CHANCE and STABILITY
Stable Distributions and their Applications

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Fluctuat nec mergitur

Contents

Foreword

This book is the third in the series of monographs “Modern Probability and Statistics” following the books

- V. M. Zolotarev, *Modern Theory of Summation of Random Variables*;
- V. V. Senatov, *Normal Approximation: New Results, Methods and Problems*.

The scope of the series can be seen from both the title of the series and the titles of the published and forthcoming books:

- Yu. S. Khokhlov, *Generalizations of Stable Distributions: Structure and Limit Theorems*;
- V. E. Bening, *Asymptotic Theory of Testing Statistical Hypotheses: Efficient Statistics, Optimality, Deficiency*;
- N. G. Ushakov, *Selected Topics in Characteristic Functions*.

Among the proposals under discussion are the following books:

- G. L. Shevlyakov and N. O. Vilchevskii, *Robust Estimation: Criteria and Methods*;
- V. E. Bening, V. Yu. Korolev, and S. Ya. Shorgin, *Compound Doubly Stochastic Poisson Processes and Their Applications in Insurance and Finance*;
- E. M. Kudlaev, *Decomposable Statistics*;
- G. P. Chistyakov, *Analytical Methods in the Problem of Stability of Decompositions of Random Variables*;
- A. N. Chuprunov, *Random Processes Observed at Random Times*;
- D. H. Mushtari, *Probabilities and Topologies on Linear Spaces*;
- V. G. Ushakov, *Priority Queueing Systems*;

- E. V. Morozov, *General Queueing Networks: the Method of Regenerative Decomposition*;
- V. Yu. Korolev and V. M. Kruglov, *Random Sequences with Random Indices*;
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- A. V. Bulinskii and M. A. Vronskii, *Limit Theorems for Associated Random Variables*;
- E. V. Bulinskaya, *Stochastic Inventory Systems: Foundations and Recent Advances*;

as well as many others.

To provide high-qualified international examination of the proposed books, we invited well-known specialists to join the Editorial Board. All of them kindly agreed, so now the Editorial Board of the series is as follows:

We hope that the books of this series will be interesting and useful to both specialists in probability theory, mathematical statistics and those professionals who apply the methods and results of these sciences to solving practical problems.

In our opinion, the present book to a great extent meets these requirements. An outbreak of interest to stable distributions is due to both their analytical properties and important role they play in various applications including so different fields as, say, physics and finance. This book, written by mathematician V. Zolotarev and physicist V. Uchaikin, can be regarded as a *comprehensive introduction* to the theory of stable distributions and their applications. It contains a modern outlook of the mathematical aspects of this theory which, as we hope, will be interesting to mathematicians. On the other hand, the authors of this book made an attempt to explain numerous peculiarities of stable distributions using the language which is understandable not only to professional mathematicians and supplied the basic material of this monograph with a description of the principal concepts of probability theory and function analysis. A significant part of the book is devoted to applications of stable distributions. A very attractive feature of the book is the material on the interconnection of stable laws with fractals, chaos, and anomalous transport processes.

V. Yu. Korolev,
V. M. Zolotarev,
Editors-in-Chief

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Introduction

In our everyday practice, we usually call random such events which we cannot completely control, for one reason or another, and therefore we cannot forecast their occurrence. If the events are connected with some numerical characteristics, then these values are spoken about as random variables. With a desire to make a theoretical basis for the study of random variables, the probability theory formalizes all possible situations with the help of a great body of mathematical models, the simplest of which is the summation of independent random variables $X_1, X_2, \ldots, X_n$. One easily apprehends the origin of this construction by considering a smooth function

$$Y = f(X_1, X_2, \ldots, X_n)$$

of random variables $X_1, X_2, \ldots, X_n$ representing small and independent actions on the system under consideration. If $f(0, \ldots, 0) = 0$, then, in the first approximation,

$$Y = \sum_{i=1}^{n} c_i X_i,$$

where $c_i = \partial f / \partial x_i$ for $x_1 = x_2 = \ldots = x_n = 0$. If all

$$c_1 = c_2 = \ldots = c_n = c,$$

then

$$Y = c \sum_{i=1}^{n} X_i.$$

Similar situations take place while analyzing observational errors arising in experiments.

P.S. Laplace and C.F. Gauss, who developed the theory of observational errors in the beginning of the XIX century, understood this well, and exactly for this reason associated the error distribution with the scheme of summation of random variables. If one assumes nothing except their smallness, the theory gives nothing interesting concerning the distribution of the sum of these variables. However, the situation changes abruptly if one supposes that the terms are independent.

The first results concerning the scheme of summation of independent random variables appeared in the famous Jacob Bernoulli's book *Ars Conjectandi* published in 1713. J. Bernoulli considered the sequence of normalized sums $\frac{1}{n}\sum_{i=1}^{n} X_i$ where independent random variables X_i take the value one with probability p and the value zero with probability $1 - p$. According to Bernoulli's theorem, for any arbitrary small but fixed $\varepsilon > 0$ the probability

$$\mathsf{P}\left\{\left|\frac{1}{n}\sum_{i=1}^{n} X_i - p\right| > \varepsilon\right\} \to 0 \qquad n \to \infty.$$

It is hard to over-estimate the practical meaning of this theorem called now the Bernoulli's form of the law of large numbers. Indeed, if we perform an experiment with a random event and want to know the probability $p = \mathsf{P}\{A\}$ of some event A interesting for us, then, generally speaking, it is impossible to obtain p from theoretical considerations. The Bernoulli theorem establishes that the value of p can be estimated experimentally to an arbitrary accuracy if one observes the frequency of occurrence of the event A in a large enough number of independent and identically performed experiments.

The law of large numbers and its various generalizations and strengthenings connect together the theory and practice and help to deduce from experiments the information sought for.

The second significant result of the XVIII century was the Moivre–Laplace theorem that extends the Bernoulli theorem. It is a particular case (related to the random variables of Bernoulli's theorem) of the central limit theorem of probability theory. We do not give it here in the general form known nowadays, and dwell only upon a special case.

Let us consider a sequence of independent random variables $X_1, X_2, \dots$ possessing one and the same distribution function (in this case the variables are referred to as identically distributed). Assuming that the mathematical expectation $a = \mathsf{E}X_i$ and the variance $\sigma^2 = \operatorname{Var} X_i$ of these variables are finite, we construct the corresponding sequence of the normalized sums $Z_1, Z_2, \dots$:

$$Z_n = \frac{\sum_{i=1}^{n} X_i - na}{\sigma\sqrt{n}}. \tag{I1}$$

Then for any $x_1 < x_2$

$$\mathsf{P}\left\{x_1 < Z_n < x_2\right\} \Rightarrow \int_{x_1}^{x_2} p^{\mathrm{G}}(x)\,dx, \qquad n \to \infty, \tag{I2}$$

where[1]

$$p^{\mathrm{G}}(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2). \tag{I3}$$

[1]The symbol $\Rightarrow$ stands for the weak convergence, $p^{\mathrm{G}}(x)$ is the density of the standard normal, or Gauss, law.

Turning back to the above-discussed observational errors problem, we see that the central limit theorem forms the basis which the rigorous error theory can be erected on. This was well understood by the founders of the theory Laplace and Gauss, as well as by their successors. We can be proud of the fact that great achievements in the theory of limit theorems for sums of random variables (both independent and obeying some kinds of dependence) is associated primarily, with the names of our outstanding compatriots P.L. Chebychov, A.A. Markov, A.M. Lyapunov, S.N. Bernstein, A.N. Kolmogorov, A.Ya. Khinchin, B.V. Gnedenko, and others.

The central limit theorem has been extended into various directions. One of them was aimed to extend the understanding of the central limit theorem with the use of not only the normal law as a limiting approximation but also some other distributions of a certain analytical structure.

The formulation looks like follows. A sequence of independent and identically distributed random variables $X_1, X_2, \dots$ is considered, without any preliminary assumptions about their distribution. With the use of sequences of real-valued constants $a_1, a_2, \dots$ and positive constants $b_1, b_2, \dots$, the sums

$$Z_n = \frac{\sum_{i=1}^{n} X_i - a_n}{b_n} \tag{I4}$$

are introduced. We assume now that the constants a_n and b_n are chosen in an appropriate way so that the distribution functions of Z_n weakly converge to some limit distribution function $G(x)$, that is,

$$\mathsf{P}\left\{Z_n < x\right\} \Rightarrow G(x), \qquad n \to \infty \tag{I5}$$

for any x which is a continuity point of the function $G(x)$. A problem thus arises: how wide is the class of the distribution functions that can play the role of the limiting law?

The class, referred to as the stable law family later on, includes the standard normal law (I3) and, generally speaking, the whole family of normal laws with different expectations and variances.

If in sums (I4) the random variables X_i are supposed to be equal to one and the same constant c with probability one and the normalizing constants are $a_n = (n-1)c$, $b_n = 1$, then the sums Z_n for any n also turn out to be equal to the constant c with probability one. The distribution function of such random variables, expressed in terms of the step function $e(x)$

$$e(x-c) = \begin{cases} 0, & x \le 0, \\ 1, & x > 0, \end{cases}$$

is called degenerate at the point c and its graph resembles the form of the step function. Moreover, a random variable taking a single value with probability one does not depend on any other random variable. Therefore, all the random variables X_j in the case under consideration are mutually independent and the limit distribution for the sums Z_n, by definition, is to belong to the stable law family. In other words, the family includes all degenerate distribution functions.

We arrive at the same conclusion if we begin with the law of large numbers, since it states nothing but the weak convergence of arithmetical means of independent random variables to the degenerate distribution concentrated at the point $x = p$. So, the two most known and important limit theorems—the law of large numbers and the central limit theorem—are proven to be related to the stable laws. This explains the interest to the stable laws as to 'kins' of normal and degenerate distributions.

The problem to describe the class of stable laws appears to be not so simple. The first results concerning the problem were obtained by the eminent Hungarian mathematician G. Polya at the same time when A.A. Markov was creating the concept of the chain dependence of random variables. Among the facts collected by Polya, one is worthwhile to notice: only the normal law (excepting, of course, the degenerate law) possesses a finite variance; the variance of any other non-degenerate stable distribution is infinite.

The next considerable step forward was made by the French mathematician P. Lévy in 1925. In his book *Calcul des Probabilites*, a special chapter was devoted to this problem. P. Lévy separated the stable law family and described its large subclass of the so-called strictly stable laws. As limit laws, they appear in (I4) while the centering constants a_n can be chosen as $a_n = nc$ with some c. While describing this subclass, some delicate analytical properties of its elements, some intrinsic properties, etc., were exposed. P. Lévy reveal a functional equation which the strictly stable laws should satisfy. We briefly cite here the idea.

The degenerate distributions are not of interest for us while describing the stable laws, and we do not discuss them. Thus, we assume the distribution function $G(x)$ in (I5) to be non-degenerate. Recalling that $a_n = nc$, we divide the sum Z_n (I4) into two parts containing k and l terms ($n = k + l$), respectively, and rewrite it as

$$Z_n = \frac{b_k}{b_n} Z_k + \frac{b_l}{b_n} Z_l' \tag{I6}$$

where

$$Z_l' = (X_{k+1} + \ldots + X_{k+l} - lc)/b_l$$

is a random variable that does not depend on Z_k and is distributed by the same law as the sum Z_l. It is convenient to rewrite this as

$$Z_l' \stackrel{d}{=} Z_l.$$

We assume that k, l, and consequently, $n = k + l$ increase without bound.

By the initial assumption,

$$Z_n \stackrel{d}{\to} Y, \qquad n \to \infty. \tag{I7}$$

This notation means that the distribution functions of the random variables Z_n weakly converge to the distribution function $G(x)$ of the random variable Y. Denoting by Y_1 and Y_2 the independent random variables distributed by the same law as Y, we obtain, on the strength of (I7),

$$Z_k \stackrel{d}{\to} Y_1, \qquad Z'_l \stackrel{d}{=} Z_l \stackrel{d}{\to} Y_2. \tag{I8}$$

>From the sequences of k and l, one can choose subsequences such that the constant factors b_k/b_n and b_l/b_n converge to non-negative c_1 and c_2 respectively. Passing to the limit in (I6), in view of (I7) and (I8), we obtain

$$Y \stackrel{d}{=} c_1 Y_1 + c_2 Y_2. \tag{I9}$$

The partition of the summands into two groups can be made in an arbitrary way. This means that one can exclude the non-interesting cases where either c_1 or c_2 is equal to zero. The detailed analysis of the behavior of the normalizing constants reveals that the constants c_1 and c_2 can take any values with the only constraint

$$c_1^\alpha + c_2^\alpha = 1,$$

where $0 < \alpha \leq 2$; α is one of the characteristic parameters of a stable law.

The right-hand side of (I9) is the sum of independent random variables; hence its distribution function is a convolution of the distribution functions of the summands

$$G(x) = G(x/c_1) * G(x/c_2). \tag{I10}$$

This equality is the functional equation for strictly stable distribution functions; its peculiarity consists in the fact that it does not identify separate laws G but, while α, c_1, c_2 vary under the above constraints, the equality separates the strictly stable laws from the whole family of distribution laws in a quite obvious way.

The description of the remaining part of the stable laws was finished more than a decade later in the middle of thirties. It was made by P. Lévy himself and the Soviet mathematician A.Ya. Khinchin independently of each other and almost simultaneously. The functional equation whose solutions form the family of stable laws was not difficult to derive; it is obtained in the same way as functional equation (I10), and differs from (I10) but little:

$$G(x - h) = G(x/c_1) * G(x/c_2). \tag{I11}$$

Here c_1, c_2 obey the above constraints, and h is a real-valued constant.

All the stable laws, except the degenerate ones, are absolutely continuous, i.e., the corresponding distribution functions $G(x)$ possess the densities

$$q(x) = G'(x),$$

but, for a few exceptions, neither the distribution functions nor the densities can be explicitly expressed in terms of elementary functions.

The stable laws are adequately described in terms of the corresponding characteristic functions

$$g(k) = \int_{-\infty}^{\infty} e^{ikx} dG(x) = \int_{-\infty}^{\infty} e^{ikx} q(x)\, dx.$$

While looking at functional equations (I10) and (I11), a suspicion arises that they become more simple in terms of characteristic functions. For example, (I11) takes the form

$$g(k)\exp(ikh) = g(c_1 k) g(c_2 k).$$

While solving this equation, it appears that the set $\mathscr{G}$ is a four-parametric family of functions. Each stable law is defined by four parameters: the characteristic $0 < \alpha \le 2$, the skew parameter $-1 \le \beta \le 1$, the shift parameter $\lambda > 0$, and the scale parameter $-\infty < \gamma < \infty$. The corresponding characteristic function can be written in the following simple form:

$$g(k) = \exp\left\{\lambda[ik\gamma - |k|^{\alpha}\omega(k;\alpha,\beta)]\right\}$$

where

$$\omega(k;\alpha,\beta) = \begin{cases} \exp[-i\beta\Phi(\alpha)\operatorname{sign} k], & \alpha \neq 1, \\ \pi/2 + i\beta \ln|k| \operatorname{sign} k, & \alpha = 1, \end{cases}$$

$$\Phi(\alpha) = \begin{cases} \alpha\pi/2, & \alpha < 1, \\ (\alpha - 2)\pi/2, & \alpha > 1, \end{cases}$$

$$\operatorname{sign} k = \begin{cases} -1, & k < 0, \\ 0, & k = 0, \\ 1, & k > 0. \end{cases}$$

After the works due to P. Lévy and A.Ya. Khinchin, many works appeared that were devoted to investigation of stable laws; now we know much more about their peculiarities. During last decades, a bulk of new ideas become evident. The concept of stable laws was extended to the case of multidimensional and even infinite-dimensional distributions.

The multidimensional analogues of stable laws turn out to be much more sophisticated than its one-dimensional predecessors. First of all, in the problem of limit approximation of distributions of the sequences $\sum_{i=1}^{n} X_i$ of independent and identically distributed m-dimensional random vectors, the two problem formulations are possible.

The first is the same as in the one-dimensional case, i.e., the sequence S_n is preliminary normalized by subtracting some constant m-dimensional vector a_n divided by a positive b_n. In other words, the distributions of the sums

$$Z_n = \left(\sum_{i=1}^{n} X_i - a_n\right) \Big/ b_n$$

are considered, and the problem is to describe all possible limit distributions of such sums as $n \to \infty$. The resulting laws form the family of the Lévy–Feldheim distributions. P. Lévy was the first who investigated them in the thirties.

The second formulation is related to another type of normalization of the sums. Namely, instead of positive constants b_n^{-1}, the sequence of non-singular matrices B_n^{-1} of order m is used, i.e., the sequences of the sums

$$Z_n = B_n^{-1}\left(\sum_{i=1}^{n} X_i - a_n\right)$$

are considered. The class of possible limit distributions obtained under such formulation is naturally wider than the class of the Lévy–Feldheim laws, and is referred to as the class of operator stable laws.

In the multidimensional case, in contrast to the one-dimensional one, the family of stable laws is determined not by a finite number of parameters, but by three parameters and some probability distribution concentrated on the unit sphere. We present here only one particular case of the stable m-dimensional distributions called the spherically symmetric distributions.

Its characteristic function is of the form

$$g(k) = \exp(-\lambda|k|^{\alpha}),$$

where k is an N-dimensional vector, $0 < \alpha \le 2$, and $\lambda > 0$. The case $\alpha = 2$ corresponds to the spherically symmetric normal law with density

$$p_N^{\mathrm{G}}(x) = (4\pi\lambda)^{-N/2} \exp(-|x|^2/4\lambda),$$

and the case $\alpha = 1$, to the N-dimensional Cauchy distribution with density

$$p_N^{\mathrm{C}}(x) = \lambda\Gamma\left((1+N)/2\right)[\pi(\lambda^2 + |x|^2)]^{-(1+N)/2},$$

where x is an N-dimensional vector again. These two cases, up to a shift of the distributions and linear transformation of their arguments, are the only ones where explicit expressions for the densities of multidimensional stable distributions are known.

The detailed information about the theory of stable laws, including the multidimensional ones, is presented in the first part of this book. The second part is devoted to their applications.

In fact, a book about the applications of the stable laws in various branches of human knowledge should be impossible by the only reason: even the normal (Gaussian) law is so widely distributed over all spheres of human activity that more or less detailed description of this phenomenon would need a good deal of volumes and for this reason it is hardly worth to do it. However, as soon as one excludes it (together with the degenerate distribution) from consideration, the field of applications becomes much narrower. Indeed, the analysis of existing publications shows that there are only a few dozens of problems, mostly in physics and the related sciences, where the stable laws have been applied. Is it not surprising? All stable laws form the single family containing an infinite number of distributions, and the normal law is the only one (within linear transformations of argument) of them!

We see three reasons for this.

We believe that the first reason, absolutely inessential from a mathematician's viewpoint but of great importance for those who use this technique to solve his own particular problems, consists of the absence of simple analytical expressions for the densities of stable laws. That is the reason why the normal law is the first of stable laws which appeared on the scene and found at once the applications in the theory of errors, then in statistical physics, and in other sciences. The next stable law that appeared in theoretical physics was the Lorentz dispersion profile of a spectral line (Lorentz, 1906) known in the probability theory as the Cauchy law, i.e., the symmetric stable distribution with parameters $\alpha = 1$ and $\beta = 0$, whose density is of the simple form

$$p^{\mathrm{C}}(x) = \frac{1}{\pi(1 + x^2)}.$$

As it was noted in (Frank, 1935) the stable distribution with parameters $\alpha = 1/2$ and $\beta = 1$, called the Lévy distribution,

$$p^{\mathrm{L}}(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2x}\right\} x^{-3/2}$$

was obtained in 1915 by Smoluchowski and Schrödinger for first-passage times in the diffusion problem. In 1919, in Holtsmark's work (Holtsmark, 1919) devoted to the same problem as the Lorentz's one but with due account of random electric fields created by nearest neighbors of a radiating atom, the three-dimensional stable Cauchy distribution ($\alpha = 1$)

$$p_3^{\mathrm{C}}(\mathbf{x}) = \frac{1}{\pi^2[1 + |\mathbf{x}|^2]^2}$$

and two more three-dimensional spherically symmetric stable laws with characteristics $\alpha = 3/2$ (the Holtsmark distribution and $\alpha = 3/4$ were presented. The Holtsmark distribution and his approach were lately widely used in astrophysical evaluations (Chandrasekhar & Neumann, 1941).

Bringing 'new land' beyond the Gaussian law into cultivation went forward slowly. One can mention the Landau distribution in the problem of ionization losses of charged particles in a thin layer (Landau, 1944) (the stable law with $\alpha = 1$ and $\beta = 1$), the Monin distribution in the problem of particles diffusion in a turbulent medium (Monin, 1955; Monin, 1956) (three-dimensional symmetric stable law with the exponent $\alpha = 2/3$), the Lifshits distribution of temperature fluctuations under action of nuclear radiation (Lifshits, 1956) (one-dimensional stable law with $\alpha = 5/3$ and $\beta = 1$) and Dobrushin's result concerning the theory of signal detection (Dobrushin, 1958) (one-dimensional stable laws with $\alpha \geq 1$ and $\beta = 1$).

