerous variants on the computer, using total memory of 3 GB.[23]

It would appear that informal proofs make full provision for the progress of mathematics—be it "ordinary" or paraconsistent mathematics.

Nevertheless, there is no slackening of efforts by mathematicians to search for increasingly rigorous proofs. Computer methods are being developed to search for formal proofs that would make it possible to verify proofs previously obtained by analytical means. Moreover, new approaches to the construction of formal proofs are considered to constitute no less than the third revolution since the birth of mathematics.[24]

Formal proofs (in which frequent use is made of the methods of the branch of mathematical logic called proof theory, which goes back to Hilbert's attempt to establish mathematical foundations by finite means) help to reorganize informal proofs in such a way that it becomes possible to extract more information from them—in particular, information about the nature of the simplest system that would permit their formalization.

What is it that compels mathematicians to examine the phenomenon of proof so intently and develop ever more "refined" methods of formal proof? Are their efforts not in vain? What hidden motives underlie them? What reasons can be identified that impart meaning (and, moreover, deep meaning) to the search for proofs of great complexity, making it more than a futile intellectual game?

It is by no means only mathematicians who take an immense interest in the phenomenon of proof. David Auburn's play *Proof* has been running at a Broadway theater since 2000, winning the Tony Award and the Pulitzer Prize. Then in 2006 the Miramax studio presented the film *Proof*,[25] directed by John Madden and starring Gwyneth Paltrow, Anthony Hopkins, and Jake Gyllenhaal. In spring 2010 the well-known Polish producer Krzysztof Zanussi presented the same play at the Russian Youth Theater in Moscow. On 16 June 2003 a conversation about proof in mathematics between Aleksandr Gordon and Academician Iurii Leonidovich Ershov was broadcast on Russian television. In short, proof is a general cultural (or even specific civilizational) phenomenon. I think, therefore, that the interpretation of proof as a social and ethical procedure has a significance that goes beyond the bounds of mathematics proper as a science of order and relations that has arisen in the process of development of the practice of calculations, measurements, and description of the forms of (real and abstract) objects and relations among them and that is based on (logical) proofs and numerical computations.

Proof in the context of "informal" logic

1.1. The fact that the concept of provable truth is narrower than that of substantive truth has been one of the reasons for reassessing the limits to the applicability of deductive methods of cognition. Well-founded doubts regarding the ability of rigorous deductive methods of reasoning to reconstruct and model the whole or even any substantial part of rational discourse (let alone intuitive judgments) have led to intensive development of the theory of argumentation and rhetoric and to attempts to construct a so-called informal logic. To a certain extent challenging the tradition of mathematical logic, it is argued that "informal errors constitute a legitimate basis for logical investigation" and that "theoretical reproduction of modes of reasoning and logical criticism in terms of informal logic has a direct payoff for such branches of philosophy as epistemology, ethics, and the philosophy of language" (retranslated from the Russian).[26] What is more, some scholars in the field of informal logic view their work as "an attempt to save logic from insanity [caused, in their opinion, by the isolation of abstract mathematical structures from reality—V.B.]."[27] Even in publications that may be considered citadels of mathematical thinking, like *Notices of the American Mathematical Society*, there have recently appeared articles insisting on the presence of important informal components in mathematical proofs that invalidate the latter's claims to absolute rigor and completeness, referring to the "writing out" of all the steps in a proof. Melvyn Nathanson asserts that mathematics resembles "a web of semi-proved (or not fully proved) theorems. . . . We mathematicians like to talk about the 'reliability' of our literature, but it is, in fact, unreliable. . . . Even in mathematics truth can be political."[28] Social pressure, in his opinion, often causes errors in mathematical proofs to be ignored, and theorems themselves can even be interpreted in terms of social systems analogous to clans or groups of people connected by ties of kinship.[29] In this sense, it does not seem extravagant to argue, as Loren Graham does, that even "mathematical equations have social attributes."[30] Graham has in view the style of mathematical thinking that applies to V.A. Fock's* presentation of the equations of gravitation theory in a form different from that used by Einstein. This form, in Graham's opinion, was quite openly determined by Fock's sociopolitical views.

