Project 1.4

Separable Equations and the Logistic Equation

If a separable differential equation is written in the form $f(y)\,dy = g(x)\,dx$, then its general solution can be written in the form

$$\int f(y)\,dy = \int g(x)\,dx + C.$$

Thus the solution of a separable differential equation reduces to the evaluation of two indefinite integrals. Hence it is tempting to use a computer algebra system such as *Maple* or *Mathematica* that can compute such integrals symbolically.

We illustrate this approach using the *logistic differential equation*

$$\frac{dx}{dt} = ax - bx^2 \tag{1}$$

that models a population $x(t)$ with births (per unit of time) proportional to x and deaths proportional to x^2. If $a = 0.01$ and $b = 0.0001$, for instance, Eq. (1) is

$$\frac{dx}{dt} = 0.01x - 0.0001x^2 = \frac{x}{10000}(100 - x). \tag{2}$$

Separation of variables leads to

$$\int \frac{dx}{x(100-x)} = \int \frac{dt}{10000} = \frac{t}{10000} + C. \tag{3}$$

Any computer algebra system gives a result of the form

$$\frac{1}{100}\ln(x) - \frac{1}{100}(x - 100) = \frac{t}{10000} + C. \tag{4}$$

You can now apply the initial condition $x(0) = x_0$, combine logarithms, and finally exponentiate in order to solve (4) for the particular solution

$$x(t) = \frac{100\,x_0 e^{t/100}}{100 - x_0 + x_0 e^{t/100}} \tag{5}$$

of (2). The direction field and solution curves shown in Fig. 1.4.11 in the text suggest that, whatever is the initial value x_0, the solution $x(t) \to 100$ as $t \to \infty$. Can you use (5) to verify this conjecture?

The sections that follow illustrate the use of *Maple*, *Mathematica*, and MATLAB to carry out the procedure outlined above. You might warm up for the investigation below by applying a computer algebra system to solve Problems 1-28 in Section 1.4 of the text.

Investigation

For your own personal logistic equation, take $a = m/n$ and $b = 1/n$ in (1), with m and n being the *largest* two distinct digits (in either order) in you student ID number.

(i) First generate a slope field for your differential equation and include a sufficient number of solution curves that you can see what happens to the population as $t \to \infty$. State your inference plainly.

(ii) Next, use a computer algebra system to solve the differential equation symbolically, and use the symbolic solution to find the limit of $x(t)$ as $t \to \infty$. Was your graphically-based inference correct?

(iii) Finally, state and solve a numerical problem using the symbolic solution. For instance, how long does it take x to grow from a selected initial value x_0 to a given target value x_1?

Using *Maple*

First we integrate both sides of our separated differential equation as in Eq. (3).

```
soln := int(1/(x*(100-x)),x) = int(1/10000,t)+C;
```

$$soln := \frac{1}{100}\ln(x) - \frac{1}{100}\ln(-100+x) = \mathrm{int}(1/10000,t) + C$$

Then we apply the initial condition $x(0) = x0$ to find the constant C.

```
C := solve(subs(x=x0,t=0,soln),C);
```

$$C := \frac{1}{100}\ln(x0) - \frac{1}{100}\ln(-100+x0)$$

We substitute this value of C and simplify.

```
soln := simplify(100*soln);
```

$$soln := \ln(x) - \ln(-100+x) = \frac{1}{100}t + \ln(x0) - \ln(-100+x0)$$

Next we exponentiate both sides of this equation.

```
soln := simplify(exp(lhs(%)) = exp(rhs(%)));
```

$$soln := \frac{x}{-100+x} = \frac{e^{\left(\frac{1}{100}t\right)} x0}{-100+x0}$$

Finally we solve explicitly for x as a function of t,

```
x(t) = solve(soln, x);
```

$$x(t) = 100\,\frac{e^{\left(\frac{1}{100}t\right)} x0}{100 - x0 + e^{\left(\frac{1}{100}t\right)} x0}$$

as in Eq. (5) above.

Using *Mathematica*

First we integrate both sides of our separated differential equation as in Eq. (3).

```
soln =
Integrate[1/(x(100-x)),x] == Integrate[1/10000,t] + c
```

$$\frac{\log(x)}{100} - \frac{1}{100}\log(x-100) == c + \frac{t}{10000}$$

Then we apply the initial condition $x(0) = x0$ to find the constant c.

```
c = First[ soln /. {t->0, x->x0} ]
```

$$\frac{\log(x0)}{100} - \frac{1}{100}\log(x0-100)$$

We substitute this value of c and simplify.

```
soln =
Expand[100*First[soln]] == Expand[100*Last[soln]]
```

$$\log(x) - \log(x-100) == \frac{t}{100} - \log(x0-100) + \frac{1}{100}\log(x0)$$

Next we exponentiate both sides of this equation.

```
soln =
Exp[First[soln]] == Exp[Last[soln]] // Simplify
```

$$\frac{x}{x-100} == \frac{e^{t/100}x0}{x0-100}$$

Finally we solve explicitly for x as a function of t,

```
soln = Solve[ soln, x ];
```

$$\left\{\left\{x \to \frac{100\,e^{t/100}x0}{e^{t/100}x0 - x0 + 100}\right\}\right\}$$

```
x = First[x /. soln]
```

$$\frac{100\,e^{t/100}x0}{e^{t/100}x0 - x0 + 100}$$

as in Eq. (5) above.

Using MATLAB

Here we solve the logistic equation in (2) using the MATLAB "symbolic toolbox" interface to the *Maple* kernel. We begin by separating variables and integrating each side of the resulting equation. However, it is more convenient now to work with "everything on one side of the equation", as in

$$\int \frac{dx}{x(100-x)} - \int \frac{dt}{10000} - C = 0.$$

So we start by "declaring" our symbolic variables and evaluating the two integrals in this equation.

```
syms x t C
soln = int(1/(x*(100-x)),x) - int(1/10000, t) - C

soln =
1/100*log(x)-1/100*log(-100+x)-1/10000*t-C
```

We are actually thinking here of the equation **`soln = 0`**, but the right-hand side zero is suppressed throughout. It simplifies the equation a bit by multiplying through by 100.

```
soln = 100*soln

soln =
log(x)-log(-100+x)-1/100*t-100*C
```

Then we apply the initial condition $x(0) = x0$ to find the constant C.

```
soln0 = subs(soln, {t,x}, {0,'x0'})

soln0 =
log(x0)-log(-100+x0)-100*C

C = solve(soln0, C)

C =
1/100*log(x0)-1/100*log(-100+x0)
```

We substitute this value of C simply by evaluating the present implicit solution.

```
soln = eval(soln)

soln =
log(x)-log(-100+x)-1/100*t-log(x0)+log(-100+x0)
```

Finally we solve explicitly for x as a function of t,

```
x = solve(soln, x);
pretty(x)

                      x0 exp(1/100 t)
             100 --------------------------
                 100 - x0 + x0 exp(1/100 t)
```

as in Eq. (5) above.