The second reason consists in that all non-degenerate stable laws, differing from the normal one, have infinite variance. As concerns the whole family of stable laws, the normal law is exceptional, something like a 'monster', an abnormal member of the family. However, from the point of view of the applied sciences, where a specific meaning is often associated with the variance, distributions with infinite variance look oddly indeed. Moreover, the random variable itself, whose distribution is under consideration, often lies in some natural limits, and hence cannot be arbitrary large (in absolute value). In this case, not only variance but all moments of higher orders are finite, and the normal law does satisfy this condition, too.

The above reasoning can lead us to the speculation that the difference between the fields of application of the normal law and of other stable laws is due to the objective reality. But a more deep brainwork inspires doubts about this conclusion. Indeed, in spite of the finiteness of all moments, the normal random variable is unbounded, and can take arbitrarily large (absolute) values, but cannot take infinite values in the sense that

$$\lim_{x\to\infty} \int_x^\infty p(x')\,dx' = \lim_{x\to-\infty} \int_{-\infty}^x p(x')\,dx' = 0.$$

In other words, the random variable distributed by the normal (as well as by any other stable) law takes finite values with probability one. Such random variables are called proper (Borovkov, 1976), and the whole family of stable laws possesses this property. The divergence of variances of stable laws different from normal is connected with the power-like behavior of the distribution tails:

$$\int_x^\infty p(x')\,dx' + \int_{-\infty}^{-x} p(x')\,dx' \propto x^{-\alpha}, \quad x \to \infty, \quad 0 < \alpha < 2.$$

The existence of finite variance of the normal law is connected with just a faster decrease of tails as compared with the others. Thus, as concerns their intrinsic structure, all the family members compete on equal terms, and only the common habit to use the variance as a characteristic of distribution makes many practical workers avoiding the mysterious infinities.

B. Mandelbrot (Mandelbrot, 1983, p. 338) wrote that to anyone with the usual training in statistics, an infinite variance seems at best scary and at worst bizarre. In fact, 'infinite' does not differ from 'very large' by any effect one could detect through the sample moments. Also, of course, the fact that a variable X has an infinite variance in no way denies that X is finite with a probability equal to one… Thus, the choice between variables should not be decided a priori, and should hinge solely on the question of which is more convenient to handle. we should accept infinite variance because it makes it possible to preserve scaling.

One more property of the normal law, important to applications, can attract our attention. It concerns the multidimensional distributions: the multidimensional spherically symmetric normal law is factorized into one-dimensional normal distributions. In other words, the components of the multidimensional vector are distributed by the one-dimensional normal law and are independent. It is easy to demonstrate that in the case of any other spherically symmetric stable law the vector components are also distributed by the one-dimensional stable law with the same characteristic α but they need not be independent. We recall that the independence property for components of normal vector is used for elementary deduction of Maxwell velocity distribution in kinetic theory of gases (Maxwell, 1860), but in the general probabilistic construction of statistical mechanics developed by A.Ya. Khinchin (Khinchin, 1949) it arises merely as a consequence of the postulated variance finiteness.

And, finally, the possibly most trivial but, at the same time, the most important reason for the slow expansion of stable laws in applied problems is the shortage of knowledge. Up to nowadays, as a rule, the information about the stable laws is absent in the textbooks on the probability theory and can be found only in some monographs (Gnedenko & Kolmogorov, 1954; Loève, 1955; Lukacs, 1960; Feller, 1966; Zolotarev, 1986) or in mathematical journals which imposes heavy demands on the reader who should be seriously experienced in mathematics. In the above-mentioned works of Lorentz, Holtsmark, Chandrasekhar, and von Neumann, Landau, Monin, the term 'stable laws' was not used, the property of stability was not emphasized, and there were no references to the mathematical works on stable laws. It looks as if authors did not know about belonging of the obtained distributions to the class of stable laws.

In the 1960s, the works (Good, 1961; Kalos, 1963) appeared, where the distributions obtained were already identified as stable laws. In 1970s, the works (Jona-Lasinio, 1975; Scher & Montroll, 1975) were published, where the authors striving for clearness and completeness of their results did not only give references but also made additions to their articles provided with some information about stable laws. This tendency to supplement the article by more or less brief information about stable laws remains good in 1980s and 1990s (Weron, 1986; Bouchaud & Georges, 1990; West, 1990; West, 1994).

Since the 1960s, the stable laws attracted the attention of scholars working in the area of economics, biology, sociology, and mathematical linguistics, due to a series of publications by Mandelbrot and his disciples (Mandelbrot, 1977;

Mandelbrot, 1983). The point is that the statistical principles described by the so-called Zipf–Pareto distribution were empirically discovered fairly long ago in all these areas of knowledge. The discrete distributions of this type are of the form

$$p_k = ck^{-1-\alpha}, \quad k \ge 1, \quad \alpha > 0,$$

while their continuous analogues (densities) are

$$p(x) = cx^{-1-\alpha}, \qquad x \ge a > 0.$$

Mandelbrot called attention to the fact that the use of the extremal stable distributions (corresponding to $\beta = 1$) to describe empirical principles was preferable to the use of the Zipf–Pareto distributions for a number of reasons. It can be seen from many publications, both theoretical and applied, that Mandelbrot's ideas receive more and more wide recognition of experts. In this way, the hope arises to confirm empirically established principles in the framework of mathematical models and, at the same time, to clear up the mechanism of the formation of these principles.

The Mandelbrot's fractal concept has found its application in the turbulence problem, anomalous diffusion, dynamical chaos and large-scale structure of the Universe (Bouchaud & Georges, 1990; Isichenko, 1992; Shlesinger *et al.*, 1993; Klafter *et al.*, 1996; Coleman & Pietronero, 1992). In the foundation of these trends, the main part is taken by the power-type laws that provide us with the scale invariance (self-similarity) of the structures and processes. Thus, the stable laws with their power-type asymptotic behavior turn out to be very suitable here.

It is quite possible that some interesting works have turned out to be beyond the scope of this book or have appeared too late to be included into it. Having no intention to give the comprehensive exposition of all that concerns the theory of stable laws and their applications, we hope, anyway, that our book will be useful for those researchers, graduate and postgraduate students who feels the need to go out the 'domain of attraction' of normal law. Experience shows that 'with probability close to one' this way leads one to shining new and interesting results.

This book consists of two parts. The first part, devoted to the theory of stable laws, is due to V.M. Zolotarev, except for Chapter 2 and Sections 6.1, 6.2, 7.4, 10.1–10.6, which were written by V.V. Uchaikin. The second part, containing actual examples of applications of stable laws to solving various problems, is due to V.V. Uchaikin, except for Sections 10.7, 10.8 13.7, 13.10, 14.1, 14.2, 14.3, 16.4, and 18.1, written by V.M. Zolotarev. Chapter 17 was composed by V.M. Zolotarev and V.Yu. Korolev.

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V. Uchaikin
V. Zolotarev

Moscow, March 1999

Part I

Theory

1

Probability

1.1. Probability space

The fundamental notions of probability theory are random variables and their probability distributions. Nevertheless, we should begin with quite simple notion of probability space (Ω, Ξ, Q), whose constituents are the space of elementary events (outcomes) Ω, the sigma algebra Ξ of Borel subsets of Ω, and some function $Q(A)$ defined on the sets of the family Ξ and taking values from the interval $[0, 1]$. Speaking of sigma algebras, we mean that both unions of a finite or countable number of elements of Ξ and their complements lie in Ξ as well. The function $Q(A)$ is a normalized countably additive measure; $Q(A) = 1$, and for any union $A = A_1 \cup A_2 \cup \dots$ of a finite or countable number of pair-wise non-overlapping elements of Ξ, the equality

$$Q(A) = Q(A_1) + Q(A_2) + \dots$$

holds. This is the basic object of probability theory.

A rich variety of probability spaces are in existence. In order to obtain a meaningful theory, one should consider a rather rich probability space (Ω, Ξ, Q); i.e., the sets Ω, Ξ should be rich, and the measure Q should take reasonably many values. For example, if we consider Ω consisting of six elements $\omega_1, \omega_2, \dots, \omega_6$, and take as Ξ the set of all 2^6 subsets of Ω (including the empty set and Ω itself), and define a measure Q on the elementary outcomes ω_i, we arrive at the model of all possible dices, both symmetric and asymmetric. It is clear that this model is too poor to be used in actual problems. Meanwhile, there exists a probability space of very simple structure which allows us to consider just about anything. Its structure is as follows: Ω is the interval $[0, 1]$ whose points are elementary events; Ξ is the system of all Borel sets of the interval Ω; Q is the so-called Lebesgue measure, which is uniquely determined by the condition that for any interval Δ in Ω the value $Q(\Delta)$ is equal to the length $|\Delta|$ of that interval.

1.2. Random variables

The probability space we constructed governs, in some sense, the random events that occur in our model.

Here we come against 'non-predictability', or 'randomness'; the following construction of the general model relates to randomness only via the probability space as the base. The emergence of randomness can be easily imagined in that special probability space we just introduced. We randomly drop a tiny 'ball' onto the interval Ω (i.e., the 'ball' can, with equal chances, be put in any point of Ω); its position is the result of a trial denoted by ω, and this is the only manifestation of 'randomness'. The trials can be repeated, with no dependence on the results of the preceding trials, and we thus arrive at the model of independent trials. But in actual experiments we observe only factors of various nature related to random trials, which we will refer to as random.

In the general model, the random variable is any real-valued function $X(\lambda)$ defined on Ω and possessing a special property called measurability.

Before turning to rigorous definition of measurability, we observe that in the real axis $\mathbb{R}^1$, as well as in the interval Ω, we can define Borel sets in the same way as we do for Ξ, on the base of all possible intervals of $\mathbb{R}^1$. Let A be some Borel set of Ξ and $X(A)$ be the set of values of the function $X(\lambda)$ while λ runs through all elements of A.

The function $X(\lambda)$ transforms the sets of Ξ and maps them onto some sets in the real axis; the function is called measurable if all sets it generates are Borel. This constraint on the structure of the functions $X(\lambda)$ playing the parts of random variables, which, at first glance, seems to be too complicated, and the constraint on the structure of the sets A from the family Ξ which stand for various random events, are necessary indeed. The point is that the notion of probability of random events related to a random variable $X(\omega)$ is defined implicitly, through the probability measure Q. If B is some Borel set from the set of values of the random variable $X(\omega)$ and $A = X^{-1}(B)$ is the corresponding pre-image (a set of Ω), then the probability that $X(\omega)$ belongs to B (this event is random and is denoted by $X(\omega) \in B$) can be calculated as follows:

$$\mathsf{P}\{X(\omega) \in B\} = \mathsf{P}\{\omega \in X^{-1}(B)\} = Q(A) \qquad (1.2.1)$$

Therefore, if the function $X(\lambda)$ representing the random variable under observation is given, then the calculation of probabilities (1.2.1) can pose computational but not fundamental obstacles.

Unfortunately, by far the most frequently encountered situation is where the function $X(\lambda)$ is unknown and moreover, we cannot even make reasonable guesses. Nevertheless, sometimes the form of the function $X(\lambda)$ can be inferred. For example, if one tosses a symmetric coin, then, assigning 0 and 1 to the sides of the coin, a trial can be considered as the observation of a random variable X taking these two values. Both of the possible outcomes are equiprobable.

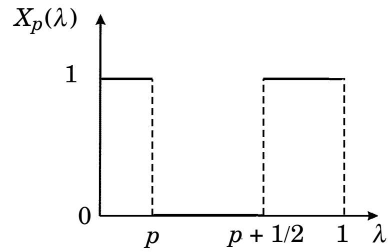


Figure 1.1. Graphs of $X_p(\lambda)$, $0 \le p \le 1/2$, simulating a symmetric coin

Thus, as $X(\lambda)$ we can take any of the functions $X_p(\lambda)$ presented in Fig. 1.1. Any of them can model the coin tossing experiment.

Let us give an insight into the idea of 'simulation' based on these models. If we take one of the functions $X_p(\lambda)$ and assign random values to λ in the interval Ω (we do not dwell on technical details of realization of such an assignment), then $X_p(\omega)$ becomes a random variable that behaves exactly as a symmetric coin.

This simple example demonstrates that even in those cases where an additional information helps us to clear up the structure of the function $X(\lambda)$, it often remains uncertain. In other words, for any random variable that we are observing in some experiment there exists a set (more exactly, an infinite set) of functions $X(\lambda)$ which can serve as its model.

This is why we do not pose the problem of reconstruction of functions $X(\lambda)$ which model random variables. While investigating actually observed random variables, the analysis does not advance beyond finding the corresponding probability distributions, i.e., sets of probabilities $\mathsf{P}\{X \in B\}$ for various Borel sets B in $\mathbb{R}^1$.

This is too difficult to be practical to deal with these sets, because it is physically infeasible to examine, item-by-item, all possible Borel sets. Some way out is provided by the so-called distribution functions of random variables. These are sets of probabilities of more simple random events $F_X(x) = \mathsf{P}\{X < x\}$, where x are real values. Knowing the distribution function $F_X(x)$, we are able to re-build unambiguously the probability distribution $\mathsf{P}\{X \in B\}$, where B are Borel sets.

1.3. Functions $X(\lambda)$

In view of the above concept of a random variable as a real-valued measurable function defined on the interval $[0, 1]$, the natural question arises: what is the set Λ_F of all functions $X(\lambda)$ that generate the random variables with a given distribution function $F(x)$? The formal answer turns out to be not too difficult.

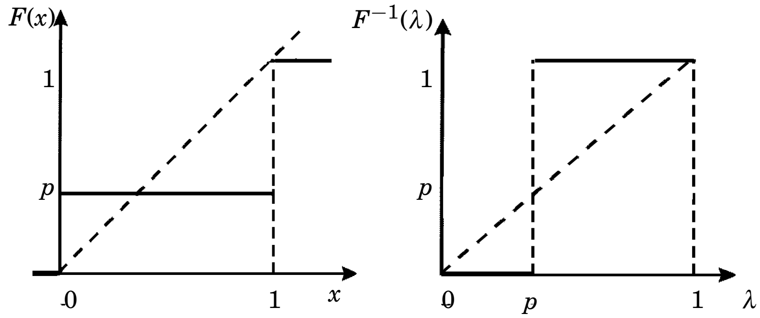


Figure 1.2. The distribution function $F(x)$ and the inverse function $F^{-1}(\lambda)$ of a random variable that takes two values: zero with probability p and one with probability $1-p$, $0 < p < 1$

There exists a universal way which allows us to construct a function generating $F(x)$. We consider the function $F^{-1}(\lambda)$ defined on the interval $[0,1]$ which is, in some sense, inverse to $F(x)$. Lest the presentation be not overcomplicated, we do not dwell on analytic definition of $F^{-1}(\lambda)$, but give a graphical illustration. Consider the reflection of the graph of the function $F(x)$ from the bissectrix of the first quadrant (Fig. 1.2).

The jumps of the function $F(x)$ become the intervals of constancy of $F^{-1}(\lambda)$, whereas the intervals where $F(x)$ is a constant become jumps of $F^{-1}(\lambda)$ It is not hard to see that

$$\mathsf{P}\left\{F^{-1}(\omega) < x\right\} = F(x)$$

i.e., $X_0(\lambda) = F^{-1}(\lambda)$ is a function of the set Λ_F.

If we assume that all non-decreasing functions on Ω are right-continuous in their discontinuity points (which does not influence the distribution functions generated), then there exists a unique $X_0(\lambda)$ in the set Λ_F.

Now we turn to the system of transformations of the 'support' function $X_0(\lambda)$ determined by the one-to-one mappings of the set Ω onto itself. This means that every point of Ω is mapped either to itself or to some other point, while two distinct points cannot be mapped to one and the same point. Together with each relocated point, the corresponding value of the function $X_0(\lambda)$ moves as well. As the result of the transformation J chosen of the set Ω, which, obviously, possesses an inverse transformation, we arrive at the extra function $X_J(\lambda)$. All measurable functions of such a kind constitute the set Λ_F.

As an illustrative example, we consider the distribution function given in Fig. 1.2. The inverse of this function plays the part of the 'support' function $X_0(\lambda)$, and the functions in Fig. 1.3 are its transformations of the above-discussed type.

To demonstrate that the above transformation of the function $X_0(\lambda)$ gives

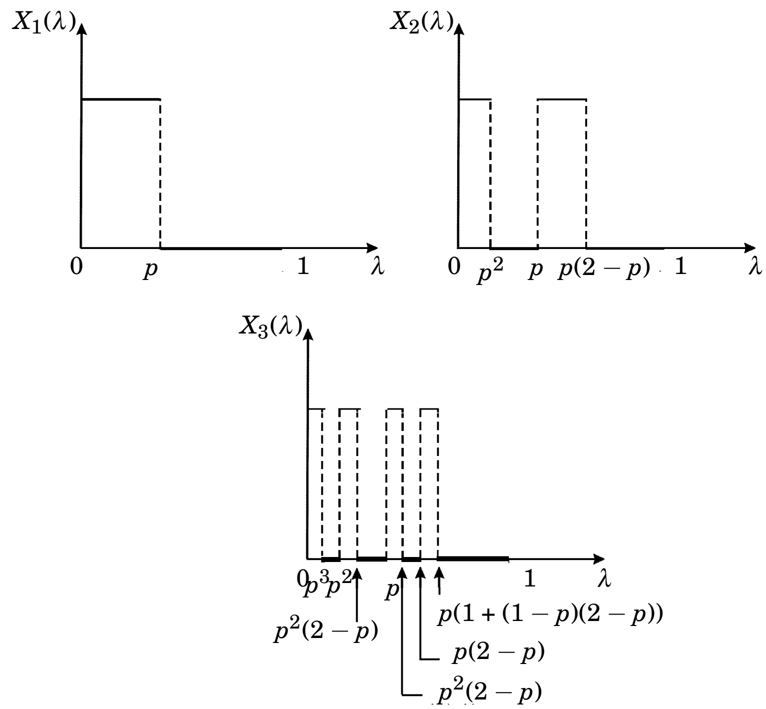


Figure 1.3. Rademacher-type functions modeling independent random variables

us the whole set Λ_F, we take some function $X(\lambda)$ from this set, and transpose elements of Ω so that the values of the resulting function do not decrease as λ grows. Because Λ_F contains only those non-decreasing functions that differ from $X_0(\lambda)$ in values at discontinuity points only, the transformation $X(\lambda)$ necessarily gives us one of non-decreasing functions of the set Λ_F.

Thus, by re-defining the support function at its discontinuity points and subsequent inverse transformation of Ω we arrive at $X(\lambda)$ starting from $X_0(\lambda)$.

In the framework of the general mathematical model, the presentation of random variables as functions $X(\omega)$ of random argument ω plays a basic part, while the whole model becomes complete and well-integrated, thus alleviating the analysis.

1.4. Random vectors and processes

Thus, a random variable is specified by its distribution and is modeled by measurable functions defined on Ω (Fig. 1.4).

If we deal with a random k-dimensional vector $\mathbf{X} = (X_1, \ldots, X_k)$, then all rea-

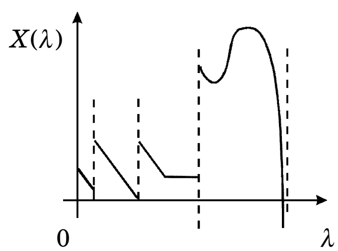


Figure 1.4. A function modeling a random variable

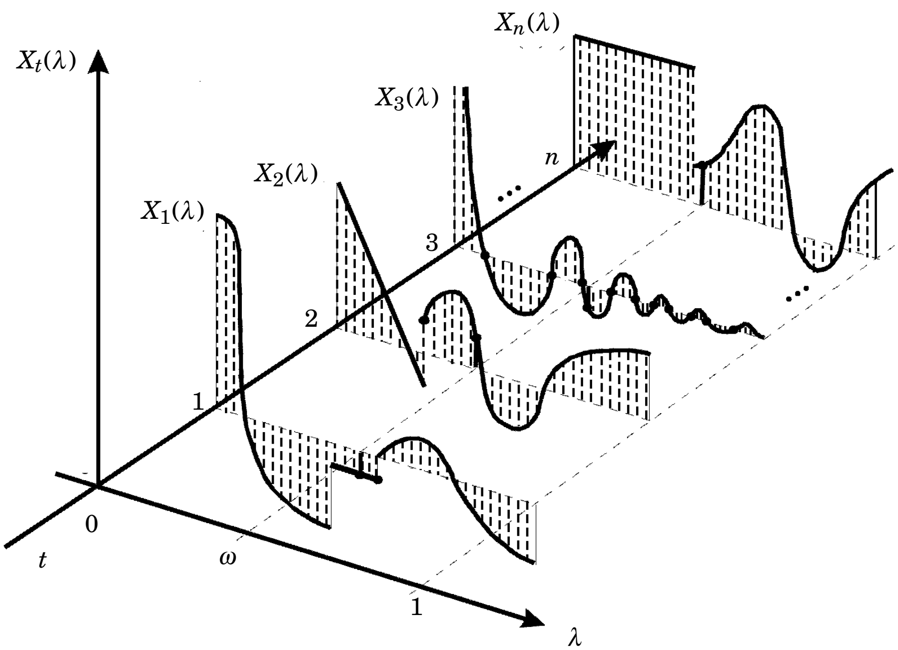


Figure 1.5. A function modeling an n-dimensional random vector

soning related to random variables can, with some complication, be extended to this case. A random vector is modeled by the set of functions $X_1(\lambda), \ldots, X_k(\lambda)$. It is convenient to present them as a single multi-layered graph (Fig. 1.5).