Investigation of the psychological aspects of informal proof shows that in a discussion the burden of proof lies, above all, on the party who initiates the discussion and is first to present his arguments, while "softer" demands may be imposed on the opposing party with regard to the strength of his arguments. This is the reverse of the situation in which the party who "makes the first move" has the advantage.

Proof as a procedure of persuasion

1.2. While proof used to be regarded as an impersonal construction (the majority of scientists are still inclined to regard it as such), in the theory of argumentation it acquires new dimensions that characterize those cognitive structures which, on the one hand, strive for "impersonal" knowledge, and, on the other hand, are oriented toward the subject or person who has obtained this knowledge, the manner of its presentation, the degree to which it is persuasive for those to whom it is addressed, and the semantic loading placed on this knowledge by its "living" source. Thus, proof is tied into a single knot with

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*Vladimir Aleksandrovich Fock (1898–1974) was a Soviet physicist, famous for his work in the areas of quantum mechanics and quantum electrodynamics.—Trans.

its social and psychological context and with ethical considerations. In other words, the theory of argumentation puts the main stress on proof not as a form of deductive knowledge but as a form of *persuasion* of the truth of a thesis, on the mechanisms and sources of this persuasion, and on *the factors thanks to which he who puts forward the thesis turns out to be right*.[31]

Although the concept of proof (in the broad sense) does not admit of exact and exhaustive definition, the essence of this concept reveals itself precisely through the persuasiveness of the mental construction that lays claim to the status of a proof—"a persuasive argument that persuades us so powerfully that with its aid we are able to persuade others."[32] The importance of the persuasiveness of a proof is indicated by many mathematicians and philosophers of mathematics: "the purpose of a proof is to persuade competent specialists"[33]; "we prove things in a social context and address them to a certain audience" that is willing to be "persuaded"[34]; "the persuasive potential acts on the psychological state of the audience."[35] Even consistent supporters and designers of methods of formal proof cannot avoid paying attention to the key role played by the factor of persuasiveness in procedures of proof.[36] Here the persuasiveness of a proof taken as a whole is determined by the persuasiveness of the weakest link in the proof.

The persuasive aspect of proof is also noted by those who insist that virtually the exclusive prerogative, motive, and justification of proof is the search for new truths and not an appeal to a scientific community. Thus, V. Ya. Perminov, who connects mathematics with the a priori character of logic, primary mathematical idealizations, and indeed mathematics in general, writes: "This proof persuades us and we conceive it as being perfectly rigorous. . . . Mathematicians persuade themselves and others of the correctness of their proofs."[37] While wholly sharing the opinion that the search for new truths is an important goal of proof (but in no way supporting Perminov's view that logic and mathematics are a priori in character), I am nonetheless inclined to think that this opinion pertains to a dimension of proof other than those dimensions to which the *motive* of proof and its *justification* pertain. Of course, distinctions among the motive, goal, and justification of proof are bound to be somewhat relative, but evidently the point is that the epistemological, social, axiological, teleonomic, and other dimensions of proof are nonetheless not identical to one another and in the general case can be distinguished (although, I repeat, the distinctions will be somewhat relative). Willy-nilly, therefore, there arises the question of the permissibility and/or impermissibility of various arguments and means of proof, and this in essence automatically poses the problem of trying to understand proof in its socioethical aspect. We can even express ourselves more strongly: the motive and justification of proof cannot be properly understood in isolation from this aspect.

It is not by chance that research in the theory of argumentation has started to take into account the socioethical dimensions of proof. This is as much due to changes in the norms of logical-mathematical proof as it is due to the ways in which these changes are connected with the ethical as well as the social and psychological aspects of the process of accepting and assessing the cognitive significance of proof.

The principle of co-feeling and proof

2. The development of increasingly complex and refined methods and techniques of proof can be attributed only in part to the effort to find more effective forms of creative work in the disciplines of logic and mathematics: proof as a procedure of formal logic is an instrument of consistent, sound reasoning and substantiation rather than a means for discovering something new. Proof is a sort of disciplining matrix for thinking, its "grammar" (knowledge of which is clearly no guarantee of success in attaining a new result). What then, it may be asked, are the incentives for making proofs more rigorous, constructive, and formalized?

