Lab 1 - Direction fields and Solution Curves, Part 1

These notes basically are from the IODE project and are written by R. Laugesen

The main object of study in this unit is the behavior of solutions to **first order ordinary differential equation**:

$$\frac{dy}{dx} = y' = f(x, y), \tag{0.1}$$

where x is the independent variable, $y(x)$ is the unknown function or solution, and $f : \mathbb{R}^2 \to \mathbb{R}$ is the given function that determines the rate of change of y at each point (x, y).

Example 0.1.

$$f(x, y) = 5, \text{ a constant} \tag{0.2}$$

What are the solutions? There are an infinite family $y(x) = 5x+C$, for any constant $C \in \mathbb{R}$.

Think of the (x, y) plane. A solution of (0.1) is a curve $(x, y(x))$ that satisfies the relation (0.1). What type of curve? If we are at the point $(x_0, y(x_0))$ on the curve, then when we increase x_0 to $x_0 + \delta x$, the curve changes in such a way that the slope at $(x_0, y(x_0))$ is given by the right side of (0.1): $f(x_0, y_0)$:

$$y(x_0 + \delta x) - y(x_0) \approx [f(x_0, y_0)(\delta x). \tag{0.3}$$

Usually there is no formula for $y(x)$ given f. But, using the idea discussed above, we can graphically construct a solution curve, at least approximately.

1. Direction fields

A solution of the ODE (0.1) is a function $y(x)$. Think of this as a curve in the (x, y)-plane. The slope of the curve at every point is given by $f(x, y)$. The **direction field** associated with the ODE (0.1) is constructed as follows: At each point (x, y) draw a *short* straight line segment *with slope given by the number* $f(x, y)$. Do this for many points in the plane. This collection of arrows is a sketch of the **direction field**.

Go to IODE and look at the *direction field* plots. Play with the form of f.

Action 1. Consider the ODE

$$y' = y, \tag{1.1}$$

so $f(x, y) = y$. Notice that f is independent of x so the small arrow at (x, y) does not depend on x.

(1) By hand, sketch the direction field. Do this at $(1, 2)$. The slope is 2. Make an arrow with its head upward. Do this at $(1, 1)$, at $(1, 1/2)$, and at $(1, 0)$. Now sketch the arrows at some points $(x, 1)$, $(x, 2)$ and $(x, 1/2)$. Since f is independent of x these are all parallel arrows.
(2) Continue to fill in the sketch of the direction field.
(3) Now imagine that $y(x)$ is a solution that passes through the point $(0, 1)$. Try to sketch the solution curve that passes through this point using the fact that the tangent line to the curve at any point (x, y) has slope $f(x, y)$. What does the curve look like?
(4) Now go to IODE and look at the direction field for this f and plot the solution. What is the exact solution in this case?

2. Plotting the Direction Field

IODE is a great tool for sketching a lot of the arrows of a direction field. This gives one an idea of how the solution curves behave without actually solving the ODE.

Action 2.

(1) Start up IODE in MatLab.
(2) Choose the first menu item Direction Fields from the Main Menu. Wait for a graphics window to open. It shows the direction field of a default ODE $y' = \sin(y - x)$.
(3) Stare at this graph. Look at the diagonal $y = x$. What is the slope there? It is zero. How do the lottle arrow lie? They are horizontal.
(4) Look at the x-axis ($y = 0$). Then, $f(x, 0) = -\sin x$. Can you see the oscillations as x varies through the interval $[-\pi, \pi]$? What happens for $x \in [\pi, 3\pi]$? And beyond?
(5) Look at the y-axis ($x = 0$) and analyze how the direction field is changing.

3. Plotting a Solution

Once one has a sketch of the direction field, one can draw in a solution curve by choosing a point (an *initial condition*) and advancing the curve so that the arrows of the direction field remain tangent to the curve. Since we are solving first order ODEs, there is one arbitrary constant so an infinite family of solution curves. You can see this by staring at the plot of the direction filed.

To fix the constant, you need to impose a condition: "I look for the solution curve that passes through the point (x_0, y_0). To sketch the curve, find a point, say $(1, 2)$ on the IODE plot of the direction field, and imagine the curve progressing from that point with the arrows tangent to the curve. IODE can do this for us!

Action 3.

(1) Go back to the Direction Fields window of IODE. Type some initial conditions into the box (the default is $(0, 0)$). Choose coordinates within the viewing window. Click on Plot Solution. You can choose the color of the solution curve.
(2) Look at the plotted solution curve. Can you see that the arrows of the direction field are tangent to the curve at each point? The graph that appears is a numerical approximation to the true solution. We will learn about numerical approximations later.
(3) IODE has another feature: If you put the cursor on a point in the plot of the direction field and click. IODE will plot the solution passing through that point.
(4) Play with the direction fields module of IODE, trying out the different control buttons and menu items, figuring out what they do. (Note. Ignore any "Other" options.) Time spent now learning the tricks of IODE will make you a "power user" later in the semester. Things you can try:
    - Plot more solutions, then right-click in the graph and undo them.
    - Left-drag the mouse to create a zoom box over part of the direction field. Then undo your zoom with a right-click.

- Enter a new ODE, using the Equation menu. See the end of this Lab for functions you can use, and for hints on valid Matlab syntax.
- Use the Equation menu to change the display parameters, in particular to increase the number of line segments. Is the direction field easier to understand with 30 segments in each direction? 50? How many seems the "right" number?
- Try relabeling the variables from (x, y) to (t, x). Many of the equations we will study are in terms of t, meaning time, and x meaning position.
- Try out the Options features, especially regarding zooming out. Then add a caption to your plot.
- Increase the step size to 0.5 or 1, and plot some solutions. Do these graphs help explain why this parameter is called "step size"?
- Print your graph, using the File Menu.
- Save your work as a file ending in .mat, then Open it up again using the File menu within the IODE window.