Here the part of probability distributions of a random vector $\mathbf{X}$ is played by the set of probabilities $\mathsf{P}\{\mathbf{X} \in B\}$, where B are Borel sets of points of the k-dimensional Euclidean space $\mathbb{R}^k$.

The 'medium' which carries the information equivalent to the probability distribution $\mathsf{P}\{\mathbf{X} \in B\}$ is the distribution function of the vector

$$F_{\mathbf{X}}(\mathbf{x}) = F_{\mathbf{X}}(x_1, \dots, x_k) = \mathsf{P}\{X_1 < x_1, \dots, X_k < x_k\},$$

where $\mathbf{x} = (x_1, \dots, x_k)$ are vectors of $\mathbb{R}^k$.

In our model, a random vector is a set of functions $X_1(\lambda), \dots, X_k(\lambda)$ defined on one and the same interval. Separate inequalities

$$X_1(\lambda) < x_1, \dots, X_k(\lambda) < x_k$$

are associated with their corresponding pre-images $A_1, \dots, A_k$ on the interval Ω, while the totality of these inequalities on Ω is associated with the set $A^{\mathbf{x}} = A_1 \cap \dots \cap A_k$ (i.e., the common part of all these sets). Thus, the function $F_{\mathbf{X}}(\mathbf{x})$ is calculated with the help of the measure Q by means of the formula $F_{\mathbf{X}}(\mathbf{x}) = Q(A^{\mathbf{x}})$.

In probability theory, the need for working on sets of random variables is quite usual. If we consider the random variables of such a set as components of a random vector, then the probability distribution of this vector, or, what is the same, the corresponding distribution function, carries the complete information about that set. It is referred to as the joint distribution (joint distribution function) of the random variables of the set.

The next stage in complicating matters is related to random processes. Let us imagine that the multi-layered graph in Fig. 1.5 continues in one or in both directions of the axis t, being supplemented by the addition of new graphs at $\dots, -1, 0, k+1, k+2, \dots$. The result is the random process $X_t(\omega)$ with discrete time t. The choice of a point ω in Ω results in an infinite sequence

$$\dots X_{-1}(\omega), X_0(\omega), X_1(\omega), \dots \tag{1.4.1}$$

or

$$X_1(\omega), \dots, X_k(\omega), X_{k+1}(\omega), \dots,$$

which is a single realization (corresponding to an elementary event ω) of the process $X_t(\omega)$.

A random process $X_t(\omega)$ with continuous time t is introduced similarly, but for it the result of a single trial, or its single realization, is not sequence (1.4.1) but a real-valued function defined on the axis t. A random process with continuous time can be visualized as in Fig. 1.6.

As we see, the model of a random process is a function of two variables t and λ defined in the strip $0 < \lambda < 1$. Choosing a point in the interval Ω, we select a length-wise section of this function through ω, being a function of t. The function is the result of a single trial, one of the possible realizations of the random process.

If we fix t and take the cross-wise section through this point, we obtain a function defined on Ω, i.e., a model of some random variable $X_t(\omega)$.

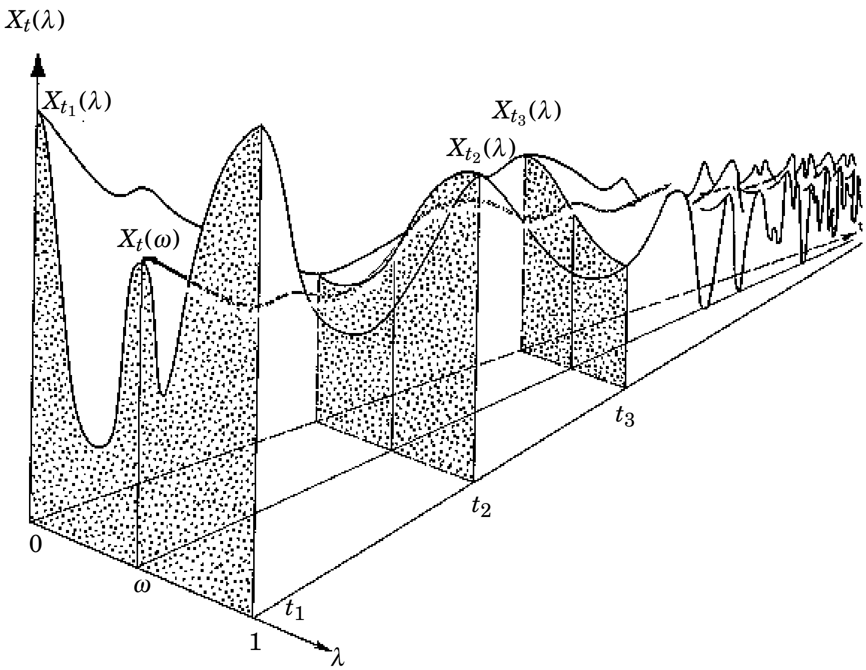


Figure 1.6. A function modeling a random process with continuous time

1.5. Independence

The notion of independence of random events, of random variables, etc. is the fundamental part of probability theory, and makes it a self-contained field of mathematics, even if evolved from set theory, measure theory, and functional analysis. Its heart consists of the following.

Let $\mathbf{X} = (X_1, \dots, X_k)$, $k \geq 2$, be some set of random variables. We denote the distribution function of the random variable X_i by $F_i(x_i)$, $i = 1, \dots, k$, and its joint distribution function by $F_{\mathbf{X}}(x_1, \dots, x_k)$. Then we say that the random variables of this set are independent, if

$$F_{\mathbf{X}}(x_1, x_2, \dots, x_k) = F_1(x_1)F_2(x_2)\dots F_k(x_k) \tag{1.5.1}$$

for any $x_1, \dots, x_k$.

Therefore, the property of mutual independence of random variables of a set $\mathbf{X}$ is some special property of its joint distributions. Property (1.5.1) is equivalent to the following more general property of probability distributions:

$$\mathsf{P}\{X_1 \in B_1, X_2 \in B_2, \dots, X_k \in B_k\} = \mathsf{P}\{X_1 \in B_1\}\mathsf{P}\{X_2 \in B_2\}\dots\mathsf{P}\{X_k \in B_k\}, \tag{1.5.2}$$

which holds for any Borel sets $B_1, \dots, B_k$ (property (1.5.1) becomes a particular case of (1.5.2) if we take the semi-axes $(-\infty, x_i)$ as B_i).

The importance of this purely mathematical notion is that it allows us to model the absence of interaction of actual random events.

Indeed, let us consider, as an example, a sequential tossing of a coin with sides labelled with zero and one. The coin may be asymmetric; then the probability p of occurrence of one is not necessarily 1/2. The modeling of divorced from each other tossings (of course, the coin itself remains the same) is carried out by means of the sequence of functions $X_1(\lambda), X_2(\lambda), \ldots$ given in Fig. 1.3. The peculiarity of these functions is that they take only two values, zero and one, whereas the ratio of lengths of neighboring intervals is $p : 1 - p$.

A direct verification demonstrates that the random variables corresponding to these functions are mutually independent, i.e., they obey property (1.5.1) or equivalent property (1.5.2). The independence of the first two random variables can be simply verified by examining the equalities

$$\mathsf{P}\{X_1 = l, X_2 = m\} = \mathsf{P}\{X_1 = l\}\mathsf{P}\{X_2 = m\} \tag{1.5.3}$$

for $l, m = 0$ or 1 (we actually need to examine only three equalities because the fourth one follows from them in view of the fact that the sum of probabilities over l and m is equal to one).

In the course of analysis, we see the following feature of property (1.5.1): it breaks down as the graph of one of the functions is slightly altered while the total length of the intervals where the function is equal to one remains the same (for example, if we shift the first interval to the right by some small distance $\varepsilon > 0$, and assign the zero value to the function in the interval $(0, \varepsilon)$ appeared).

The sense of the abovesaid is that the property of independence of random variables is a very special structural property of the functions that model them. The mathematical notion of independence used in probability theory is of particular importance because it is able to model the situations where the events do not cause one another. Nevertheless, the notion of independence inherent in probability theory is much wider than its ordinary interpretation evolved from practice. This becomes evident while we discuss independence of the events related to one and the same trial. So, it is hard to imagine that, while tossing a symmetric dice, the occurrence of a multiple of three (event A) and the occurrence of an even number (event B) are independent, whereas the occurrence of a prime number (event C, i.e., the occurrence of 2, 3, or 5) depends on event B but is independent of event A. Moreover, events A and B can be made dependent and events B and C can be made independent, by suitable deformation of the dice.

Too rigorous understanding of dependence of random events in the foundation of theoretical analysis is not good because this downgrades our inference potentialities. Nevertheless, while we consider random objects (events, variables, processes, etc.) that are, to some extent, inter-related, we drop the independence condition in favor of one or other type of weak dependence, so eliminating the above excessive rigor.

A remarkable property of independent random variables $X_1, \ldots, X_k$, $k \geq 2$, consists in the fact that any measurable functions of these random variables

$$Y_1 = g_1(X_1), \ldots, Y_k = g_k(X_k) \tag{1.5.4}$$

are again independent random variables.

Another eye-catching property of independent random variables consists of multiplicativity of means:

$$\mathsf{E}[g_1(X_1)\ldots g_k(X_k)] = \mathsf{E}g_1(X_1)\ldots \mathsf{E}g_k(X_k). \tag{1.5.5}$$

In particular,

$$\mathsf{E}X_iX_j = \mathsf{E}X_i\, \mathsf{E}X_j, \qquad 1 \leq i < j \leq k, \tag{1.5.6}$$

where, as in (1.5.5), the means of random variables on the right-hand side are assumed to be finite.

The last relation remains valid if we change the mutual independence of the random variables $X_1, \ldots, X_k$ for the weaker condition that they are pair-wise independent, i.e., X_i and X_j are independent if $i \neq j$.

The fact that the mutual independence of all k random variables in (1.5.6) can be weakened and changed for pair-wise independence is an evident example of extending the idea of independence of a family of random variables. In modern probability theory, such a phenomenon comes into evidence in the form of the so-called martingale dependence.

1.6. Mean and variance

In the general case, the probability distribution of a random variable X, or its distribution function, is a rather complex characteristic. This propelled us into hunting for more simple and convenient but yet informative characteristics of random variables. Searching along these lines revealed such characteristics as the mean and the mean root square deviation of a random variable X, which also are referred to as the mathematical expectation and the standard deviation, whereas the square of the latter is called the variance.

Under our interpretation of a random variable X as a function $X(\omega)$ of random argument, these characteristics are defined as follows:

$$\mathsf{E}X = \int_0^1 X(\lambda)d\lambda \tag{1.6.1}$$

(the mean of X),

$$\operatorname{Var} X = \int_0^1 (X(\lambda) - \mathsf{E}X)^2 d\lambda \tag{1.6.2}$$

(the variance of X).

The property

$$\mathsf{E}\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} \mathsf{E}X_i \tag{1.6.3}$$

is, in view of definition (1.6.1), the ordinary property of additivity of the definite integral which holds under quite weak constraints on the summands $X_i(\lambda)$. The similar property of additivity of the variances

$$\mathrm{Var}\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} \mathrm{Var}\, X_i. \tag{1.6.4}$$

is not so universal as (1.6.3).

Usually, (1.6.4) is presented under the mutual independence of summands, but actually it is true as soon as the variances on the right-hand sides exist and the summands are pair-wise independent, i.e., (1.5.6) holds. The variance $\mathrm{Var}\, X$ does not change if some constant C is added to X, because

$$\mathsf{E}(X + C) = \mathsf{E}X + C$$

and therefore,

$$\mathrm{Var}(X + C) = \mathrm{Var}\, X.$$

Thus, considering property (1.6.4), without loss of generality we can take the means $\mathsf{E}X_i$ equal to zero, so (1.6.4) takes the form

$$\mathsf{E}S_n^2 = \mathsf{E}S_{n-1}^2 + \mathsf{E}X_n^2, \tag{1.6.5}$$

where

$$S_n = \sum_{i=1}^{n} X_i.$$

If we represent the left-hand side of (1.6.5) as

$$\mathsf{E}(S_{n-1} + X_n)^2 = \mathsf{E}S_{n-1}^2 + 2\mathsf{E}S_{n-1}X_n + \mathsf{E}X_n^2,$$

then it becomes clear that (1.6.4) is equivalent to

$$\mathsf{E}(S_{n-1}X_n) = \mathsf{E}(S_{n-1}\mathsf{E}(X_n \mid S_{n-1})) = 0,$$

where $\mathsf{E}(X_n \mid S_{n-1})$ stands for the conditional mathematical expectation of X_n under the condition that S_{n-1} takes a certain value.

The sign of outer expectation shows that we carry out the averaging over all possible values of S_{n-1}.

The last equality, obviously, holds if $\mathsf{E}(X_n \mid S_{n-1}) = 0$, but this is exactly the condition called the martingale dependence between the summands of S_n (we say that the sequence S_n forms a martingale).

The conditional mathematical expectation $\mathsf{E}(X \mid Y)$ of a random variable X under the condition that another random variable Y takes a value y is understood as

$$\mathsf{E}(X \mid Y) = \int x\, d_x \left(d_y F_{XY}(x,y)/d_y F_Y(y)\right), \tag{1.6.6}$$

where $F_{XY}(x,y)$ is the joint distribution function of the pair of random variables X, Y, and $F_Y(y)$ is the distribution function of the random variable Y. In the case where these distribution functions possess densities $p_{XY}(x,y)$ and $p_Y(y)$, the integrand takes a simpler form

$$\mathsf{E}(X \mid Y) = \int x \frac{p_{XY}(x,y)}{p_Y(y)}\, dx.$$

The above definition of the martingale is somewhat generalized, and is called the Lévy martingale. Usually, the name of martingale is related to the condition

$$\mathsf{E}(X_n \mid X_1, \ldots, X_{n-1}) = 0.$$

As a simple example of the latter is the pitch-and-toss game with independent tossing while the bets in each round are set by one of the players with consideration for his payoff in the preceding rounds. Although in this game the random variables X_i take, as before, two values M_n and $-M_n$ with probabilities 1/2 each, the value of M_n is determined by one of the players with regard to values of X_i in the preceding rounds, which makes the random variables $X_1, X_2, \ldots$ dependent. Nevertheless, no matter what strategy is chosen, the martingale condition always holds.

1.7. Bernoulli theorem

In 1713, eight years after the death of Jacob Bernoulli, his book *'Ars Conjectandi'* came into the light, and marked the beginning of probability theory as a branch of science. The author, a magnificent Swiss scientist, famed for his works in mathematics, mechanics, and physics, went down in history as the patriarch of probability theory who established the first and, as will soon become evident, fundamental limit theorem, the law of large numbers[1].

This remarkable mathematical fact that reflects one of the most important laws of the Universe, consists in the following.

[1]Although we know this famous theorem by the publication of 1713, in fact this result was obtained by Jacob Bernoulli about 1693.

Let us consider a sequence of non-related (i.e., independent) trials under stable environmental conditions. The outcome of a trial is either an event A we are interested in but whose occurrence we cannot forecast, or the contrary event $\overline{A}$. After n trials, we count the number $N(A)$ of occurrences of event A. The ratio

$$\nu_n(A) = N(A)/n, \qquad (1.7.1)$$

is interpreted as the frequency of event A in the first n trials. If we repeat sequences of trials of length n, the frequences should not necessarily coincide: the frequency is random. But, as even some predecessors of J. Bernoulli noticed, while n grows, the frequency 'stabilizes' around certain value. This empirically strengthened fact allows us to interpret the frequency for large enough n as the measure of 'uncertainty' of event A. This law which lies in the heart of many actual applications should, of course, be theoretically validated. The Bernoulli theorem is exactly the validation we need.

Let $X_1, X_2, \ldots$ be a sequence of independent[2] random variables, each taking two values, 0 and 1, provided that $p = \mathsf{P}\{X_n = 1\}$ for all n (which implies that $\mathsf{P}\{X_n = 0\} = 1 - p$ remains the same for all random variables).

We consider the arithmetical means of the first n random variables

$$\mu_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad n = 1, 2, \ldots \qquad (1.7.2)$$

If we associate $X_i = 1$ with the occurrence of event A in the ith trial, and $X_i = 0$, with the contrary event $\overline{A}$, then μ_n becomes exactly the frequency of event A during n trials, i.e., a variable equal to $\nu_n(A)$. The frequencies $\nu_n(A)$ are 'stabilized' near some constant which characterizes the measure of randomness of occurrence of event A in the trials. It is natural that the part of such a measure has to be played by the probability p. Thus, the arithmetic mean μ_n should match p. This is the way that Jacob Bernoulli prepared; the result can be formulated as follows.

THEOREM 1.7.1 (Jacob Bernoulli). *Let a positive ε be arbitrarily small; then*

$$\mathsf{P}\{|\mu_n - p| \geq \varepsilon\} \to 0, \qquad n \to \infty. \qquad (1.7.3)$$

Since

$$\varepsilon_n = \mu_n - p \xrightarrow{\mathsf{P}} 0,$$

i.e., converges to zero in probability as $n \to \infty$, the law of large numbers, that is, Bernoulli's theorem can be formulated as follows:

$$\mu_n = p + \varepsilon_n, \qquad n \geq 1. \qquad (1.7.4)$$

[2]Speaking of an infinite sequence of independent random variables, we mean that the set of the first n random variables, no matter what n is, consists of independent random variables.

The sense of this assertion consists exactly in that it confirms the experimental observation that the frequency of occurrence of event A in a sequence of independent trials approaches the a priori probability of occurrence of this event in a single trial. We put stress on the fact that none of mathematical proofs, no matter how pretty they are, can act for a ground of actually observed data. So, the Bernoulli theorem cannot serve as the proof of the existence of a limit of the actually observed frequencies $\nu_n(A)$.

The mathematical proof of the fact that the frequency μ_n of the occurrences of event A tends to the probability p of the occurrence of event A in a single trial is valid in the framework of a certain mathematical model, i.e., under somewhat idealized conditions that are in some correspondence to the conditions under which the actual trials take place. In other words, the mathematical assertion established by Bernoulli should not be mixed with the widespread and empirically validated law of stability of frequencies. The mathematical interpretation of the law of large numbers and its manifestation in actual processes are similar but nevertheless not identical.

The Bernoulli theorem is of great importance indeed, because it is the primary connecting link between probability theory and its applications. There are many parameters and functional characteristics in models used in probability theory that cannot be identified without invoking experimental data.

As a simple example, let us consider deformed, asymmetric coin, for which it is not feasible to calculate the probability of occurrence of a chosen side in tossing. But this probability can be evaluated by experiment, by observing a large enough number of tossings and finding the corresponding frequency ν_n. The Bernoulli theorem provides us with grounds to expect that ν_n and p are close to each other if the number of tossings is large enough. But it remains unclear which number of trials should be observed to ensure that the difference between ν_n and p becomes no greater than an ε given.

Bernoulli, being not an only outstanding mathematician but deeply engaged in practical work, (which was not rare in XVII–XVIII centuries) was not satisfied with assertion (1.7.3) of qualitative nature, and gave explicit bounds for the probability entering into that expression. In more recent times, these bounds were refined. One of the refined bounds is given by the relation

$$\mathsf{P}\{|\mu_n - p| \ge \varepsilon\} \le \beta = 2\exp\left(-n\varepsilon^2/2\right), \tag{1.7.5}$$

see (Uspensky, 1937).

If β is small, for example, if $\beta = 0.05$ or $\beta = 0.01$, then with probability $\alpha = 1-\beta$ close to one the validity of inequality $|\mu_n - p| \le \varepsilon$ should be considered as a certain event. Based on bound (1.7.4), we obtain

$$\mu_n - \varepsilon < p < \mu_n + \varepsilon, \tag{1.7.6}$$

where

$$\varepsilon = \sqrt{\frac{2}{n}\ln\frac{2}{\beta}}.$$

For β chosen, practically certain events are determined by the levels of probabilities no less than 0.95 and 0.99. The corresponding bounds for ε become, respectively,

$$\varepsilon = 2.72/\sqrt{n}, \qquad \varepsilon = 3.25/\sqrt{n}.$$

This is the traditional presentation and interpretation of the Bernoulli theorem. Now, let us take a second look at this theorem from the viewpoint of modeling of random variables treated at some length in the preceding sections.

We have demonstrated how to model a sequence of independent random variables taking the value 1 with probability p, $0 < p < 1$, and the value 0 with probability $1 - p$ in Fig. 1.3, where particular functions $X_1(\lambda), X_2(\lambda), X_3(\lambda)$ of Rademacher type[3] were pictured. Their arithmetic mean

$$\mu_n(\lambda) = \frac{1}{n}\sum_{i=1}^{n} X_i(\lambda) \tag{1.7.7}$$

is again a function defined in the interval $[0, 1]$. As n grows, the functions $\mu_n(\lambda)$ whimsically change and become much more complicated, but μ_n gather around the point p.