2.1. The discovery of something new (including, inter alia, new methods of reasoning and argumentation and new techniques of proof) is usually (but not always) the result of a leap in the quality of mental activity—a leap that often occurs as a sudden flash of insight and is perceived as such. Here the scientist is guided chiefly by his intuitive conceptions; he relies on his substantive, unformalized experience. He intuitively senses that his insight, idea, or emerging theory is correct. His actions in this situation are subordinated to the "principle of sympathy (intuition)" in heuristics. And at this stage the idea or theory can be accepted only by those who have the same "feeling," whose intuition suggests the same result. S.V. Meien emphasized: "This situation is described by the principle of sympathy or co-feeling (co-intuition).— If the insight of the originator of an idea always met with the 'co-insight' of his contemporaries, if what was intuitively clear to one person became 'co-intuitively' clear to others, then there would not be many obstacles to the spread of 'lunatic ideas' and 'ideas of the century'." However, "strict requirements of complete rigor, if not axiomatization, are immediately imposed on interesting ideas."[38]

Proof and an elaborated system of argumentation enter into their rights only after the image of the thesis to be proved has been outlined with some degree of clarity and completeness. It is reasonable to ask whether the latter judgment does not conceal a contradiction: if the process of proof and argumentation serves only to order, substantiate, and systematize new knowledge, then is there perhaps no really vital need for the elaboration of a highly complex system of argumentation and for proof as such? What then is their purpose and real meaning? What is the justification for efforts to develop refined techniques of proof—above all, of course, in mathematics? What, let us say, justified and motivated the proofs of Burnside's two hypotheses in the theory of finite groups (concerning the existence of nondecidable finite groups of odd order), each of which took up about five hundred pages? And what of the hundred mathematicians from six countries who worked for several decades to produce a classification of complete finite groups, presented in no fewer than twenty volumes containing 15,000 pages of journal paper format? Were their titanic efforts not in vain?

The ethical nature of mathematical proof

2.2. It is not at all coincidental that techniques of proof have been perfected and new methods of argumentation invented, nor is it coincidental that knowledge obtained through insight has in general been accepted by a scientific community and assigned the status of scientific knowledge, recognized as reliable, certain, and objectively true, only when it has passed through numerous "filters" of proof and been reproduced by many generations of scientists, when a solid foundation has been laid under it in the form of an indisputable (at a given stage) system of argumentation.

A proof is none other than a form of appeal to a scientific community. It is immersed in a realm of opinions, assessments, stereotypes, and norms of discourse and accomplished with the aid of means of argumentation accepted in a scientific community. Outside of this realm and independently of these means, a proof is evidently lacking in substance and in any case unpersuasive. It suffices to recall the fate of the numerous discoveries that were ahead of their time and did not receive the recognition of contemporaries, that were not—as it were—drawn into the stream of scientific thought, which only later "matured" to the point at which it was able to appreciate those discoveries or was forced to generate them anew. The imaginary logic of N.A. Vasil′ev is an example of this.* In other words, a multifaceted investigation and comprehension of the phenomenon of proof and of argumentation as a whole are hardly possible without taking into account their socioethical and/or psychological dimensions, their social and cultural context.

A specific scientific community carries and gives expression to concrete norms and standards of proof and supports specific means and procedures of argumentation that are implicitly fused into the picture of reality drawn by and

*The logician (and poet) Nikolai Aleksandrovich Vasil′ev (1880–1940) pioneered the idea of a non-Aristotelian logic in the years preceding World War I. His work had little resonance at the time but was rediscovered in the early 1960s.—Trans.

given general significance in theory and incorporated into the philosophical and methodological outlooks of scientists, into their style of thinking. Here an analogy suggests itself with the principle of relativity with respect to type of interaction or means of observation (V.A. Fock). This analogy can be extended to the formulation of a corresponding principle of relativity with respect to means of argumentation.[39]

Generally speaking, an appraisal of one or another scientific proposition should include an appraisal of the character of its argumentation. A proposition may sometimes be so firmly "attached" to its argumentational context that it simply loses all meaning if this context is left out of account. It is precisely from this point of view that we must evidently appraise the discussion between the representatives of the two fundamental conceptions of "lines" in the composition of scientific knowledge—Bohr and Einstein.