We consider the ε-neighborhood of p, i.e., the strip between the levels $p - \varepsilon$ and $p + \varepsilon$, $0 < \varepsilon < \min(p, 1 - p)/2$, and denote it by B_ε. In the interval $[0, 1]$ we choose λ such that $\mu_n(\lambda) \in B_\varepsilon$. The set $A_{n,\varepsilon}$ of those λ is a Borel set of Ω; therefore

$$\mathsf{P}\{\mu_n(\omega) \in B_\varepsilon\} = \mathsf{P}\{p - \varepsilon < \mu_n(\omega) < p + \varepsilon\} = \mathsf{P}\{|\mu_n(\omega) - p| < \varepsilon\} = Q(A_{n,\varepsilon}).$$

The Bernoulli theorem states that for any fixed ε the measure $Q(A_{n,\varepsilon})$ of the set $A_{n,\varepsilon}$ tends to one as $n \to \infty$.

This is illustrated in Fig. 1.7 in the case $p = 1/2$.

Since the effect that the bulk of the values of the functions $\mu_n(\lambda)$ approaches the level p holds not only for the functions $X_i(\lambda)$ of special structure (Fig. 1.5) but also for a wide class of other functional sequences, it looks very curious and even intriguing from the viewpoint of the function theory and analysis. The main contribution to this effect is due to the special interconsistent nature of the functions $X_i(\lambda)$ which we refer to as 'independence of random variables'.

We can also discuss how the Bernoulli theorem describes the behavior of the distribution functions $G_n(x)$ of the random variables μ_n. Let $F(x)$ stand for the common distribution function of the summands X_i (given in Fig. 1.2). All $G_n(x)$ are step-functions and vary from zero to one on the interval $[0, 1]$, as well as the function $F(x)$. If for $n = 1$ there are two jumps ($G_1(x)$ coincides

[3]The German mathematician G. Rademacher considered a sequence of functions $r_k(\lambda)$ related to the functions $X_k(\lambda)$, $p = 1/2$, by means of the equality $r_k(\lambda) = 2X_k(\lambda) - 1$, $k = 1, 2, \dots$ The main property of these functions is that all of them can be considered as mutually inddpendent random variables $X_1, X_2, \dots, X_n, \dots$

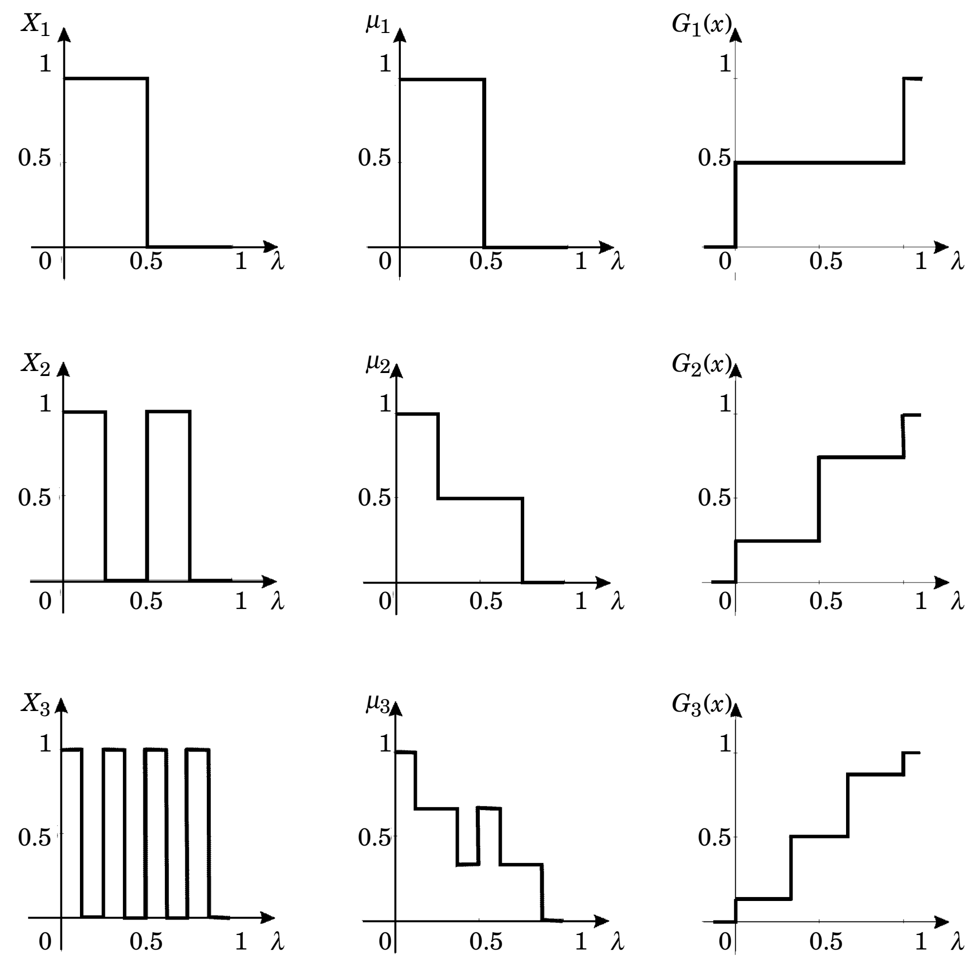


Figure 1.7. Evolution of the functions $\chi_n(\lambda)$, $\mu_n(\lambda)$, and $G_n(x)$ as n grows

with $F(x)$), then for $n = 2$ the number of jumps increases to three, for $n = 3$, to four, etc. (see Figures 1.7 and 1.8). The function $G_n(x)$ at the points $x_k = k/n$, $k = 0, \dots, n$, possesses jumps of amplitudes

$$\Delta G_n(x_k) = \binom{n}{k} 2^{-n}.$$

The Bernoulli theorem states that μ_n converges in probability to the constant p as $n \to \infty$ (the limiting relation (1.7.6) describes exactly this type of convergence). The random variable equal to some constant p possesses the one-step

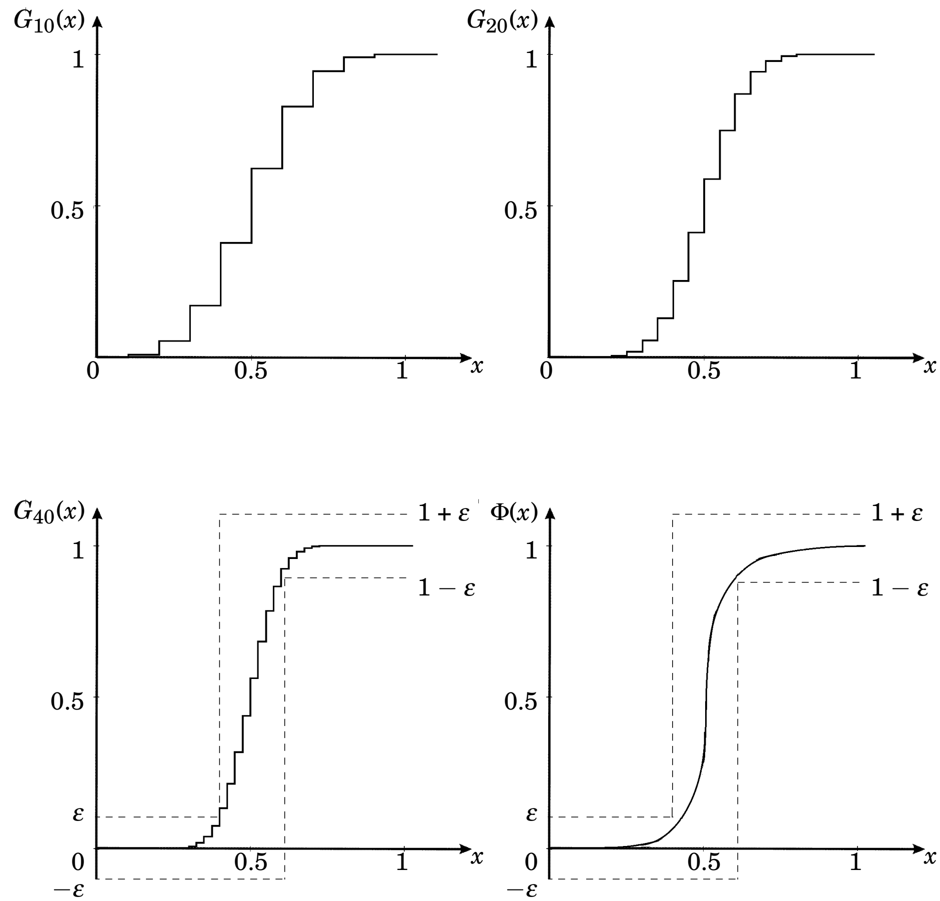


Figure 1.8. Further evolution of the function $G_n(x)$

distribution function

$$e(x-p) = \begin{cases} 0, & x \le p, \\ 1, & x > p, \end{cases}$$

with unit jump at the point $x = p$ (such a distribution function is called degenerate at the point p).

It is well known that the convergence in distribution of any sequence of random variables to a constant is equivalent to the weak convergence of the distribution functions of these variables to the corresponding degenerate distribution function (the convergence takes place at any point except the jump point of the latter function). Denoting this convergence by the symbol ⇒, we rewrite the assertion due to Bernoulli in the form

$$G_n(x) \Rightarrow e(x-p) \qquad n \to \infty. \tag{1.7.8}$$

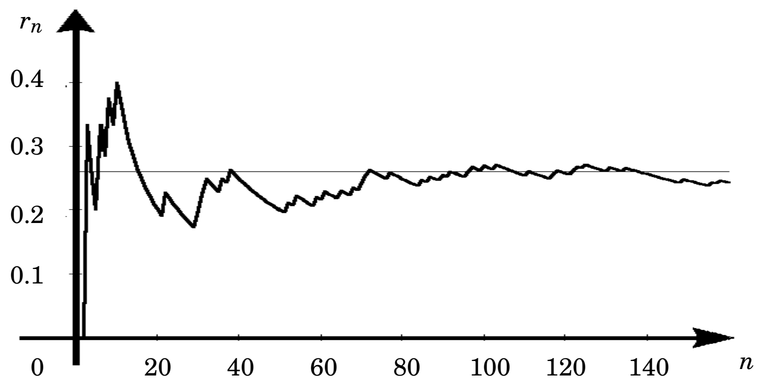


Figure 1.9. The behavior of the head occurrence frequency after 100 tossings of an asymmetric coin, $p = 0.26$

The graphical interpretation of this convergence is simple. If we enclose the graph of the function $e(x - p)$ (the vertical line at the jump point included) in a strip of width 2ε, then, whatever ε is, beginning with some n the graph of the function $G_n(x)$ appears to be lying inside the ε-neighborhood of the graph of $e(x - p)$ (Fig. 1.8). Here $\Phi(x)$ is the graph of the distribution function of the standard normal law. As n grows, the graphs of $G_n(x)$ approximate to $\Phi(x)$, which is exactly the graphical interpretation of the Moivre–Laplace theorem, the simplest version of the central limit theorem.

An impression can be formed that the Bernoulli theorem fits only simple problems of finding a probability of an individual event. It is formally so, but the potentialities of the law discovered by Bernoulli are much wider. In some form it manifests itself in a great body of problems of mathematical statistics.

We conclude our presentation of the Bernoulli theorem, which is now frequently referred to as the Bernoulli law of large numbers, with the demonstration of behavior of the head occurrence frequency in a long sequential tossing of an asymmetric coin given in Fig. 1.9.

1.8. The Moivre–Laplace theorem

The law of large numbers and the bounds for the unknown probability p resulting from it did not satisfy the mathematicians who fervently desired to refine them. One of such refinements was due to Moivre, and recently is referred to as the Moivre–Laplace theorem.

Under the modern notation, the Moivre result can be formulated as follows. Instead of the arithmetic means μ_n of the independent random variables

$X_1, X_2, \ldots$, we consider the sequence of normalized sums

$$Z_n = \frac{1}{b_n}\left(\sum_{i=1}^{n} X_i - a_n\right), \qquad n = 1, 2, \ldots, \tag{1.8.1}$$

where

$$a_n = np, \qquad b_n^2 = np(1-p),$$

and denote the corresponding distribution functions by $W_n(x)$.

We fix a pair of arbitrary numbers x and y so that $y < x$; then, as $n \to \infty$,

$$\mathsf{P}\{y \le Z_n < x\} = W_n(x) - W_n(y) \sim \frac{1}{\sqrt{2\pi}} \int_y^x \exp(-z^2/2)\, dz \tag{1.8.2}$$

uniformly in y and x. If we let $y \to -\infty$ on the right-hand side of (1.8.2), then the limit obtained, which depends only on x, appears to be some distribution function. It is usually denoted by $\Phi(x)$ and called the standard normal distribution, or the Gauss law.

The right-hand side of (1.8.2) is equal to $\alpha = \Phi(x) - \Phi(y)$. Since $\Phi(x) \to 1$ as $x \to \infty$, and $\Phi(y) \to 0$ as $y \to \infty$, by choosing a sufficiently large $x > 0$ and $|y|$, $y < 0$, we can always make α so close to one as desired. So, setting $x = -y = 1.96$, we obtain $\alpha = 0.95$. Therefore, for large n the inequality

$$-1.96 < Z_n < 1.96 \tag{1.8.3}$$

becomes practically certain (if we say that those are events of probability no less than 0.95).

In view of definition (1.8.1) of normalized sums Z_n, we derive from inequalities (1.8.3) the practically certain bounds

$$\mu_n - 1.96\sqrt{p(1-p)/n} < p < \mu_n + 1.96\sqrt{p(1-p)/n}, \tag{1.8.4}$$

which are asymptotically (i.e., for n large enough) more precise than bounds (1.7.6) following from the recent refinement (1.7.5) of the Bernoulli theorem. It should be said in all fairness that the advantage of bounds (1.7.5) over (1.8.4) is that the former are true for all n, whereas the latter, for sufficiently large n (without further specifying).

The Moivre–Laplace theorem represents a considerable step forward, which was not outperformed in more than gross of years, although such great mathematicians as Laplace, Gauss, and Chebyshov were developing this field.

The generalization of the Bernoulli law of large numbers turned out to be a more easygoing problem.

1.9. The law of large numbers

In 1830, Siméon Denis Poisson formulated a theorem that extended the law of large numbers. He did not provide a rigorous proof; it was given sixteen years later by Chebyshov. Poisson considered the case where independent random variables $X_1, X_2, \dots$ take values 0 and 1 but do not need to be identically distributed, i.e., the probabilities $p_i = \mathsf{P}\{X_i = 1\}$ can differ for different i. The theorem asserts that for an arbitrary small positive ε

$$\mathsf{P}\left\{\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n} p_i\right| \geq \varepsilon\right\} \to 0$$

as $n \to \infty$.

Poisson also introduced the notion of 'law of large numbers'. The theorem due to him is usually referred to as the Poisson law of large numbers. The next advance, in 1867, was due to Chebyshov. He considered a very general case where the independent random variables $X_1, X_2, \dots$ possess means $\alpha_i = \mathsf{E}X_i$ and variances $\sigma_i^2 = \operatorname{Var} X_i$, without any further constraints. The theorem due to Chebyshov looks as follows.

THEOREM 1.9.1 (Chebyshov). *For any $T \geq 1$ the probability that*

$$\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n} \alpha_i\right| \leq T\sqrt{\frac{1}{n}\sum_{i=1}^{n} \sigma_i^2}$$

is no less than $1 - 1/T^2$.

This theorem immediately implies a sufficient condition for the validity of the Chebyshov law of large numbers, which consists in the convergence in probability of the arithmetic means of the random variables X_i and the corresponding mathematical expectations α_i. This condition itself consists in the validity of the limit relation

$$\left(\sigma_1^2 + \dots + \sigma_n^2\right)/n^2 \to 0 \qquad n \to \infty. \tag{1.9.1}$$

We put stress on the fact that the Chebyshov theorem was proved by a remarkable simple approach by making use of the general inequality

$$\mathsf{P}\{|Y - \mathsf{E}Y| \geq T\} \leq \frac{\operatorname{Var} Y}{T^2}, \tag{1.9.2}$$

which holds for any $T > 0$ and any random variable possessing a finite expectation and a finite variance. Indeed, substituting the arithmetic mean of the independent random variables X_i for Y, we arrive at an inequality which implies the sufficiency of condition (1.9.1).

Some very interesting but now widely known results are due to Bernstein, who also dealt with the Poisson law of large numbers. Bernstein considered a sequence of random variables $X_1, X_2, \dots$ taking two values, 0 and 1. No extra assumption on independence was posed; nevertheless, for each of the random variables X_i,

$$0 < p_i = \mathsf{P}\{X_i = 1\} < 1.$$

In this case, obviously, $q_i = \mathsf{P}\{X_i = 0\}$ possesses the same property. To formulate the criterion of validity of the law of large numbers, one makes use of the conditional probabilities

$$p_{ij} = \mathsf{P}\{X_i = 1 \mid X_j = 1\} = \frac{\mathsf{P}\{X_i = 1, X_j = 1\}}{\mathsf{P}\{X_j = 1\}},$$
$$q_{ij} = \mathsf{P}\{X_i = 1 \mid X_j = 0\} = \frac{\mathsf{P}\{X_i = 1, X_j = 0\}}{\mathsf{P}\{X_j = 0\}},$$

The Bernstein criterion is expressed as follows. For an arbitrary small $\varepsilon > 0$,

$$\mathsf{P}\left\{\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n} p_i\right| > \varepsilon\right\} \to 0 \qquad n \to \infty \tag{1.9.3}$$

if and only if

$$\max_{1\le j\le n} p_j q_j \left|\frac{1}{n}\sum_{i=1}^{n} p_{ij} - \frac{1}{n}\sum_{i=1}^{n} q_{ij}\right| \to 0 \qquad n \to \infty.$$

Turning back to the scheme of summation of independent identically distributed random variables $\{X_i\}_{i=1}^{\infty}$, we present the following result known as the Khinchin law of large numbers

THEOREM 1.9.2 (Khinchin). *If the mathematical expectation* $\mathsf{E}X_i$ *exists and is equal to a, then the normalized sums* $Z_n = \sum_{i=1}^{n} X_i/n$ *converge in probability to* μ *as* $n \to \infty$*:*

$$Z_n \xrightarrow{\mathsf{P}} \mu, \qquad n \to \infty.$$

This means that for any $\varepsilon > 0$

$$\mathsf{P}\{|Z_n - \mu| > \varepsilon\} \to 0$$

as $n \to \infty$. In other words, the distribution of the random variable Z_n converges to the degenerate distribution concentrated in μ. The density of this distribution is represented by the Dirac δ-function $\delta(x - \mu)$, a generalized function which is zero for all $x \neq a$ and satisfies the normalization condition [4]

$$\int \delta(x - \mu)\, dx = 1.$$

As we see, we do not need the finiteness of variances any more; only the mathematical expectations should be finite.

1.10. Strong law of large numbers

We turn back to the Bernoulli law of large numbers in its second form, and consider the functions $X_1(\lambda), X_2(\lambda), \ldots$ which model sequential tossing of a coin (which does not need to be symmetric, and the probability p can be any number lying in the interval $(0, 1)$). The first three functions of such a form are presented in Fig. 1.3.

With the use of the functions $X_n(\lambda)$, we construct a multilayered graph similar to that in Fig. 1.5. Such a graph, as we have said, corresponds to some random process $X_t(\omega)$ with discrete time $t = 1, 2, \ldots$. We transform the function $X_t(\lambda)$ by the rule

$$\bar{X}_t(\lambda) = \frac{1}{n} \sum_{i=1}^{n} X_i(\lambda). \qquad (1.10.1)$$

The multilayered graph of $\bar{X}_1(\lambda)$ in Fig. 1.10 corresponds to some random process with discrete time. For each fixed outcome ω, which corresponds to the length-wise section of the graph of $\bar{X}_1(\lambda)$ passing through $\lambda = \omega$, we obtain a graph given in Fig. 1.11. What does the Bernoulli theorem say about the behavior of these functions corresponding to different ω as n grows? It turns out that the information available is very limited. The point is that relation (1.7.3) concerns the probability of an event associated with a single moment $t = n$, i.e., in the multilayered graph of $\bar{X}_t(\lambda)$ one considers only one section and concludes that the set of those λ which are beyond the strip $|X_n(\lambda) - p| < \varepsilon$ is of vanishing measure as n grows.

The Bernoulli theorem itself cannot help us to clear up the behavior of the functions emerging as length-wise sections of $\bar{X}_t(\lambda)$. We should invoke some general principle of probability theory which relates the probabilities of single events in a long series of trials and the probabilities of their simultaneous occurrences. This would allow us to consider several cross-sections of the graph of $\bar{X}_t(\lambda)$ at a time.

Such a principle which is able to help us is the well-known Borel–Cantelli lemma, more exactly, the following its part.

[4]Formally speaking, the Dirac δ-function can be considered as the derivative of the step function $e(x)$.