2.3. To prove is in a broad sense to persuade. Nevertheless, persuasion is not only the result of proof but also its starting point. Proof rests on persuasion; it is the form in which persuasion is actualized; however, the initial result of persuasion is inevitably adjusted after the completion of proof, and sometimes—in the most outstanding, fundamental cases—it undergoes qualitative changes. Persuasion is a predicate not only of the personal but also of the suprapersonal. Therefore a chain of reasoning is regarded as a proof on the condition that there occurs the social act of its adoption and recognition by a scientific community. The history of science is full of illustrations of this statement. Kline writes: "A proof is considered acceptable if it obtains the approval of the leading specialists of its time or if it is built on principles whose use is fashionable at a given moment."[40]

Consequently, the socioethical nature of proof and argumentation consists in *the transformation of an insight from an intrapersonal phenomenon into a phenomenon of general significance that can be reproduced and verified (at least in principle) by any sufficiently competent member of a scientific community*.

In saying this, I am of course abstracting from the time factor and from the technical difficulties that play no small role in the procedures for verifying one or another result. V.A. Uspenskii observes: "Proofs gradually shift from the realm of individual experience to that of collective experience. The very concept of persuasiveness [of a proof—V.B.] begins to lose its individualized nuance and increasingly acquires the character of 'collective persuasiveness'."[41]

3. As a rule, a logical-mathematical proof is merely a brief outline of the formal derivation. The current state of the logical-mathematical proof (and/or proof saturated with logical-mathematical content) is characterized by a marked complication of the procedures for demonstrating its reliability, and

this imposes strict requirements with regard to making intermediate stages of the formal derivation explicit. Proof is necessary when and where the path paved toward new knowledge is being tested for reliability, optimality, and compactness.

We have arrived at a period in science when the reproduction (= verification) of a proof is often a laborious task. Thus, the verification of proofs constructed with the aid of a computer requires the compilation of programs more complex and "capacious" than the program being verified, and in order to repeat analytical calculations it is in fact necessary almost fully to reconstruct the formal derivation. "I get the impression," Uspenskii comments, "that with the development of mathematics (and the appearance of ever lengthier and more complicated proofs) proofs are losing their chief property—that of persuasiveness. It is hard to see what then remains of proof, for persuasiveness is, as it were, part of its definition. In addition, as proof becomes more complicated its subjective element also grows. . . . With excessive expansion in the length of proofs, the very concept of proof becomes diffuse—just as the concept of natural number becomes diffuse as one progresses to extremely large numbers."[42]

Some mathematicians have noted the curious fact that it is often easier to prove general propositions than to prove special cases contained in them.

3.1. By offering a proof and elaborating a system of argumentation, a scientist *takes responsibility* for the certainty, reliability, and (in principle) reproducibility (within the limits of accepted abstractions, idealizations, and means of argumentation) of the new knowledge obtained by him. Knowledge that for one reason or another is nonreproducible is, so to say, unique; it lies outside of science and, in essence, outside of morality until it can be proved or derived with the aid of standards used by a scientific community, until means of argumentation encompass its conceptual structure.

Overcoming the gap between a new result and its acquisition of the *status of a proof* in the eyes of a scientific community is a difficult and contradictory process. One lever (but not the only one, and perhaps not the most effective one) that helps to integrate a fundamentally new result into the conceptual structure of a science is dialogue, which takes the form of debate, controversy, and the exchange of opinions. Of course, dialogue is not always an effective channel of communication and mutual understanding, but in any case it adequately clarifies points of agreement and disagreement. The opposing sides may be separated by different types of worldview, by differences in means of argumentation and in styles of thinking, and by the abstractions used being of different orders. In general, however, levels of abstraction and generalization being of the same order is no guarantee of the consolidation of opposed positions in the process of dialogue.

It is precisely discussions of this kind (in particular, at seminars), conducted in a spirit of critical reflection, that make it possible to fill in gaps and to discover and correct errors, which can be made even by very prominent mathematicians. Thus, by correcting proofs the mathematical community in fact shares responsibility with their authors.

As is well known, "*in mathematics errors play no less of a role than proofs*: by analyzing their causes and ways of overcoming them, it is possible to make faster progress than by obtusely trying to move forward in a little-studied direction."[43]

New proofs of already known results are also motivated by considerations of further improvement (the search for a more compact and/or elegant procedure or for a different form associated, for instance, with new techniques).

Shifts in the criteria for appraising mathematical proofs: The Simon syndrome

3.2. As a consequence of methods of proof becoming more complicated, the center of gravity in the process of verifying the reliability of a supposedly proved result is being transferred to *indirect* considerations—for example, appraisals of how closely the result corresponds to general expectations, to the developmental tendencies of a theory or research program, or to such criteria as the continuity and simplicity of knowledge, the reputation of the school to which the mathematician concerned is affiliated, and the authority of his scientific leader or of reviewers. In other words, there is a shift under way in the appraisals used in the procedures for verifying proved results—a shift toward social and ethical aspects. This, in particular, is why "the proofs of one generation are viewed by another generation as a heap of logical errors. . . . In our time the concept of rigor depends on which school a mathematician belongs to."[44]

All this means, obviously, a considerable increase in the responsibility borne by the author of a proof. *The external requirements that are imposed on a proof by a scientific community through a complex chain of mediations are being transformed into the internal basis of the motivation of the (potential) author of the proof*. To the aggregate of methodological, heuristic, and paradigmatic regulators of the search for scientific truth are added such regulators as the scientist's sense of inner responsibility, his conscience, capacity for self-criticism, and lack of bias. These qualities are all the more important in view of the fact that "mathematicians are much more concerned with proving their own theorems than with searching for errors in the proofs of others. . . . In reality, a mathematician does not rely on rigorous proof as much as people usually think."[45]

However, the circumstance that there is now a tendency to verify the correctness of a proof with the aid of indirect considerations not only adds new socioethical regulators of the search for scientific truth, but also poses serious socioethical problems whose methodological implications need to be understood. One of these problems is conventionally known as the "Simon syndrome."

The acuity of this problem was, perhaps, fully felt for the first time after the computer-aided proof of the four-color theorem. In fact, this was the first important theorem proved with the use of machinery. While the premachine (precomputer) part of the proof consisted of an analytical procedure expressed in a visible albeit very complex text, the machine (computer) part of the proof—certainly the decisive link—was generated autonomously from the rest of the proof and could not be printed out as a visible text. The operator was unable to give a guarantee that during the computations (the proof of the four-color theorem took up over 1,200 hours of machine time in 1976–77) there had been no glitches that could have distorted the final result. Thus, a proof analogous in nature to a proof of the four-color theorem turned out to be essentially nonreproducible—that is, it did not satisfy a requirement that applies to any scientific or mathematical result.[46]

The "Simon syndrome" is a consequence of precisely that situation (an unfortunate one, probably, from the classical point of view) in which a proof (or part of a proof) is nonreproducible. This situation enables a scientist who has once acquired authority through strong results obtained with the aid of traditional—reproducible—proofs to rely no longer on methods that permit direct verification and whose reliability is generally recognized, but on—let us say—his own authority (or the reputation of the school with which he was formally affiliated). "*We should never allow ourselves . . . to be hypnotized by authorities; the essence of the matter is more important than the authoritative status of a classical formulation!*" Arnold emphasizes.[47] Then, for example, he may omit a proof in the full knowledge that his own past reputation will ensure that the scientific community accepts a thesis (theorem, calculation, model, etc.) that he puts forward. A similar situation arises when an authoritative scientist gives a positive review of a manuscript or dissertation that he has not studied carefully or on the basis of other extrascientific considerations (for instance, personal inclination or trust). Here we have scientific dishonesty that leads in the final analysis to a declining level of professionalism, to devaluation of the criteria of scientific truth, and to erosion of the contours of the scientific community. As a result, the only permissible considerations in any appraisal of a scientist—considerations connected with objective truth—are pushed aside. There arises the phenomenon of shadow science.[48] As is well known, truth is also a moral value.