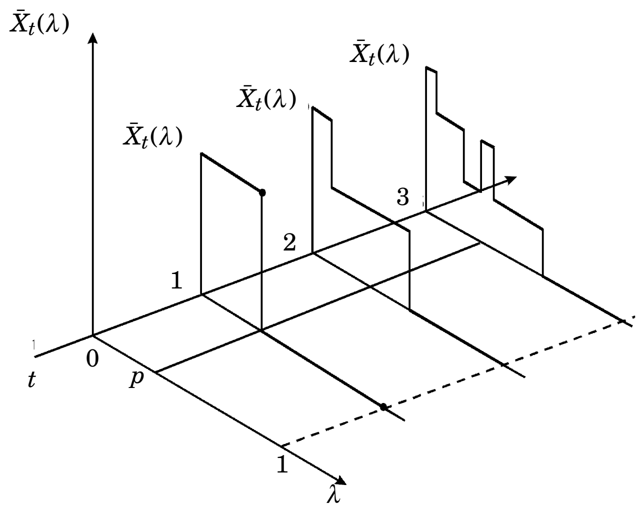


Figure 1.10.

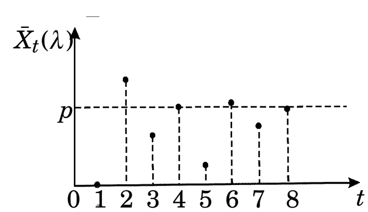


Figure 1.11.

We consider a sequence of arbitrary trials (not assuming that they are independent, etc.) with two possible outcomes each. This means that in the trial numbered n either the event A_n or its negation $\bar{A}_n$ occurs.

Let $C_1, C_2, \ldots$ be a chain of events where each C_n coincides with either A_n or its negation $\bar{A}_n$. We consider the set $C = C_1 \cap C_2 \cap \ldots$ consisting of simultaneous occurrence of the events $C_1, C_2, \ldots$ We divide all events C into two groups; the former group Γ is constituted by those events where at least one infinite subsequence of the events A_n exists, whereas the latter group $\bar{\Gamma}$ consists of the remaining events. The events C of the latter group are characterized by the property that, beginning with some place, their 'tails' contain only $\bar{A}_n$.

According to the Borel–Cantelli lemma, in the case where the series $\mathsf{P}\{A_1\} + \mathsf{P}\{A_2\} + \ldots$ converges, the probability of any event $C \in \Gamma$ is equal to zero.

As A_n, we take

$$A_n = \{\omega : |\mu_n(\omega) - p| \geq n^{-1/4}\},$$

and apply inequality (1.7.5) to estimating $Q(A_n)$; then

$$Q(A_n) \le 2\exp(-\sqrt{n/2}).$$

Hence it follows that the series of probabilities $Q(A_n)$ converges, i.e., that in a sequence of tossings of one and the same coin the inequality $|\mu_n - p| < n^{-1/4}$ can be not true infinitely many times only with zero probability. In other words, if in the graph of $X_t(\lambda)$ we take length-wise sections along different points λ, they can be divided into two non-overlapping sets B and $\bar{B}$. For $\lambda \in B$, the inequalities

$$\left| \frac{1}{n} \sum_{i=1}^{n} X_i(\lambda) - p \right| < n^{-1/4} \tag{1.10.2}$$

hold beginning with some n_0 (which, of course, need not be one and the same for different λ). The points of the set B do not possess this property. Moreover, the measure Q of the set $\bar{B}$ is equal to 1, whereas that of B is equal to 0. Dropping randomly a point ω, we almost surely find it in the set B. Saying 'almost sure', we mean the occurrence of an event with probability one, which is a synonym to certainty.

Property (1.10.2) immediately yields $\mu_n(\lambda) \to p$ as $n \to \infty$ for any $\lambda \in B$. Those λ for which the convergence of $\mu_n(\lambda)$ to p takes place, can find themselves in the set $\bar{B}$ as well, but for them relation (1.10.2) is systematically violated.

The regularities found with the help of the Borel–Cantelli lemma are much more rich in content than the Bernoulli law of large numbers, because we are able to trace the behavior of $\mu_n(\lambda)$ as n grows for almost all λ. This can be represented as follows:

$$\mathsf{P}\left\{ \lim_{n\to\infty} \mu_n(\omega) = p \right\} = 1. \tag{1.10.3}$$

This refinement of Bernoulli's theorem plays an important role in applications, because provides impressive evidence that the frequency of occurrence of a random event in a long series of independent trials is adequate to the probability of this event.

Property (1.10.3) is referred to as the strong law of large numbers, which was first formulated by Émile Borel in 1909. Later, many scientists dealt with generalizing various forms of the law of large numbers. Let us cite some of the breakthroughs in this field.

In 1927–1929, Khinchin studied sequences of dependent random variables $X_1, X_2, \dots$ with finite means $\alpha_i = \mathsf{E}X_i$ and variances $\sigma_i^2 = \operatorname{Var} X_i$, and suggested sufficient conditions for the strong law of large numbers, i.e., the validity with probability one of the limiting relation

$$\lim_{n\to\infty} \left| \frac{1}{n} \sum_{i=1}^{n} X_i - \frac{1}{n} \sum_{i=1}^{n} \alpha_i \right| = 0. \tag{1.10.4}$$

In particular, those conditions imply that in the case of identically distributed random variables with finite means $\alpha = \mathsf{E}X_i$ and variances $\sigma_i^2 = \operatorname{Var} X_i$, the equality

$$\lim_{n\to\infty} \left| \frac{1}{n} \sum_{i=1}^{n} X_i - \alpha \right| = 0 \tag{1.10.5}$$

holds with probability one.

Later on, Kolmogorov established (1.10.5) under the only condition that the mathematical expectation $\mathsf{E}|X_1|$ exists.

In 1933, Kolmogorov suggested a rather wide sufficient condition for the validity of (1.10.4) for sequences of independent random variables X_i possessing finite means α_i and variances σ_i^2: the series

$$\sigma_1^2 + \sigma_2^2/4 + \ldots + \sigma_n^2/n^2 + \ldots$$

should converge.

The question on necessary and sufficient conditions for the validity of the strong law of large numbers appeared to be quite difficult, and was solved only forty years after the arrival of astonishing elegant result due to Kolmogorov.

The solution of this problem is mainly due to two famous Russian mathematicians, Yu. Prokhorov and S. Nagaev.

First, Prokhorov attempted to unravel this problem; the results he obtained in the fifties set the stage for the Nagaev's quest later.

It turns out that the criterion sought for cannot be formulated in terms of means and variances of independent random variables.

Some evidences exist that the usage of moments of various orders fares poorly, and some more complex characteristics become a necessity.

Set, say, $f_n(s,\varepsilon) = \mathsf{E}\exp(sX_n)\mathbf{1}(|X_n| \le \varepsilon n)$ [5], where $s > 0$, $\varepsilon > 0$, and define $h_r(\varepsilon)$ as the unique solution of the equation

$$\sum\nolimits^{r} \frac{\partial}{\partial s} \ln f_n(s,\varepsilon) = \varepsilon n_r, \qquad n_r = 2^{r+1},$$

where $\sum^r$ stands for the summation over n from 2^r to 2^{r+1}.

The theorem due to Nagaev asserts that the strong law of large numbers, i.e., the validity of (1.10.4) with probability one, holds true if and only if for any $\varepsilon > 0$ the series

$$\sum_{n=1}^{\infty} \mathsf{P}\{X_n > \varepsilon n\}, \qquad \sum_{r=1}^{\infty} \exp\left(-\varepsilon h_r(\varepsilon) n_r\right)$$

converge.

[5]The function $\mathbf{1}(A)$ is the indicator of the event A; it takes value 1 if A occurs, and 0 otherwise

Not only the development of that briefly cited branch of probability theory was inspired by the Borel–Cantelli lemma. The strong law of large numbers, as a deepened version of the usual law of large numbers, attracted the attention of many mathematicians and propelled them into hunting for similar tendencies that hold with probability one in other models.

1.11. Ergodicity and stationarity

We cannot give here even a short survey of the results related to the Borel–Cantelli lemma. The Borel's strong law of large numbers manifests itself in schemes of summation of dependent random variables, random vectors, random processes with continuous time, random fields and schemes associated with various group operations.

We consider in more detain one result related to special types of random processes.

A process $X_t(\omega)$ is called narrow-sense stationary if for any finite set of times $t_1 < t_2 < \ldots < t_n$ and any real τ the joint distributions of the vectors

$$\bigl(X_{t_1}(\omega), X_{t_2}(\omega), \ldots, X_{t_k}(\omega)\bigr)$$

and

$$(X_{t_1+\tau}(\omega), X_{t_2+\tau}(\omega), \ldots, X_{t_k+\tau}(\omega))$$

coincide.

Let us recall Figures 1.5 and 1.6 which represent random processes with discrete and continuous time respectively. Choosing some times, we dissect the graph of $X_t(\lambda)$ cross-wise at the corresponding points. The set of the sections forms the set of functions, i.e., a model of a random vector. Shifts of the sections by some τ, of course, change the graphs, but, due to special structure of the function $X_t(\lambda)$, do not alter the distributions generated by the resulting graphs.

In particular, the distributions generated by the sections are identical, whereas the numerical characteristics such as the moments $\mathsf{E}X_t(\omega)$, $\operatorname{Var} X_t(\omega)$, etc., do not depend on t.

In 1931, a famed American mathematician G. Birkhoff obtained a very important result of the theory of dynamic systems, known as the Birkhoff's ergodic theorem.

In 1938, Khinchin discovered a direct inter-relation between the dynamic systems and narrow-sense stationary random processes. The analog of Birkhoff's theorem in the theory of stationary processes is the following assertion.

THEOREM 1.11.1 (Birkhoff–Khinchin). *If for a stationary process $X_t(\omega)$ the* *mathematical expectation* $\mathsf{E}\,|X_t(\omega)|$ *exists, then the limit*

$$\lim_{T-S\to\infty} \frac{1}{T-S} \sum_{t=S+1}^{T} X_t(\omega) = \hat{X} \tag{1.11.1}$$

for a process with discrete time, or the limit

$$\lim_{T-S\to\infty} \frac{1}{T-S} \int_S^T X_t(\omega)\,dt = \hat{X} \tag{1.11.2}$$

for a process with continuous time, exists with probability one.

Since the sequences $X_1, X_2, \dots$ of independent identically distributed random variables are particular cases of narrow-sense stationary processes with discrete time, property (1.11.1) becomes an immediate extension of the strong law of large numbers in the Kolmogorov's formulation above. The existence of the mathematical expectation is not sufficient for $\hat{X} = m = \mathsf{E}X_t(\omega)$, though. In this sense, Kolmogorov's theorem is stronger. In the case of narrow-sense stationary process, the extra condition that guarantees the validity of this equality consists in the existence of the variance $\sigma^2 = \operatorname{Var} X_t(\omega)$ and the property

$$\lim_{T\to\infty} \frac{1}{T} \sum_{\tau=0}^{T-1} B(\tau) = 0$$

for a process with discrete time, or

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T B(\tau)\,d\tau = 0$$

for a process with continuous time, where

$$B(\tau) = \mathsf{E}\left[X_{t+\tau}(\omega)X_t(\omega)\right] - m^2$$

is the correlation function of the process $X_t(\omega)$. It is hard to imagine a typical function modeling a stationary process, because its structure is very complicated indeed. Some degree of the knowledge how a stationary process looks like can be obtained from Fig. 1.6.

One of the properties of this process is the equality of areas under the graphs of the functions resulting from cross-wise sectioning of the surface of $X_t(\lambda)$, which corresponds to the independence of the mean $a = \mathsf{E}X_t(\omega)$ of time t.

Fig. 1.12 presents the functions $X_t(\lambda)$ and $\bar{X}_t(\lambda)$ modeling a stationary process X and its averaging $\bar{X}$. The easily observable peculiarity of the function $X_t(\lambda)$ is a gradual (as t grows) flattening of its graph.

In conclusion, we give an example. Consider a $\lambda \in (0,1)$ and represent it as the infinite decimal fraction

$$\lambda = 0.k_1k_2\dots k_n\dots,$$

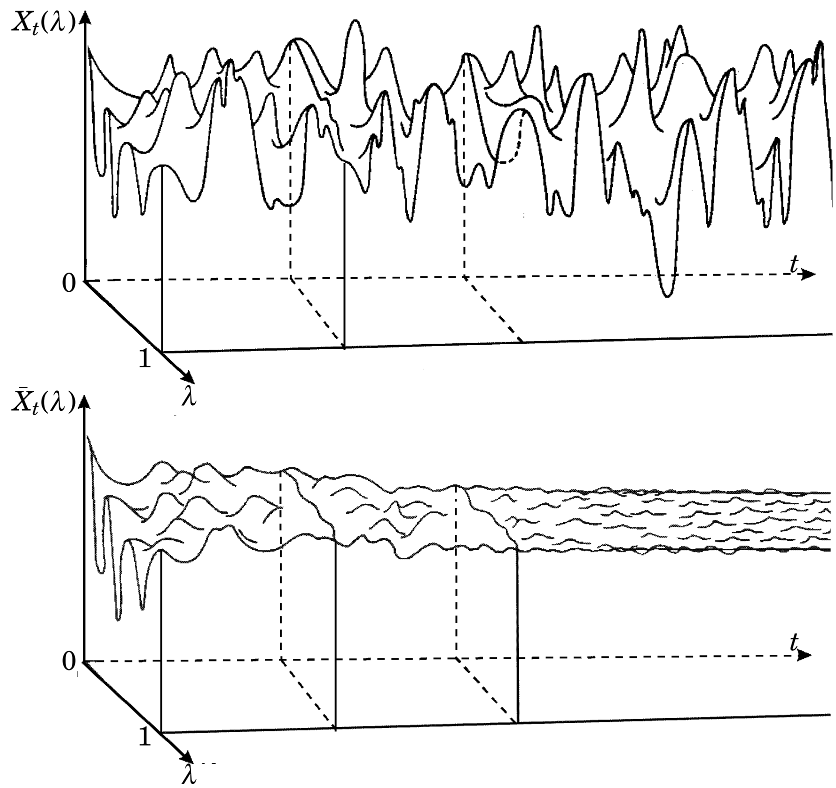


Figure 1.12.

allowing for infinite sequences of zeros but not allowing for infinite sequences of nines; then we become able to represent any λ uniquely as a decimal fraction. We introduce the function $X_t(\lambda)$, $0 < \lambda < 1$, $t = 1, 2, \ldots$, setting $X_n(\lambda) = k_n$, where k_n is the nth digit in the decimal representation of λ. As we know, the function $X_t(\lambda)$ models some random process with discrete time. The realizations of the process are length-wise sections of the function $X_t(\lambda)$ through the points $\lambda = 0, k_1, k_2, \ldots$ which are sequences $(k_1, k_2, \ldots)$. Their behavior is very diversified.

So, if λ is a rational number, then its representation $\lambda = 0, k_1, k_2, \ldots$ always, beginning with some place, contains a periodically repeating group of digits (the length of the period can be equal to one). Let us consider the sequence of arithmetical means of the first n digits in the decimal representation of λ, i.e.,

$$z_n(\lambda) = \frac{1}{n}(k_1 + \ldots + k_n), \qquad n = 1, 2, \ldots$$

In view of the just mentioned periodicity, as n grows, the sequence

$z_n(\lambda)$ possesses a limit that is equal to the arithmetical mean of the digits constituting the shortest periodically repeating group. For example, for $\lambda = 10/99 = 0.10101\ldots$ and $\lambda = 1/4 = 0.25000\ldots$, as $n \to \infty$, we obtain

$$z_n(\lambda_1) \to 1/2, \qquad z_n(\lambda_2) \to 0.$$

The part of limits for sequences z_n corresponding to rational λ can be played by any rational number lying in the interval $[0, 9)$.

For irrational λ, the description of the behavior of $z_n(\lambda)$ becomes much more tangled. First of all, in some cases there exist no limits of the numbers, which is illustrated by the example of $\lambda = 0.1122\ldots211\ldots122\ldots21\ldots$, where alternating groups of ones and twos grow very quickly: they are of lengths $2^1, 2^4, 2^9, 2^{16}, \ldots$

It is not hard to construct irrational λ such that $z_n(\lambda)$ tends to a prescribed rational q lying in the interval $(0, 9)$. To this end it suffices to take a rational μ such that $z_n(\mu) \to q$ and 'dilute' its decimal fraction representation by growing blocks of digits so that the total volume of these blocks among the first n digits increases as $o(n)$ while $n \to \infty$ and so that these extra blocks break down the periodicity. In a slightly more complicated manner we can construct irrational λ such that the corresponding sequence $z_n(\lambda)$ has a prescribed irrational number q lying in the interval $[0, 9]$ as its limit.

The above brainwork allows us to imagine how complicated is the function $X_t(\lambda)$. Nevertheless it is possible to prove that it models a narrow-sense stationary random process.

It is not hard to see that for the process $X_t(\omega)$ there exist finite mean and variance, and

$$\mathsf{E}X_t(\omega) = (0 + 1 + \ldots + 9)/10 = 4.5.$$

It is easily seen that for the process $X_t(\omega)$ the Birkhoff–Khinchin ergodic theorem holds; moreover, the role of the limit $\hat{X}$ in relation (1.11.1) is played by the mean $\mathsf{E}X(1, \omega) = 4.5$. It remains to note that

$$z_n(\omega) = \frac{1}{n}\sum_{i=1}^{n} X_t(\omega).$$

Thus, the limit relation

$$\lim z_n(\omega) = 4.5$$

holds with probability one.

1.12. The central limit theorem

We recall that the above-cited Moivre–Laplace theorem also obeys the law of large numbers in the sense that as $n \to \infty$, the distribution of the arithmetical means tends to a degenerate one. But, in contrast to the law of large numbers, this theorem asserts that, prior to degeneration, the distribution turns into Gaussian. It is natural to assume that the use of dichotomic random variable that takes two values, 0 and 1, is not necessary for such a behavior. Investigations concerning the generalization of the Moivre–Laplace theorem constitute an important part of probability theory and provide us with a great body of limit theorems and related results. Let us present the most widely known (for those who are not specialists in probability theory) version of the central limit theorem.

THEOREM 1.12.1 (central limit theorem). *Let $X_1, X_2, \ldots$ be independent identically distributed random variables with mean μ and variance $\sigma^2 < \infty$. Then, as $n \to \infty$,*

$$\mathsf{P}\left\{\frac{\sum_{i=1}^n X_i - n\mu}{\sigma\sqrt{n}} < x\right\} \Rightarrow \Phi(x) \equiv \int_{-\infty}^{x} p^{\mathrm{G}}(x)\,dx, \qquad (1.12.1)$$

where

$$p^{\mathrm{G}}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Since $\Phi(x)$ is continuous, actually, the convergence in (1.12.1) is uniform in x.

This theorem essentially extends the domain of applications of the normal distribution. Indeed, the random summands are now arbitrary, provided that

- the summands are identically distributed;
- the variance is finite.

Further investigation demonstrates that these conditions need not be present to arrive at the normal distribution. The first constraint is eliminated by means of the following theorem proved in 1922 by Lindeberg.

THEOREM 1.12.2 (Lindeberg). *Let $X_1, X_2, \ldots$ be independent random variables with zero means and variances $\sigma_1^2, \sigma_2^2, \ldots$. We set*

$$B_n^2 = \sum_{k=1}^{n} \sigma_k^2 = \operatorname{Var} \sum_{i=1}^{n} X_i.$$

If for any $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{B_n^2} \sum_{k=1}^{n} \int_{|x|>\varepsilon B_n} x^2 dF_k(x) = 0,$$

then the distributions of $\sum_{i=1}^n X_i/B_n$ weakly converge to the normal law.

The second constraint can be weakened due to the following assertion.

THEOREM 1.12.3 (on convergence to the normal law). *The distribution of the normalized sum*

$$Z_n = \frac{\sum_{i=1}^n X_i - a_n}{b_n}$$

of independent identically distributed random summands with distribution function $F(x)$, with some a_n, $b_n > 0$, converges to the normal law if and only if

$$\lim \frac{x^2 \int_{|y|>x} dF(y)}{\int_{|y|<x} x^2 dF(y)} = 0. \qquad (1.12.2)$$

The normalizing constants b_n may either grow as $\sqrt{n}$, which occurs if and only if the variance of the summands is finite, or differ from $\sqrt{n}$ by a slowly varying factor. In particular, for

$$\int_{|y|>x} dF(y) \propto x^{-2}, \qquad x \to \infty,$$

the variance of the summands is infinite, but (1.12.2) holds and $B_n \propto \sqrt{n \ln n}$ as $n \to \infty$.

Under various additional constraints, we are able to obtain better results. In particular, if independent identically distributed summands X_i with zero mean and variance σ^2 possess the finite third moment $\mathsf{E}|x_1|^3$, then there exists an absolute constant C such that

$$\left| \mathsf{P}\left\{ \frac{\sum_{i=1}^n X_i}{\sigma\sqrt{n}} < x \right\} - \Phi(x) \right| \le C \frac{\mathsf{E}|x_1|^3}{\sigma^3 \sqrt{n}}, \qquad -\infty < x < \infty;$$

C was estimated as 7.59, 2.9 (Esseen); 2.05 (Walles); 0.9051, 0.8197 (Zolotarev); 0.7975 (van Beek); 0.7655 (Shiganov) (see (Zolotarev, 1997; Senatov, 1998). It is known that $C \ge 1/\sqrt{2\pi}$.