The specific procedural characteristics of mathematical proofs

4. Although they share a universal foundation, proofs differ in terms of their procedures—more precisely, their "procedural" components.

We may distinguish the following components of a proof:

—a *declaratory* component, which pertains to the *object* of the proof;

—a *narrowly procedural* component, which is determined by *how and by what means* the proof is constructed;

—an *explanatory* component, which shows *for what "reasons" and on what grounds* the given theorems are true;

—an *enthymematic* component, which is characterized by the "gaps" (deliberate and/or unintended) in the proof; and

—a *rigoristic* component, which is given by the level of rigor adopted in the proof and is most susceptible to revision in the course of time (this, of course, also entails the revision of details in the narrowly procedural component).

It is especially important to note the role played by didactic considerations in the *narrowly procedural* components of the search for and acquisition of new mathematical knowledge.

The history of mathematics shows quite clearly that major breakthroughs, and sometimes also revolutions, in mathematics are often connected with the writing of textbooks. We may recall the history of the discovery of non-Euclidean geometry by Nikolai Lobachevsky or of the establishment of analysis by Augustin Cauchy. And this is natural: didactic goals demand a presentation that is as accessible and transparent as possible—one in which there is no place for "gaps" in proofs, discrepancies, vagueness, and so on. The authors of textbooks are therefore compelled to think proofs through more deeply and thoroughly than usual, and this not infrequently leads to the emergence of new points of growth of mathematical knowledge. While apparently pursuing purely didactic goals, they in fact accomplish a task of supreme scientific importance—the search for new knowledge.

These considerations, however, do not touch on the mutual relations of teacher and pupil, which are not confined within the bounds set by the transmission of skills and knowledge in the narrow sense but are shaped by a multitude of personal factors. Thus, Charles Hermite, who was—as it is put nowadays—the scientific leader of Henri Poincaré, was extremely displeased by Poincaré's unwillingness to take heed of his advice and to polish and publish complete proofs. Hermite even removed Poincaré from his position as a lecturer.

It is an irony of history that the name of Hermite already belongs in many ways to the past of mathematics, while many branches of mathematics in the second half of the twentieth and the early twenty-first century have developed within the context of Poincaré's ideas.

This is no cause for surprise: norms of proof have changed since the nineteenth century, as has the sociopsychological context within which mathematics develops. Nevertheless, I think that despite their direct dependence on a scientific community the ethical meaning and purpose of proof are extratemporal in character.

Notes

1. G.A. Brutian, "Argumentatsiia," *Voprosy filosofii*, 1982, no. 11, p. 47.

2. I.T. Kasavin, *Tekst. Diskurs. Kontekst* (Moscow, 2008), p. 64.

3. M. Klain [Kline], *Matematika. Utrata opredelennosti* (Moscow, 1984), p. 359.

4. See V.V. Tselishchev, "Epistemicheskie kriterii dokazatel′stva," *Filosofiia nauki*, 2006, no. 4, pp. 20–44; V.V. Tselishchev, *Epistemologiia matematicheskogo dokazatel′stva* (Novosibirsk: Parallel′, 2006); A. Jaffe, "Proof and Evolution of Mathematics," *Synthèse*, 1997, vol. 3, no. 2, pp. 133–46; J. Lucas and M. Redhead, "Truth and Provability," *British Journal for the Philosophy of Science*, 2007, vol. 58, pp. 331–32; G. Lolli, "Experimental Methods in Proof," in R. Lupacchini and G. Corsi, ed., *Deduction, Computation, Experiment: Exploring the Effectiveness of Proof* (Dordrecht: Springer, 2008); M. MacEvoy, "The Epistemic Status of Computer-Assisted Proofs," *Philosophia Mathematica*, 2008, vol. 16, pp. 374–87.

5. J. Glimm, "Mathematical Perspectives: Reflections and Prospects," *Bulletin of the AMS*, 2010, vol. 47, no. 1, pp. 127–36.

6. P.J. Cohen, "Skolem and Pessimism About Proof in Mathematics," *Philosophical Transactions of the Royal Society A (Mathematical, Physical and Engineering Sciences)*, 2005, vol. 363, pp. 2414, 2418.

7. L.J. Rips, *The Psychology of Proof* (Cambridge: MIT Press, 1994); L.J. Rips, "Logical Approaches to Human Deductive Reasoning," in J.E. Adler and L.J. Rips, ed., *Reasoning: Studies of Human Inference and Its Foundations* (New York: Cambridge University Press, 2008), pp. 187–205.

8. I. Anellis, "Mathematical Proof *vs*. Logical Proof," in *Filosofiia matematiki. Aktual′nye problemy. Tezisy vtoroi mezhdunarodnoi konf. 28–30 maia 2009 v MGU* (Moscow: Maks-Press, 2009), pp. 154–55; A. Bundy, M. Jamnik, and A. Fugard, "What Is Proof?" *Philosophical Transactions of the Royal Society A (Mathematical, Physical and Engineering Sciences)*, 2005, vol. 363, pp. 2377–91; B. Lowe and T. Muller, *Degrees of Belief and Knowledge in Mathematics*. Preprint X-2004-03 (Amsterdam: ILLC Publications, 2004).

9. U.A.O. Kuain [W.V.O. Quine], "Osnovaniia matematiki," in *Matematika v sovremennom mire* (Moscow, 1967), p. 108.

10. Quoted from J.C. Webb, *Mechanism, Mentalism and Metamathematics: An Essay on Finitism* (Dordrecht: Springer, 1980), p. ix.

11. V.A. Uspenskii, "Sem′ razmyshlenii na temy filosofii matematiki," in *Zakonomernosti razvitiia matematiki* (Moscow, 1987), p. 139; see also V.A. Uspenskii, *Trudy po nematematike* (Moscow: OGI, 2002), p. 95; V.A. Uspenskii, *Prosteishie primery matematicheskikh dokazatel′stv* (Moscow: Izd-vo MTsNMO, 2009), p. 7.

12. V.I. Arnold, *Teoriia katastrof* (Moscow, 1983), p. 10.

13. R. Thom, "Comments," *Bulletin of the AMS*, 1993, no. 1, p. 25.

14. Klain, *Matematika*, pp. 193, 197.

15. J. Grabiner, "Is Mathematical Truth Time-Dependent?" *Mathematical Monthly*, 1974, vol. 81, no. 4, p. 358.

16. I. Lakatos, "Deduktivistskii *versus* evristicheskii podkhod," *Epistemologiia i filosofiia nauki*, 2009, vol. 20, no. 2, pp. 211, 213; see also V.A. Bazhanov, "Neizvestnyi Lakatos," *Epistemologiia i filosofiia nauki*, 2009, vol. 20, no. 2, pp. 204–209.

17. I. Lakatos, "Protsedury dokazatel′stva v sovremennom matematicheskom analize," *Voprosy filosofii*, 2009, no. 8, p. 97; see also V.A. Bazhanov and I. Pereosmyslivaia, "Lakatosa zanovo," *Voprosy filosofii*, 2009, no. 8, pp. 92–7.

18. Quoted from Y. Rav, "A Critique of a Formalist-Mechanist Version of the Justification of Arguments in Mathematicians' Proof Practices," *Philosophia Mathematica*, 2007, vol.15, p. 291.

19. Lowe and Muller, *Degrees of Belief*, p. 3.

20. R. Hersh, "Proving is Convincing and Explaining," *Educational Studies in Mathematics*, 1993, vol. 24, no. 4, p. 391.

21. D. Fallis, "Intentional Gaps in Mathematical Proofs," *Synthesis*, 2003, vol. 134, pp. 45–69.

22. See G. Priest, "60% Proof: Lakatos, Proof, and Paraconsistency," *Australian Journal of Logic*, 2007, no. 5, p. 89.

23. It is worth noting that at that point in the development of information technology 3 GB was an impressive volume. This proof already has an analogue in the form of a formal proof.

24. J. Harrison, "Formal Proof—Theory and Practice," *Notices of the AMS*, 2008, vol. 55, no. 11, pp. 1413–14. The first revolution, in Harrison's opinion, resulted from the discovery of the procedure of proof as such in ancient Greece and the appearance of Euclid's *Elements*. The second revolution consisted in the rigorous substantiation of analysis by Augustin-Louis Cauchy.