The above theorems, as well as many others, answer the question under which hypotheses and with which rate the distributions of the appropriately normalized sums of independent random variables converge to the normal law. It is natural to pose the question in a different form: if normalized partial sums have a certain limiting distribution, what is this limit? Is it necessarily a Gaussian distribution? May some other law appear? This question was answered by Paul Lévy in the beginning of 1930s, so enriching probability theory immeasurably.

Those readers who are not familiar with stable distributions can take the following theorem due to Lévy as the definition.

THEOREM 1.12.4 (Lévy). *Let $X_1, X_2, \ldots$ be independent identically distributed random variables, and let there exist constants $b_n > 0$ and a_n such that*

$$\mathsf{P}\left\{ \frac{\sum_{i=1}^n X_i - a_n}{b_n} < x \right\} \Rightarrow G(x), \qquad n \to \infty, \qquad (1.12.3)$$

for some function $G(x)$ which is not degenerate. Then $G(x)$ is a stable law.

What are the stable laws differing from the normal one? Which constraints should be imposed on the distributions of the summands in order to find appropriate coefficients a_n and b_n, and how do they look? Which scientific and applied problems inspire such laws and what are their sequels? We will answer these questions in the following chapters of the present book.

2

Elementary introduction to the theory of stable laws

2.1. Convolutions of distributions

We begin with a simple example. Let X_1 and X_2 be independent random variables (r.v.'s) uniformly distributed on $[0,1]$. Their common distribution function and density function are

$$F_X(x) \equiv \mathsf{P}\{X < x\} = \begin{cases} 0, & x \le 0, \\ x, & 0 < x \le 1, \\ 1, & x > 1, \end{cases}$$

$$p_X(x) = F'_X(x) = \begin{cases} 0, & x < 0, \\ 1, & 0 < x < 1, \\ 0, & x > 1. \end{cases} \qquad (2.1.1)$$

The value of $p_X(x)$ at the points $x = 0$ and $x = 1$ is a matter of convention, say, $p_X(0) = p_X(1) = 0$. The sum of these variables $X_1 + X_2$ is a r.v. distributed on $[0,2]$. To find its distribution function

$$F_{X_1+X_2}(x) = \mathsf{P}\{X_1 + X_2 < x\}, \qquad (2.1.2)$$

look at Fig. 2.1.

The unit square contains all possible positions of the random point P with independent uniformly distributed coordinates X_1, X_2. The probability for such a point to fall into any part of the given square is equal to the area of that part. The event $X_1 + X_2 < x$ corresponds to the domain A lying below the line $X_2 = x - X_1$. The dependence of its area on x gives us the distribution function

$$F_{X_1+X_2}(x) = \begin{cases} x^2/2, & 0 \le x \le 1, \\ 2x - x^2/2 - 1, & 1 < x \le 2. \end{cases}$$

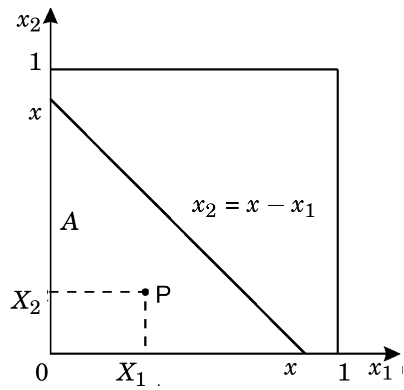


Figure 2.1.

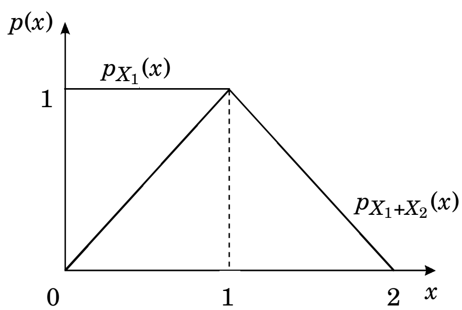


Figure 2.2.

Differentiating it with respect to x, we obtain the probability density of the sum:

$$p_{X_1+X_2}(x) = F'_{X_1+X_2}(x) = \begin{cases} x, & 0 \le x \le 1, \\ 2 - x, & 1 < x \le 2. \end{cases} \tag{2.1.3}$$

The density function of the summand (2.1.1) is in the shape of a square, the density of the sum of two terms (2.1.3) takes the form of a triangle (Fig. 2.2). Note also that the change of the form is accompanied by its shift and spread.

In the case of summing two arbitrarily distributed r.v.'s, probability (2.1.2) is determined by the probability measure of the domain $x_1 + x_2 < x$:

$$F_{X_1+X_2}(x) = \iint_{x_1+x_2<x} p_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2, \tag{2.1.4}$$

where

$$p_{X_1,X_2}(x_1,x_2)\,dx_1\,dx_2 = \mathsf{P}\{X_1 \in dx_1 \cap X_2 \in dx_2\}.$$

Introducing the Heaviside step function

$$e(x) = \begin{cases} 0, & x \le 0, \\ 1, & x > 0, \end{cases} \tag{2.1.5}$$

we can rewrite (2.1.4) as

$$F_{X_1+X_2}(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e(x - x_1 - x_2) p_{X_1,X_2}(x_1,x_2)\,dx_1\,dx_2. \tag{2.1.6}$$

Differentiating this equality with respect to x and keeping in mind that the derivative of $e(x)$ is the Dirac δ-function,

$$e'(x) = \delta(x), \tag{2.1.7}$$

we obtain

$$p_{X_1+X_2}(x) = \int_{-\infty}^{\infty} p_{X_1,X_2}(x - x', x')\,dx'. \tag{2.1.8}$$

For independent r.v.'s X_1 and X_2, we have

$$p_{X_1,X_2}(x_1,x_2) = p_{X_1}(x_1)p_{X_2}(x_2);$$

therefore we obtain

$$p_{X_1+X_2}(x) = \int_{-\infty}^{\infty} p_{X_1}(x - x')p_{X_2}(x')\,dx'. \tag{2.1.9}$$

Changing the integration variable, we can transform (2.1.9) to the equivalent form

$$p_{X_1+X_2}(x) = \int_{-\infty}^{\infty} p_{X_2}(x - x')p_{X_1}(x')\,dx'. \tag{2.1.10}$$

The operation defined by (2.1.9) and (2.1.10) is referred to as a convolution of distribution densities, and is denoted by the symbol $*$:

$$\int_{-\infty}^{\infty} p_{X_1}(x - x')p_{X_2}(x')\,dx' \equiv p_{X_1}(x) * p_{X_2}(x). \tag{2.1.11}$$

The same notation can also be applied to distribution functions:

$$\begin{aligned} F_{X_1+X_2}(x) &= \int_{-\infty}^{\infty} F_{X_1}(x - x')\,dF_{X_2}(x') \\ &= \int_{-\infty}^{\infty} F_{X_2}(x - x')\,dF_{X_1}(x') \\ &\equiv F_{X_1}(x) * F_{X_2}(x). \end{aligned}$$

For non-negative summands, the integration limits are changed in an appropriate way:

$$p_{X_1+X_2}(x) = \int_0^x p_{X_1}(x-x')p_{X_2}(x')\,dx'. \tag{2.1.12}$$

Applying (2.1.12) to the sum of uniformly distributed on (0, 1) r.v.'s,

$$p_{X_1+X_2}(x) = \int_0^1 p_X(x-x')\,dx' = \int_{x-1}^x p_X(x')\,dx'$$

and substituting (2.1.1), we again arrive at distribution (2.1.3).

We turn back to (2.1.9), and integrate both its parts with respect to x:

$$\int_{-\infty}^{\infty} p_{X_1+X_2}(x)\,dx = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} p_{X_1}(x'')\,dx''\right]p_{X_2}(x')\,dx' = \int_{-\infty}^{\infty} p_{X_2}(x')\,dx' = 1.$$

This is natural: a distribution density function should be normalized to 1. The corresponding expressions for the first and the second moments of the distribution $p_{X_1+X_2}(x)$ are

$$\begin{aligned}
\int_{-\infty}^{\infty} xp_{X_1+X_2}(x)dx &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}(x''+x')p_{X_1}(x'')dx''\right]p_{X_2}(x')\,dx' \\
&= \int_{-\infty}^{\infty} x'p_{X_1}(x')\,dx' + \int_{-\infty}^{\infty} x'p_{X_2}(x')\,dx', \\
\int_{-\infty}^{\infty} x^2p_{X_1+X_2}(x)\,dx &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}(x''+x')^2p_{X_1}(x'')\,dx''\right]p_{X_2}(x')\,dx' \\
&= \int_{-\infty}^{\infty} x^2p_{X_1}(x)\,dx + 2\int_{-\infty}^{\infty} xp_{X_1}(x)\,dx\int_{-\infty}^{\infty} xp_{X_2}(x)\,dx \\
&\quad + \int_{-\infty}^{\infty} x^2p_{X_2}(x)\,dx.
\end{aligned}$$

Introducing the notation

$$\mathsf{E}X \equiv \int_{-\infty}^{\infty} xp_X(x)\,dx, \qquad \operatorname{Var}X = \mathsf{E}X^2 - (\mathsf{E}X)^2,$$

we can present the abovesaid as follows:

$$\mathsf{E}(X_1+X_2) = \mathsf{E}X_1 + \mathsf{E}X_2, \tag{2.1.13}$$

$$\operatorname{Var}(X_1+X_2) = \operatorname{Var}X_1 + \operatorname{Var}X_2. \tag{2.1.14}$$

Formulae (2.1.13) and (2.1.14) lie at the heart of the probability theory: mathematical expectation of a sum of r.v.'s is equal to the sum of expectations of summands (2.1.13), and the variance of a sum of independent r.v.'s is equal to the sum of variances (2.1.14).

However, the assertions given here are not rigorous. They should be supplemented by the condition that the summands on the right-hand side of (2.1.13), (2.1.14) exist. In the case of unbounded in x distributions, this means the existence of the improper integrals

$$\int_{-\infty}^{\infty} x p_X(x)\,dx = a, \qquad a = \mathsf{E}X,$$
$$\int_{-\infty}^{\infty} x^2 p_X(x)\,dx = a^2 + \sigma^2, \qquad \sigma^2 = \operatorname{Var} X.$$

It is clear that if a r.v. is distributed in a bounded domain, then $p_X(x)$ is equal to zero outside this domain and no problem with the existence of the moments arises. Such a problem arises in the only case where the domain of values of a r.v. is infinite, and the density function decreases not so fast at large distances from the origin. Such cases are to be found in physics, but experienced physicists always discover the reason why the tail of such a distribution may be (and even must be) truncated on large distances and then all moments, of course, exist.

Our book, however, is devoted to those distributions for which the integrals representing variances, or even expectations, are divergent. As it will be understood from the following, such distributions can be extremely useful in a series of physical, and not only physical, applications.

2.2. The Gauss distribution and the stability property

The most popular distribution in various physical and engineering applications is the normal distribution (Gauss law):

$$p^{\mathrm{G}}(x; a, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-a)^2}{2\sigma^2}\right\}, \quad -\infty < x < \infty, \quad \sigma > 0. \quad (2.2.1)$$

In view of (2.1.9), the density function of the sum of two normally distributed independent r.v.'s with parameters a_1, σ_1, a_2, σ_2, respectively, is of the form

$$p_{X_1+X_2}(x) = \frac{1}{2\pi\sigma_1\sigma_2} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x-x'-a_1)^2}{2\sigma_1^2} - \frac{(x'-a_2)^2}{2\sigma_2^2}\right\} dx'. \quad (2.2.2)$$

Let us evaluate this integral. We present the integrand as $\exp\{-\psi(x,x')/2\}$, where

$$\psi(x, x') = \frac{(x-x'-a_1)^2}{\sigma_1^2} + \frac{(x'-a_2)^2}{\sigma_2^2}.$$

Changing the variables

$$x = \sigma z + a_1 + a_2, \quad x' = \sigma y + a_2, \quad \sigma = \sqrt{\sigma_1^2 + \sigma_2^2},$$

removing the brackets and rearranging the terms, we obtain

$$\psi(x, x') = \frac{\sigma^4}{\sigma_1^2 \sigma_2^2} \left[y - \left(\frac{\sigma_2}{\sigma} \right)^2 z \right]^2 + z^2.$$

We substitute this function in the exponent in (2.2.2), and obtain

$$p_{X_1+X_2}(x) = \frac{1}{2\pi} \frac{\sigma \exp\left\{-z^2/2\right\}}{\sigma_1 \sigma_2} \int_{-\infty}^{\infty} \exp\left\{ -\frac{\sigma^4}{\sigma_1^2 \sigma_2^2} \left[y - \left(\frac{\sigma_2}{\sigma} \right)^2 z \right]^2 \right\} dy.$$

Taking into account that the integral of the density (2.2.1) is equal to one by means of normalization and turning back to the former variable, we arrive at

$$p_{X_1+X_2}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{(x - a_1 - a_2)^2}{2\sigma^2} \right\}, \qquad \sigma^2 = \sigma_1^2 + \sigma_2^2.$$

This means that the sum of two independent normally distributed r.v.'s with parameters (a_1, σ_1) and (a_2, σ_2) is a r.v. which is normally distributed with parameters $(a_1 + a_2, \sqrt{\sigma_1^2 + \sigma_2^2})$:

$$p_{X_1+X_2}(x) = p^{\mathrm{G}}(x; a_1 + a_2, \sqrt{\sigma_1^2 + \sigma_2^2}).$$

In other words, the convolution of normal distributions is again a normal distribution:

$$p^{\mathrm{G}}(x; a_1, \sigma_1) * p^{\mathrm{G}}(x; a_2, \sigma_2) = p^{\mathrm{G}}(x; a_1 + a_2, \sqrt{\sigma_1^2 + \sigma_2^2}). \tag{2.2.3}$$

In Fig. 2.3, two initial distributions and their convolution are presented. As well as in the case of rectangular distributions (Fig. 2.2) the resulting distribution does not coincide with the initial ones: while summing, the barycentre of distribution is shifted along the x-axes and the distribution becomes more 'dissipated'.

The basic distinction between these cases consists of that in the first of them, the form of distribution changes, whereas it is not so in the second case. It is necessary, however, to determine what is meant by the form of distribution, or, at least, to establish the sense of expressions of a type 'the form of distribution changes' , 'the form of distribution does not change'.

Let us introduce the equivalence relation: we say that two r.v.'s X and Y are equivalent if their distributions coincide, and write

$$X \stackrel{d}{=} Y;$$

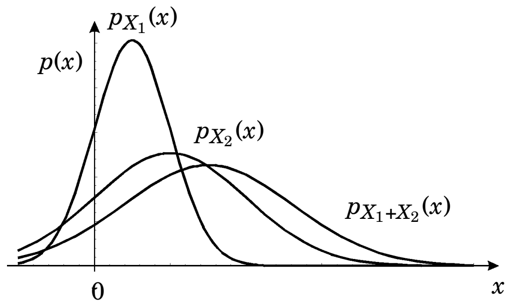


Figure 2.3.

this hence means

$$p_X(x) = p_Y(x).$$

We does not assume that X and Y are pair-wise independent; if X is a uniformly distributed on (0, 1) r.v., then

$$1 - X \stackrel{d}{=} X,$$

and for any symmetrically distributed (about zero) Y,

$$-Y \stackrel{d}{=} Y.$$

We also introduce the similarity relation: we say that two r.v.'s X and Y are similar and write

$$Y \stackrel{s}{=} X$$

if there exist constants a and $b > 0$ such that

$$Y \stackrel{d}{=} a + bX.$$

Otherwise, the distributions of two r.v.'s are of the same form, if there exists a linear transformation which brings them into coincidence. Indeed,

$$\begin{aligned} p_Y(x)dx = p_{a+bX}(x)dx &= \mathsf{P}\{x < a + bX \le x + dx\} \\ &= \mathsf{P}\left\{\frac{x-a}{b} < X \le \frac{x-a}{b} + \frac{dx}{b}\right\} \\ &= p_X\left(\frac{x-a}{b}\right)\frac{dx}{b}. \end{aligned} \tag{2.2.4}$$

As concerns distribution functions, we have

$$F_{a+bX}(x) = F_X\left(\frac{x-a}{b}\right).$$

As we can see from (2.2.1)), if we set

$$p^{\mathrm{G}}(x) \equiv p^{\mathrm{G}}(x;0,1) = \frac{1}{\sqrt{2\pi}} \exp\left\{-x^2/2\right\}, \tag{2.2.5}$$

then the distribution of any normal r.v. is expressed in terms of the standard normal distribution (2.2.5) by means of relation (2.2.4) as follows:

$$p^{\mathrm{G}}(x;a,\sigma) = \frac{1}{\sigma} p^{\mathrm{G}}\left(\frac{x-a}{\sigma}\right).$$

We write Y^{G} for the normal r.v. distributed by law (2.2.5); then (2.2.3) can be rewritten as

$$\sigma_1 Y_1^{\mathrm{G}} + \sigma_2 Y_2^{\mathrm{G}} \stackrel{d}{=} \sqrt{\sigma_1^2 + \sigma_2^2}\, Y^{\mathrm{G}}.$$

where Y_1^{G} and Y_2^{G} are independent random variables with the same distribution as Y^{G}. Assuming $\sigma_1 = \sigma_2 = 1$ and applying this relation to an arbitrary number of summands, we obtain

$$\sum_{i=1}^{n} Y_i^{\mathrm{G}} \stackrel{d}{=} \sqrt{n}\, Y^{\mathrm{G}} \tag{2.2.6}$$

or

$$\sum_{i=1}^{n} Y_i^{\mathrm{G}} \stackrel{s}{=} Y^{\mathrm{G}}, \quad a = 0, \quad b = \sqrt{n}.$$

Property (2.2.3) can be re-formulated now as follows: the sum of independent normal r.v.'s is similar to the summands. This property of the normal distribution lies at the heart of the general definition of stable distributions.

DEFINITION OF STABLE R.V.'S. A random variable Y is referred to as stable if

$$\sum_{i=1}^{n} Y_i \stackrel{s}{=} Y \tag{2.2.7}$$

for any n, i.e., if there exist constants $b_n > 0$ and a_n such that

$$\sum_{i=1}^{n} Y_i \stackrel{d}{=} a_n + b_n Y, \tag{2.2.8}$$

where $Y_1, Y_2, \dots$ are independent random variables each having the same distribution as Y.

DEFINITION OF STRICTLY STABLE R.V.'S. A stable r.v. is called strictly stable if (2.2.8) holds with $a_n = 0$:

$$\sum_{i=1}^{n} Y_i \stackrel{d}{=} b_n Y, \tag{2.2.9}$$

where $Y_1, Y_2, \dots$ are independent random variables with the same distribution as Y.

The normal r.v. Y^{G} is strictly stable with

$$b_n^{\mathrm{G}} = \sqrt{n}. \tag{2.2.10}$$

Applying the mathematical expectation operator to both parts of (2.2.8)

$$n\mathsf{E}Y \stackrel{d}{=} a_n + b_n \mathsf{E}Y$$

and assuming $\mathsf{E}Y = 0$, we obtain

$$a_n = 0.$$

Thus, if the expectation of a stable r.v. exists and is zero, this variable is strictly stable. Note that (2.2.6) rewritten in the form

$$\frac{1}{n}\sum_{i=1}^{n} Y_i^{\mathrm{G}} \stackrel{d}{=} Y^{\mathrm{G}}/\sqrt{n} \tag{2.2.11}$$

manifests itself as the law of large numbers: the larger n, the closer (in the probabilistic sense) the arithmetic mean is to the mathematical expectation $\mathsf{E}Y^{\mathrm{G}} = 0$.

Actually, as was shown by P.Lévy, a r.v. is stable as soon as (2.2.7) is true for $n = 2$ and 3 (Feller, 1966). Keeping this in mind, one can make use of another definition of stable r.v. that is equivalent to the one given above.

DEFINITION OF STABLE R.V.'S. A random variable Y is stable if and only if for any arbitrary constants b' and b'' there exist constants a and b such that

$$b'Y_1 + b''Y_2 \stackrel{d}{=} a + bY, \tag{2.2.12}$$

where Y_1 and Y_2 are independent and

$$Y_1 \stackrel{d}{=} Y_2 \stackrel{d}{=} Y.$$

Distribution functions and densities of stable (strictly stable) r.v.s are called stable (strictly stable) and denoted by $G(x)$ and $q(x)$ respectively. Definition (2.2.12) can expressed in terms of $G(x)$ and $q(x)$ as

$$G\left(\frac{x}{b'}\right) * G\left(\frac{x}{b''}\right) = G\left(\frac{x-a}{b}\right) \tag{2.2.13}$$

and

$$\frac{1}{b'b''} q\left(\frac{x}{b'}\right) * q\left(\frac{x}{b''}\right) = \frac{1}{b} q\left(\frac{x-a}{b}\right). \tag{2.2.14}$$

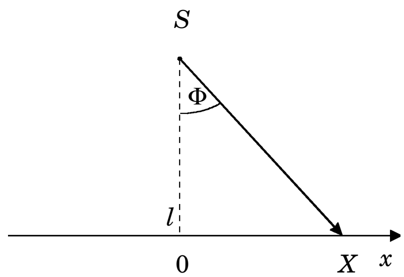


Figure 2.4.

Definition (2.2.9) of strictly stable r.v.'s is equivalent to

$$q(x) * q(x) = (1/b_2)q(x/b_2), \tag{2.2.15}$$

$$q(x) * q(x) * q(x) = (1/b_3)q(x/b_3), \tag{2.2.16}$$

and so on. In particular, distribution (2.2.5) is a strictly stable distribution satisfying (2.2.15)–(2.2.16) with coefficients $b_2 = \sqrt{2}$, $b_3 = \sqrt{3}$.

2.3. The Cauchy and Lévy distributions

The Gaussian distribution is not the only distribution law which possesses the stability property. We give here two more examples of stable distributions.

At some point S, let an emitter of particles be placed, and at the distance l away from it, let the screen be installed (Fig. 2.4). The particles are emitted in a plane which is perpendicular to the screen, and the angle Φ between the plane anf the normal to the screen is a random variable which is uniformly distributed on $(-\pi/2, \pi/2)$. Let us find the distribution of the random coordinate $X = l \tan \Phi$ on the screen, assuming that the particles fly along straight lines. By virtue of the monotonous dependence X on Φ,

$$F_X(x) = \mathsf{P}\{X < x\} = \mathsf{P}\{\Phi < \phi(x)\}, \qquad \phi(x) = \arctan(x/l).$$

Since

$$\mathsf{P}\{\Phi < \phi(x)\} = F_\Phi(\phi(x)) = 1/2 + \phi(x)\pi,$$

we arrive at

$$p_X(x) = F'_X(x) = \frac{\phi'(x)}{\pi} = \frac{l}{\pi(l^2 + x^2)}. \tag{2.3.1}$$

Distribution (2.3.1) is called the Cauchy law. It is a symmetrical 'bell-shaped' function like the Gaussian distribution (Fig. 2.5), but it differs from that in the behavior of their tails: the tails of the Cauchy density decrease as

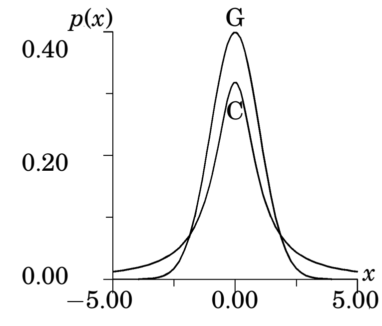


Figure 2.5.

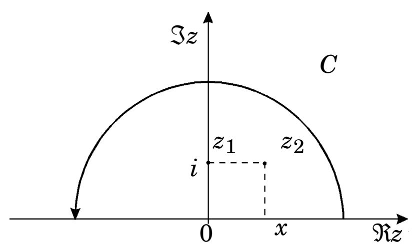


Figure 2.6.

x^{-2}; the variance does not hence exist. The parameter l plays the role of scale factor. In what follows, we use the representation where l is assumed to be equal to 1:

$$p^{\mathrm{C}}(x) = \frac{1}{\pi(1+x^2)}. \tag{2.3.2}$$

We denote the corresponding r.v. by Y^{C}.

If this distribution were stable, it should be strictly stable due to its symmetry about $x = 0$; to prove its stability, we have to make sure that relations (2.2.6)–(2.2.7) hold with $a = a' = a'' = 0$. We rewrite the expression for $p_2(x)$ as

$$p_2(x) = p^{\mathrm{C}}(x) * p^{\mathrm{C}}(x) = (1/\pi)^2 \int_{-\infty}^{\infty} \frac{dz}{[1+(x-z)^2][1+z^2]}. \tag{2.3.3}$$

This integral can be transformed to an integral along a closed contour C in the complex plane (Fig. 2.6), which encloses the two poles of the integrand $z_1 = i$

and $z_2 = x + i$. Applying the residue method, we obtain

$$
\begin{aligned}
p_2(x) &= (1/\pi)^2 \oint_C \frac{dz}{(z-i)(z+i)[z-(x+i)][z-(x-i)]} \\
&= (2i/\pi)\left\{\frac{1}{2i(-x)(-x+2i)} + \frac{1}{x(x+2i)2i}\right\} \\
&= \frac{1}{2\pi[1+(x/2)^2]},
\end{aligned}
$$

i.e.,

$$
p^{\mathrm{C}}(x) * p^{\mathrm{C}}(x) = \frac{1}{2} p^{\mathrm{C}}\left(\frac{x}{2}\right).
$$

Evaluating in the same way the convolution $p^{\mathrm{C}}(x) * p^{\mathrm{C}}(x) * p^{\mathrm{C}}(x)$, we see that the Cauchy distribution is strictly stable, and the scale factors are

$$
b_2^{\mathrm{C}} = 2, \qquad b_3^{\mathrm{C}} = 3. \tag{2.3.4}
$$

In terms of r.v.'s, this property takes a quite unexpected form

$$
(Y_1^{\mathrm{C}} + \ldots + Y_n^{\mathrm{C}})/n \stackrel{d}{=} Y_1^{\mathrm{C}} \tag{2.3.5}
$$

meaning that the arithmetic mean of the Cauchy r.v.'s is distributed as an individual term of the sum. It is worthwhile to give another remarkable property of the Cauchy random variable X^{C}:

$$
1/Y^{\mathrm{C}} \stackrel{d}{=} Y^{\mathrm{C}}. \tag{2.3.6}
$$

Indeed, for $x > 0$

$$
\begin{aligned}
F_{1/Y^{\mathrm{C}}}(x) &= \mathsf{P}\{1/Y^{\mathrm{C}} < x\} \\
&= \tfrac{1}{2} + \mathsf{P}\{0 < 1/Y^{\mathrm{C}} < x\} \\
&= \tfrac{1}{2} + \mathsf{P}\{Y^{\mathrm{C}} > 1/x\} \\
&= \tfrac{3}{2} - F_{Y^{\mathrm{C}}}(1/x).
\end{aligned} \tag{2.3.7}
$$

After differentiating and making use of (2.3.2), this yields

$$
p_{1/Y^{\mathrm{C}}}(x) = p_{Y^{\mathrm{C}}}(x). \tag{2.3.8}
$$

Now we consider a non-symmetric one-sided (concentrated on the positive semiaxis) distribution named the Lévy distribution. Let X be a r.v. distributed by the normal law with density

$$
p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-x^2/2\right\}, \tag{2.3.9}
$$

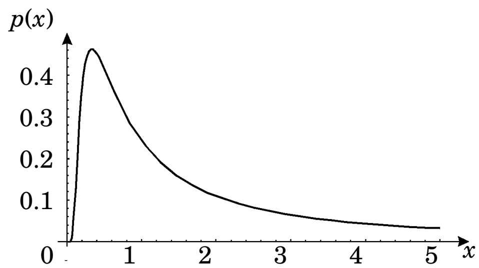


Figure 2.7.

and set $Y = X^{-2}$. By definition, the distribution function of Y is

$$F_Y(x) = \mathsf{P}\{Y < x\} = \mathsf{P}\left\{1/X^2 < x\right\} = 2\mathsf{P}\{X > 1/\sqrt{x}\}$$
$$= \sqrt{2/\pi} \int_{1/\sqrt{x}}^{\infty} \exp\left\{-y^2/2\right\} dy.$$

Differentiating this expression with respect to x, we obtain

$$p_Y(x) \equiv p^{\mathrm{L}}(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2x}\right\} x^{-3/2}, \qquad x > 0. \tag{2.3.10}$$

This is exactly the well-known Lévy distribution[1]. Its graph is presented in Fig. 2.7. In view of the existence of the inverse value of the argument of the exponent, all derivatives of the Lévy density at the origin are equal to zero. The density attains its maximum at the point $x = 1/3$, and its tail is

$$p^{\mathrm{L}}(x) \sim \frac{1}{\sqrt{2\pi}} x^{-3/2}, \qquad x \to \infty.$$

Both the variance and expectation of this distribution are infinite.

The direct evaluation of the convolution of distributions (2.3.10) is rather a tiresome procedure. We obtain the distribution of $Y_1 + Y_2$ using the well-known connection with the normal law specified above.

Let X_1 and X_2 be independent r.v.'s with density (2.3.9). The probability for the point P with these coordinates to fall into $dA = dx_1\, dx_2$ is

$$\mathsf{P}\{P \in dA\} = \frac{1}{2\pi} \exp\left\{-\frac{x_1^2 + x_2^2}{2}\right\} dA. \tag{2.3.11}$$

[1]The reader can meet in the literature some formulae for the Cauchy and Lévy distributions which differ from those cited here like the standard normal distribution (with unit variance) differs from (2.2.5). Various forms of representations of the stable laws will be considered in some of the following chapters. The stable distributions considered in the present chapter are given in form *A*.

Passing to the polar coordinates r, φ,

$$x_1 = r\cos\varphi, \qquad x_2 = r\sin\varphi,$$
$$X_1 = R\cos\Phi, \qquad X_2 = R\sin\Phi,$$

instead of (2.3.11) we obtain

$$\mathsf{P}\{P \in dA\} = \mathsf{P}\{R \in dr,\ \Phi \in d\varphi\} = \exp\left\{-r^2/2\right\} r\,dr\frac{d\varphi}{2\pi}. \qquad (2.3.12)$$

It can easily be seen from this formula that the distance $R = \sqrt{X_1^2 + X_2^2}$ and the azimuth angle Φ are independent. This is an important property of the bivariate normal distribution (2.3.11).

Now we consider the sum of two independent r.v.'s distributed by the Lévy law

$$Z = Y_1 + Y_2 = \frac{1}{X_1^2} + \frac{1}{X_2^2}. \qquad (2.3.13)$$

Its distribution function is

$$F_Z(z) = \mathsf{P}\{Z < z\} = \mathsf{P}\left\{R^2 \sin^2 2\Phi > 4/z\right\}$$
$$= \frac{1}{2\pi}\iint_{A_z} \exp\left\{-r^2/2\right\} r\,dr\,d\varphi, \qquad (2.3.14)$$

where A_z means the domain of values of the variables r, φ (Fig. 2.8) defined by the inequality

$$r^2 \sin^2 2\varphi > 4/z.$$

Integrating with respect to r while φ is fixed,

$$\int_{2/(\sqrt{z}|\sin 2\varphi|)}^{\infty} e^{-r^2/2} r dr = \exp\{-2[z\sin^2 2\varphi]^{-1}\},$$

and integrating then with respect to φ, we arrive at

$$F_Z(z) = \frac{1}{2\pi}\int_0^{2\pi} \exp\{-2[z\sin^2 2\varphi]^{-1}\}\,d\varphi.$$

The density

$$p_Z(z) = \frac{4}{\pi z^2}\int_0^{\pi/2} \exp\{-2[z\sin^2 2\varphi]^{-1}\}\frac{d\varphi}{\sin^2 2\varphi}$$

by the change of variables $t = \cot 2\varphi$ is transformed to

$$p_Z(x) = \sqrt{2/\pi}\exp\left\{-2/z\right\} z^{-3/2}.$$

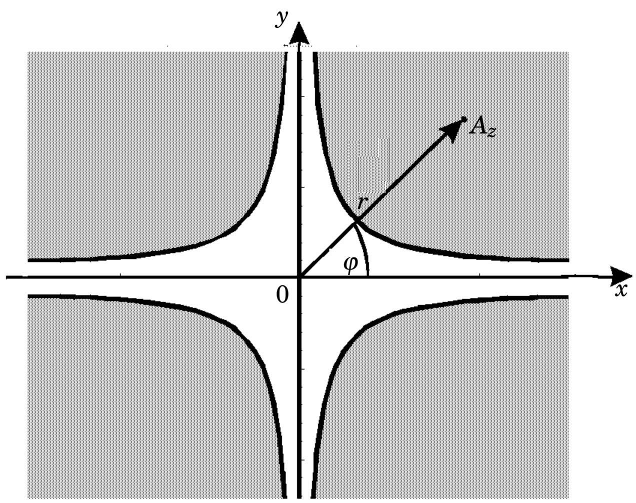


Figure 2.8.

Comparing this result with (2.3.10), we see that the convolution of Lévy distributions is of the form

$$p^{\mathrm{L}}(x) * p^{\mathrm{L}}(x) = (1/4)p^{\mathrm{L}}(x/4),$$

which is consistent with (2.2.11) as

$$b_2^{\mathrm{L}} = 4. \tag{2.3.15}$$

Repeating the above procedure, we are able to validate (2.2.12) and thus complete the proof of the strict stability of the Lévy distribution. The scale factor is

$$b_3^{\mathrm{L}} = 9. \tag{2.3.16}$$

Writing out Y^{L} for the r.v. with distribution density (2.3.9) taking into account that distribution (2.3.8) corresponds to the r.v. $Y^{\mathrm{G}}/\sqrt{2}$, we can represent the results obtained above as follows:

$$Y^{\mathrm{L}} \stackrel{d}{=} [Y^{\mathrm{G}}]^{-2}, \tag{2.3.17}$$

$$(Y_1^{\mathrm{L}} + Y_2^{\mathrm{L}})/2 \stackrel{d}{=} 2Y_1^{\mathrm{L}}. \tag{2.3.18}$$

The latter relation is even more surprising than (2.3.5): the arithmetic mean of Y_i^{L} has a more ‘dissipated’ distribution than an individual term.

In view of these unusual properties of stable random variables, the problem of their summation deserves a more close look.

Before doing that, we give one more useful relation between the Gauss and Cauchy random variables. Let

$$Z = Y_1^{\mathrm{G}}/Y_2^{\mathrm{G}}$$

with independent normal variables Y_1^{G} and Y_2^{G}. It is clear that Z is a symmetrical random variable and its distribution function can be written in the form (for $z > 0$)

$$\begin{aligned}
F_Z(z) &= \mathsf{P}\{Y_1^{\mathrm{G}}/Y_2^{\mathrm{G}} < z\} \\
&= \tfrac{1}{2}\mathsf{P}\{Y_1^{\mathrm{G}}/Y_2^{\mathrm{G}} < z \mid Y_1^{\mathrm{G}} > 0, Y_2^{\mathrm{G}} > 0\} \\
&= \tfrac{1}{2}\int_0^\infty dx_1 \int_{x_1/z}^\infty dx_2 p_{Y_1^{\mathrm{G}},Y_2^{\mathrm{G}}}(x_1, x_2 \mid Y_1^{\mathrm{G}} > 0, Y_2^{\mathrm{G}} > 0) \\
&= \frac{1}{\pi}\int_0^\infty dx_1 e^{-x_1^2/2} \int_{x_1/z}^\infty e^{-x_2^2/2}\, dx_2.
\end{aligned}$$

Differentiating the equality with respect to z, we obtain

$$p_Z(z) = \frac{1}{\pi z^2}\int_0^\infty e^{-[1+z^{-2}]x_1^2/2} x_1 dx_1 = \frac{1}{\pi(1+z^2)};$$

therefore

$$Y_1^{\mathrm{G}}/Y_2^{\mathrm{G}} \stackrel{d}{=} Y^{\mathrm{C}}. \tag{2.3.19}$$

2.4. Summation of strictly stable random variables

In view of (2.2.8), the problem of summing stable r.v.'s, i.e., finding the distribution of their sum S_n, can be reduced to finding a_n, b_n. For a strictly stable r.v., $a_n = 0$, and (2.2.8) takes the form

$$S_n = \sum_{i=1}^n Y_i \stackrel{d}{=} b_n Y. \tag{2.4.1}$$

This problem is most easily solved for the normal distribution which is the only stable distribution with finite variance. Calculating variances of both sides of (2.4.1), we obtain

$$n \operatorname{Var} Y = b_n^2 \operatorname{Var} Y, \tag{2.4.2}$$

which, together with $\operatorname{Var} Y \neq 0$, immediately yields

$$b_n \equiv b_n^G = n^{1/2}. \tag{2.4.3}$$

We consider now the general case of summation of strictly stable r.v.'s. Rewriting (2.4.1) as the sequence of sums

$$
\begin{aligned}
Y_1 + Y_2 &\stackrel{d}{=} b_2 X\\
Y_1 + Y_2 + Y_3 &\stackrel{d}{=} b_3 X\\
Y_1 + Y_2 + Y_3 + Y_4 &\stackrel{d}{=} b_4 X\\
&\dots
\end{aligned}
\tag{2.4.4}
$$

we consider only those sums which contain 2^k terms, $k = 1, 2, \dots$:

$$
\begin{aligned}
Y_1 + Y_2 &\stackrel{d}{=} b_2 Y\\
Y_1 + Y_2 + Y_3 + Y_4 &\stackrel{d}{=} b_4 Y\\
Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 + Y_7 + Y_8 &\stackrel{d}{=} b_8 Y\\
&\dots\\
Y_1 + Y_2 + \dots + Y_{2^k-1} + Y_{2^k} &\stackrel{d}{=} b_{2^k} Y\\
&\dots
\end{aligned}
$$

Making use of the first formula, we transform the second one as follows:

$$
S_4 = (Y_1 + Y_2) + (Y_3 + Y_4) \stackrel{d}{=} b_2(Y_1 + Y_2) \stackrel{d}{=} b_2^2 Y.
$$

Here we keep in mind that $X_1 + X_2 \stackrel{d}{=} X_3 + X_4$. Applying this reasoning to the third formula, we obtain

$$
\begin{aligned}
S_8 &= (Y_1 + Y_2) + (Y_3 + Y_4) + (Y_5 + Y_6) + (Y_7 + Y_8)\\
&\stackrel{d}{=} b_2(Y_1 + Y_2) + b_2(Y_5 + Y_6)\\
&\stackrel{d}{=} b_2^2 Y_1 + b_2^2 Y_5\\
&\stackrel{d}{=} b_2^2(Y_1 + Y_5) = b_2^3 Y.
\end{aligned}
$$

For the sum of 2^k terms, we similarly obtain

$$
S_{2^k} \stackrel{d}{=} b_{2^k} Y \stackrel{d}{=} b_2^k Y.
$$

Comparing this with (2.4.1), with $n = 2^k$, we obtain:

$$
b_n = b_2^k = b_2^{(\ln n)/\ln 2};
$$

hence

$$
\ln b_n = [(\ln n)/\ln 2] \ln b_2 = \ln n^{(\ln b_2)/\ln 2}.
$$

Thus, for the sequence of sums we obtain

$$b_n = n^{1/\alpha_2}, \quad \alpha_2 = (\ln 2)/\ln b_2, \quad n = 2^k, \quad k = 1, 2, \dots \tag{2.4.5}$$

Choosing now from (2.4.4) those sums which contain 3^k terms, and repeating the above reasoning, we arrive at

$$b_n = n^{1/\alpha_3}, \quad \alpha_3 = (\ln 3)/\ln b_3, \quad n = 3^k, \quad k = 1, 2, \dots \tag{2.4.6}$$

In the general case,

$$b_n = n^{1/\alpha_m}, \quad \alpha_m = (\ln m)/\ln b_m, \quad n = m^k, \quad k = 1, 2, \dots \tag{2.4.7}$$

We set $m = 4$. By virtue of (2.4.7),

$$\alpha_4 = (\ln 4)/\ln b_4,$$

whereas (2.4.5) with $k = 2$ yields

$$\ln b_4 = (1/\alpha_2)\ln 4.$$

Comparing the two last formulae, we conclude that

$$\alpha_2 = \alpha_4.$$

By induction, we come to the conclusion that all α_m are equal to each other:

$$\alpha_m = \alpha.$$

The following expression hence holds for the scale factors b_n:

$$b_n = n^{1/\alpha}, \qquad n = 1, 2, 3, \dots \tag{2.4.8}$$

whereas (2.4.1) takes the form

$$S_n = \sum_{i=1}^{n} Y_i \stackrel{d}{=} n^{1/\alpha} Y. \tag{2.4.9}$$

Substituting (2.4.8) into (2.4.2), we obtain

$$n \operatorname{Var} Y = n^{2/\alpha} \operatorname{Var} Y,$$

and see that for non-degenerate ($\operatorname{Var} Y \neq 0$) distributions with finite variance the index α must be equal to 2. If $\alpha \neq 2$, this relation can be formally satisfied only with $\operatorname{Var} Y = \infty$. Indeed, all stable distributions, except normal, have infinite variances, and some of them, as we will see below, have also infinite mathematical expectations. We will see also that there exist no distributions with $\alpha > 2$, so $\alpha \in (0, 2]$. The parameter α called the characteristic exponent of a stable distribution is its most important characteristic determining the rate of decrease of its tails.

Like the normal law, all other stable distributions remain stable under linear transformations, and one can choose some standard values of the shift and scale parameters. In this case we deal with reduced stable densities which are characterized by one more parameter—the skewness parameter $\beta \in [-1, 1]$. It characterizes the degree of asymmetry of the distributions being different from the normal law.

Stable distributions with $\beta = 0$ are symmetric (for example, the Cauchy distribution):

$$\mathsf{P}\{Y > x\} = \mathsf{P}\{Y < -x\}.$$

If $\beta \neq 0$, the symmetry is violated:

$$\mathsf{P}\{Y > 0\} > \mathsf{P}\{Y < 0\}$$

for $\beta > 0$ and vice versa for $\beta < 0$. In the extreme cases where $\beta = 1$ and $\beta = -1$, the corresponding probabilities $\mathsf{P}\{Y > 0\}$ attain their maximum values depending on α. If $\alpha \leq 1$, the maximal probabilities become equal to 1 and we deal with one-sided stable distributions concentrated on the positive ($\beta = 1$) or on the negative ($\beta = -1$) semiaxes only. The Lévy distribution is an example of such a kind: it corresponds to the parameters $\alpha = 1/2$ and $\beta = 1$. The reflected Lévy distribution $p^{\mathrm{L}}(-x)$ gives another example corresponding to the values $\alpha = 1/2$ and $\beta = -1$.