25. See D. Ullman's review of the film: "Review of *Proof*," *Notices of AMS*, 2006, vol. 53, no. 3, pp. 340–42. The prototype for the chief hero of the play *Proof*, as for Sylvia Nasar's book and Ron Howard's sensational film *A Beautiful Mind* (in the Russian version—*Games of the Mind* [Igry razuma]), was John Nash, a Nobel Laureate in Economics and brilliant mathematicians who suffered from schizophrenia.

26. *Informal Logic* (Inverness, CA), 1980, p. x.

27. M. Scriven, "The Philosophical and Pragmatic Significance of Informal Logic," *Informal Logic* (Inverness, CA: Edgepress), 1980, p. 148. See also I.N. Griftsova, *Logika kak teoreticheskaia i prakticheskaia distsiplina* (Moscow: URSS, 1998).

28. M. Nathanson, "Desperately Seeking Mathematical Truth," *Notices of the AMS*, 2008, no. 7, p. 773.

29. M. Nathanson, "Desperately Seeking Mathematical Truth," *The Mathematical Intelligencer*, 2009, vol. 31, no. 2, pp. 9–10.

30. See L. Grekhem [Graham], "Imeiut li matematicheskie uravneniia sotsial′nye atributy?" *Naukovedenie*, 2004, no. 4, pp. 121–31.

31. See also V.A. Bazhanov, "Logiko-matematicheskoe dokazatel′stvo v sotsial′nom kontekste," in *Chelovek, filosofiia, kul′tura*, issue 3 (Moscow, 1984), pp. 56–60; V.A. Bazhanov, "Argumentatsiia, dokazatel′stvo i normy nauki. Eticheskii i psikhologicheskii podtekst diskussii Bora i Einsteina," in *Filosofskie problemy argumentatsii* (Yerevan, 1986), pp. 427–36.

32. V.A. Uspenskii, "Sem′ razmyshlenii na temy filosofii matematiki," in *Zakonomernosti razvitiia matematiki* (Moscow, 1987), p. 140.

33. R. Hersh, "Proving Is Convincing and Explaining," *Educational Studies in Mathematics*, 1993, vol. 24, no. 4, p. 389.

34. W.P. Thurston, "On Proof and Progress in Mathematics," *Bulletin of the AMS*, 1994, vol. 30, no. 2, pp. 170, 173.

35. E.R. Weintraub and T. Gayer, "Equilibrium Proofmaking," *Journal of the History of Economic Thought*, 2001, vol. 23, no. 4, p. 440.

36. See, for example: J. Harrison, "Formal Proof—Theory and Practice," *Notices of the AMS*, 2008, vol.55, no. 11, p. 1403.

37. V.Ia. Perminov, *Razvitie predstavlenii o nadezhnosti matematicheskogo dokazatel′stva* (Moscow, 1986), pp. 5, 151.

38. S.V. Meien, "Printsip sochuvstviia," in *Puti v neznaemoe. Pisateli rasskazyvaiut o nauke* (Moscow, 1977), p. 418.

39. See V.A. Bazhanov, *Nauka kak samopoznaiushchaia sistema* (Kazan, 1991); V.A. Bazhanov, "Proof as an Ethical Procedure," in E. Agazzi and F. Minazzi, ed., *Science and Ethics: The Axiological Contexts of Science* (New York: Peter Lang, 2008), pp. 185–93.

40. Klain, *Matematika*, p. 363.

41. Uspenskii, "Sem′ razmyshlenii," pp. 147–48.

42. Ibid., pp. 150–51.

43. Arnold, *Teoriia katastrof*, p. 46; emphasis added.

44. Klain, *Matematika*, p. 367.

45. Ibid., p. 361.

46. In 2004 the four-color theorem was proved again using a computer—this time in the form of a formal proof.

47. V.I. Arnold, *Chto takoe matematika?* (Moscow, 2004), p. 70.

48. See Bazhanov, *Nauka*, ch. 4.2; A.V. Iurevich, "Tenevaia nauka.ru," *Vestnik RAN*, 2006, vol. 76, no. 3, pp. 234–41.