Introducing the notation $q(x; \alpha, \beta)$ for the reduced stable densities, we can rewrite the distributions mentioned above as

$$\begin{aligned} p^{\mathrm{C}}(x) &= q(x; 1, 0), \\ p^{\mathrm{L}}(x) &= q(x; 1/2, 1), \\ p^{\mathrm{L}}(-x) &= q(x; 1/2, -1). \end{aligned}$$

As far as the normal distribution is concerned, it turns out to be more convenient to accept as the reduced form the distribution with variance 2:

$$q(x; 2, \beta) = p^{\mathrm{G}}(x; 0, \sqrt{2}) = \frac{1}{2\sqrt{\pi}} e^{-x^2/4}.$$

The distribution function corresponding to the density $q(x; \alpha, \beta)$ is denoted by $G(x; \alpha, \beta)$:

$$G(x; \alpha, \beta) = \int_{-\infty}^{x} q(x; \alpha, \beta)\, dx$$

and the stable variable itself, by $Y(\alpha, \beta)$.

As will be shown in the following chapter, all stable distributions with $\alpha < 1$ are strictly stable, and the stable distributions with $\alpha > 1$ are easily transformed to strictly stable ones since $\mathsf{E}Y = a$ exists and is used for centering r.v. The distribution $q(x; 1, \beta)$ is strictly stable if only $\beta = 0$. The general relation of equivalence for sums S_n of independent identically distributed strictly stable r.v.s Y_i is of the form

$$\sum_{i=1}^{n} Y_i(\alpha, \beta) \stackrel{d}{=} n^{1/\alpha} Y(\alpha, \beta). \tag{2.4.10}$$

As was noticed by Feller, these results have important and unexpected consequences. Let us consider, for example, a stable distribution with $\alpha < 1$. The arithmetic mean $(X_1 + \ldots + X_n)/n$ has the same distribution as $X_1 n^{-1+1/\alpha}$. Meanwhile, the factor $n^{-1+1/\alpha}$ tends to infinity as n grows. Without pursuing the rigor, we can say that the average of n variables X_k turns out considerably greater than any fixed summand X_k. This is possible only in the case where the maximum term

$$M_n = \max\{X_1, \ldots, X_n\} \tag{2.4.11}$$

grows extremely fast and gives the greatest contribution to the sum S_n. The more detailed analysis confirms this speculation.

We do not perform detailed analysis here, using only simple arguments of rather heuristic nature.

The distribution function for the maximum (2.4.11) can immediately be written. Indeed, the event $\{M_n < x\}$ implies $\{Y_1 < x, \ldots, Y_n < x\}$ and vice versa; therefore,

$$F_{M_n}(x) = \mathsf{P}\{M_n < x\} = \mathsf{P}\{Y_1 < x, \ldots, Y_n < x\}.$$

Since Y_i are independent, the right-hand side of this expression can be transformed to the product of probabilities of individual events $\{Y_i < x\}$. In view of coincidence of the distributions of Y_i, it is the mere nth power of this probability:

$$F_{M_n}(x) = \mathsf{P}^n\{X_i < x\} = F_Y^n(x). \tag{2.4.12}$$

If the maximum gives a primary contribution to the sum, the following relation should be satisfied:

$$\bar{F}_{S_n}(x) \sim \bar{F}_{M_n}(x), \qquad x \to \infty. \tag{2.4.13}$$

where

$$\bar{F}(x) \equiv 1 - F(x).$$

By virtue of (2.4.9),

$$\bar{F}_{S_n}(x) = \mathsf{P}\{S_n \geq x\} = \mathsf{P}\left\{Y \geq n^{-1/\alpha}x\right\} = \bar{F}_Y(n^{-1/\alpha}x). \qquad (2.4.14)$$

Substituting (2.4.14) into the left-hand side of (2.4.13), and (2.4.12) into the right-hand one, we obtain

$$\bar{F}_Y(n^{-1/\alpha}x) \sim 1 - F_Y^n(x);$$

changing the variable $n^{-1/\alpha}x = y$, we arrive at

$$\begin{aligned}\bar{F}_Y(y) &\sim 1 - [1 - \bar{F}_Y(yn^{1/\alpha})]^n \\ &\sim n\bar{F}_Y(yn^{1/\alpha}), \qquad y \to \infty.\end{aligned}$$

The solution of this equation, provided that $\beta \neq -1$, is

$$\bar{F}_Y(x) \sim cx^{-\alpha}, \qquad x \to \infty. \qquad (2.4.15)$$

Similarly we obtain

$$F_Y(x) \sim d|x|^{-\alpha}, \qquad x \to \infty. \qquad (2.4.16)$$

Differentiating (2.4.15) and (2.4.16) with respect to x, we obtain the following asymptotic expressions for the density:

$$\begin{aligned}p_Y(x) &= -\bar{F}'_Y(x) \sim \alpha c x^{-\alpha-1}, \qquad x \to \infty, \\ p_Y(x) &= F_Y(x) \sim \alpha d|x|^{-\alpha-1}, \qquad x \to -\infty.\end{aligned}$$

It is clear that this does not correspond to the normal law ($\alpha = 2$), and that there are no $\alpha > 2$: the normal law is the only stable law with finite variance. Thus, the characteristic exponent α takes values from the interval $(0, 2]$; for $\alpha = 2$ we have the normal law (2.2.5); with $\alpha < 2$ we have stable laws whose probabilities of large deviations are power functions (2.4.15)–(2.4.16). In some cases, only one of the tails may exist.

2.5. The stable laws as limiting distributions

Were it not for one circumstance, the problem of summaton of stable r.v.'s itself would not be of much interest. The matter is that the sums S_n of independent r.v.'s, which do not belong to the family of stable laws, in some sense can become stable beginning with some, large enough, number of summands. This circumstance extremely expands the area of application of stable law theory indeed. The stable laws with $\alpha < 2$ play the same role in summation of r.v.'s with infinite variances as the normal law does in the case of finite variances.

First, let us turn back to summation of uniformly distributed r.v.'s (2.1.1). They have finite mathematical expectations that are equal to 1/2 and can easily be transformed to r.v.'s with zero mean:

$$X^0 = X - 1/2.$$

The distribution of the r.v.'s (2.1.3) is shifted by -1 along the x-axis, and we obtain (by dropping the superscript 0), for the centered r.v.'s,

$$p_X(x) = 1, \qquad -1/2 < x < 1/2 \tag{2.5.1}$$

and

$$p_{X_1+X_2}(x) = \begin{cases} x+1, & -1 < x < 0, \\ 1-x, & 0 < x < 1. \end{cases} \tag{2.5.2}$$

Outside the specified areas, the densities are zero.

Calculate now the distribution of the sum of three independent r.v.'s of identical type. To this end, we use formula (2.1.10) where distribution (2.5.1) is $p_{X_1}(x)$, and instead of $p_{X_2}(x)$ we take the distribution $p_{X_2+X_3}(x)$ determined by (2.5.2):

$$\begin{aligned} p_{X_1+X_2+X_3}(x) &= \int_{-\infty}^{\infty} p_{X_2+X_3}(x-x')p_{X_1}(x')\,dx' = \int_{x-1/2}^{x+1/2} p_{X_2+X_3}(x')dx' \\ &= \begin{cases} (x+3/2)^2/2, & -3/2 < x < -1/2, \\ 3/4 - x^2, & -1/2 < x < 1/2, \\ (x-3/2)^2/2, & 1/2 < x < 3/2. \end{cases} \end{aligned}$$

This distribution is constituted by three pieces, but they are adjusted to each other better than in the case of two terms, and form a smooth 'bell-shaped' curve.

All three distributions (2.5.1)–(2.5) are symmetric about the origin but their forms differ. The difference is due to the natural 'dissipation' of distributions during the summation process. Their variances grow linearly as the number of summands grow,

$$\operatorname{Var} \sum_{i=1}^{n} X_i = n \operatorname{Var} X = n/12.$$

To eliminate this effect, we re-normalize the sums, assigning identical variances to them, say, 2 (the standard normal distribution in form (2.2.5) has such a variance):

$$Z_n = \frac{1}{b_n} \sum_{i=1}^{n} X_i,$$

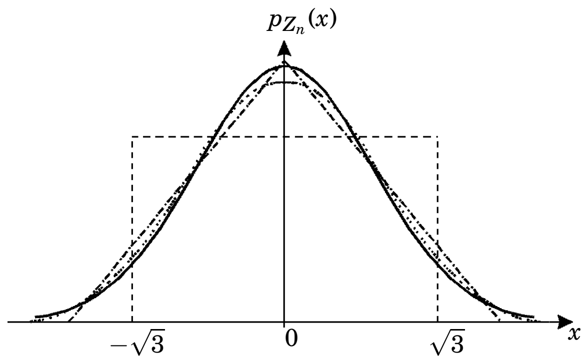


Figure 2.9.

where

$$b_n = \sqrt{n \operatorname{Var} X} = b_1\sqrt{n}, \qquad b_1 = 1/\sqrt{12}. \tag{2.5.3}$$

Figure 2.9 pictures the densities of distributions Z_2, Z_3, and the normal distribution. Their behaviour as n grows can serve as an illustration to one of the main theorems of the probability theory, the central limit theorem, whose assertion can be presented as

$$\sum_{i=1}^{n} X_i \overset{d}{\sim} b_1 n^{1/2} Y^{\mathrm{G}}, \qquad n \to \infty.$$

The proof of this theorem can be found in any textbook on probability theory. We note here only some points that are important for understanding the further presentation.

First, the limiting distribution of normalized sums is the normal law. It belongs to the family of stable laws, and, as it can easily be seen, it appears in the central limit theorem just for this reason. Indeed, let $X_1, X_2, \ldots$ be an infinite sequence of independent identically distributed r.v.'s, and let n be large enough for the normalized sum Z_n to be considered as already distributed according to the limiting law. The same holds true for the sum of the following n terms of the given sequence Z'_n as well, and, of course, for the sum of $2n$ terms Z_{2n}. Hence,

$$Z_{2n} \overset{s}{=} Z_n + Z'_n,$$

which implies that Z_n should belong to the family of stable r.v.'s.

Second, we see here that the normalizing factors are determined by the sequence $b_n = b_1\sqrt{n}$, rather than $b_n = \sqrt{n}$ while summing r.v.'s satisfying (2.2.5); b_1 depends on a single characteristic of X, the variance, as

$$b_1 = \sqrt{\operatorname{Var} X}.$$

As far as Var X exists, other properties of X do not alter the form of the limiting distribution.

Before turning to discussing other stable laws, it is pertinent to say that Feller noticed that the important role which the normal distribution F^{G} plays in probability theory is based on the central limit theorem. Let $X_1, \dots, X_n$ be independent r.v.'s distributed by the law F_X with zero mean and variance one. We set $S_n = X_1 + \dots + X_n$. The central limit theorem asserts that the distributions of $S_n n^{-1/2}$ converge to F^{G}. If the distributions have no variances, $n^{-1/2}$ cannot play the part of normalizing constants, other choice of a normalizing constant can still bring the limit into existence. It is rather interesting that all stable laws, and only they, can be obtained as such limits. Let us introduce the following terminology which makes the further discussion of the problem in question more simple.

DEFINITION OF A DOMAIN OF ATTRACTION. We say that the common distribution F_X of independent r.v.'s X_k belongs to the domain of attraction of the distribution F if there exist some normalizing constants a_n, $b_n > 0$ such that the distribution of $\left(\sum_{i=1}^{n} X_i - a_n\right)/b_n$ converges to F.

The above-mentioned declaration (that stable laws, and only they, play that part of limiting laws) can be now reformulated as follows: the distribution F possesses a domain of attraction if and only if it is stable.

What are the conditions which the distribution of summands should satisfy in order to belong to the domain of attraction of a given stable law? Or, in other words, how to find the stable law whose domain of attraction contains a given distribution?

Before answering this question, we try to predict the answer, based only on the abovesaid and the analysis of an example.

First, in order to belong to the domain of attraction of a stable law differing from the normal law, it is necessary for the summands to have infinite variance, because otherwise the central limit theorem is true.

As in the normal case, we can assume that the dependence b_n on n is of the same form for these sums as for sums of stable r.v.'s. But in case of stable r.v.s the dependence

$$b_n = b_1 n^{1/\alpha}$$

is related to the tails of the summands distribution that behave as a power function. We can assume that this relation remains valid in the general case as well. Thus, it seems likely that the general scheme of summation is similar to summation of stable r.v.'s, but in the asymptotic sense only, as $n \to \infty$.

Denoting by $X_i(\alpha, \beta)$ a random variable belonging to the domain of attraction of the strictly stable law $q(x; \alpha, \beta)$, by analogy with (2.4.10) we write

$$\sum_{i=1}^{n} X_i(\alpha, \beta) \stackrel{d}{\sim} b_1(\alpha) n^{1/\alpha} Y(\alpha, \beta), \qquad n \to \infty. \tag{2.5.4}$$

It follows herefrom that for $\beta \neq -1$

$$\mathsf{P}\left\{\sum_{i=1}^{n} X_i/b_n > x\right\} \sim \mathsf{P}\{X/b_1 > x\}, \qquad x \to \infty,$$

or

$$\bar{F}_{\sum_{i=1}^{n} X_i}(b_n x) \sim n\bar{F}_X(b_n x) \sim \bar{F}_X(b_1 x). \tag{2.5.5}$$

where

$$\bar{F}_X(x) \sim cx^{-\alpha}, \qquad x \to \infty.$$

From (2.5.5) it follows that

$$b_n \sim b_1 n^{1/\alpha}, \tag{2.5.6}$$

where b_1 is determined by means of

$$\bar{F}_X(b_1 x) \sim 1 - G(x; \alpha, \beta), \qquad x \to \infty. \tag{2.5.7}$$

To make the situation more clear, we give the following example. Let X be distributed by the symmetric Zipf–Pareto law, that is,

$$p_X(x) = \begin{cases} cx^{-2}, & |x| > \varepsilon, \\ 0, & |x| < \varepsilon, \end{cases} \tag{2.5.8}$$

where the positive constants c and ε are determined from the normalization condition as

$$\int_{-\infty}^{\infty} p_X(x)\, dx = 2c \int_{\varepsilon}^{\infty} x^{-2} dx = 2c/\varepsilon = 1.$$

The distribution of the sum of two independent r.v.'s is given by the convolution

$$\begin{aligned} p_{X_1+X_2}(x) &= \int_{-\infty}^{\infty} p_X(x - x_1) p_X(x_1)\, dx_1 \\ &= c\left\{\int_{-\infty}^{-\varepsilon} p_X(x - x_1) x_1^{-2}\, dx_1 + \int_{\varepsilon}^{\infty} p_X(x - x_1) x_1^{-2}\, dx_1\right\} \\ &= c\{I_+(x) + I_-(x)\}, \end{aligned} \tag{2.5.9}$$

where

$$I_+(x) = \int_{\varepsilon}^{\infty} p_X(x + x_1) x_1^{-2}\, dx_1, \tag{2.5.10}$$

$$I_-(x) = \int_{\varepsilon}^{\infty} p_X(x - x_1) x_1^{-2}\, dx_1. \tag{2.5.11}$$

In view of the symmetry, it is sufficient to calculate distribution (2.5.9) for positive x.

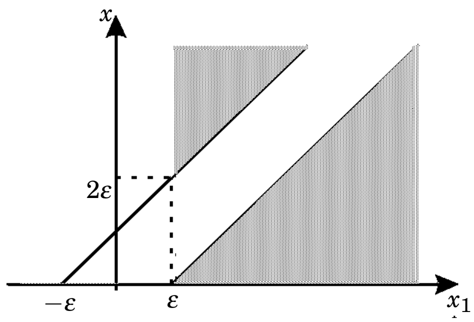


Figure 2.10.

We notice that the condition $|x| > a$ from (2.5.8) does not influence the integration domain in (2.5.10) for positive x:

$$I_+(x) = c\int_\varepsilon^\infty \frac{dx_1}{(x+x_1)^2x_1^2} = c\int_0^{1/\varepsilon} \frac{z^2dz}{(xz+1)^2}.$$

After one more change of the variable $xz + 1 = y$, we obtain

$$I_+(x) = \frac{c}{x^3}\int_1^{x/(\varepsilon+1)} [(y-1)^2/y^2]\,dy = \frac{c}{x^3}[(1+x/\varepsilon) - 2\ln(1+x/\varepsilon) - (1+x/\varepsilon)^{-1}].$$

The integration domain in the second integral (5.12) is the intersection of the complement $\overline{(x-\varepsilon, x+\varepsilon)}$ with the ray (ε, ∞) (Fig 2.10). If $x < 2\varepsilon$, the integral

$$I_-(x) = c\int_{\varepsilon+x}^\infty \frac{dx_1}{(x-x_1)^2x_1^2} = c\int_\varepsilon^\infty \frac{dx_1}{x_1^2(x+x_1)^2}$$

coincides with $I_+(x)$; if $x > 2\varepsilon$, the integral

$$I'_-(x) = c\int_\varepsilon^{x-\varepsilon} \frac{dx_1}{(x-x_1)^2x_1^2}$$

is added, which can be computed in a similar way:

$$I'_-(x) = -\frac{\varepsilon}{x^3}[(1-x/\varepsilon) - 2\ln(|1-x/\varepsilon|) - (1-x/\varepsilon)^{-1}].$$

As a result, for (2.5.9) we obtain

$$p_{X_1+X_2}(x) = \begin{cases} \varepsilon I_+(|x|), & |x| < 2\varepsilon, \\ \varepsilon I_+(|x|) + (\varepsilon/2)I'_-(|x|), & |x| > 2\varepsilon. \end{cases} \tag{2.5.12}$$

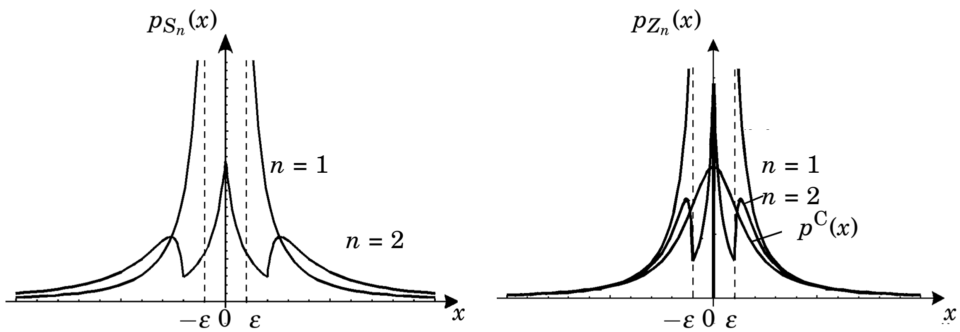


Figure 2.11.

The densities of distributions (2.5.8) and (2.5.12) are shown in Fig. 2.11.

The distribution of the sum already fills the gap $(-\varepsilon, \varepsilon)$ which was open for one term. The density at the central point of the distribution is obtained by removing the indeterminacy in $I_+(x)$ by the L'Hôpital rule:

$$p_{X_1+X_2}(0) = \varepsilon \lim_{x\to 0} I_+(x) = \frac{1}{6\varepsilon}. \qquad (2.5.13)$$

For large values of x,

$$p_{X_1+X_2}(x) \sim 2cx^{-2},$$
$$\bar F_{X_1+X_2}(x) \sim 2cx^{-1}. \qquad (2.5.14)$$

We know a symmetric stable distribution with such tails, and it is the Cauchy distribution:

$$p^{\rm C}(x) \sim (1/\pi)x^{-2}. \qquad (2.5.15)$$

From (2.5.8) and (2.5.15) it follows that

$$\bar F_X(x) \sim cx^{-1}, \qquad F^{\rm C}(x) \sim (1/\pi)x^{-1}.$$

Using (2.5.5)–(2.5.7) and taking (2.5.14) into account, we obtain

$$b_1 = \pi c, \qquad b_2 = 2b_1 = 2\pi c. \qquad (2.5.16)$$

This agrees with formula (2.5.6) where $\alpha = 1$. (Factors a_n do not appear here due to the symmetry).

The density functions for normalized sums of r.v.'s $Z_1 = X/b_1$, $Z_2 = (X_1 + X_2)/b_2$ and the Cauchy stable density are shown in Fig. 2.11. Of course, Fig. 2.11 is no more than illustration to other theorem that extends the central limit theorem to the case of summation of r.v.s with infinite variances. The form given here is not the most general one (actually, we allow c and d to be slowly varying functions), but it is nevertheless sufficient for many applications.

GENERALIZED CENTRAL LIMIT THEOREM. *Let $X_1, \dots, X_n$ be independent identically distributed random variables with the distribution function $F_X(x)$ satisfying the conditions*

$$1 - F_X(x) \sim cs^{-\mu}, \qquad x \to \infty, \tag{2.5.17}$$

$$F_X(x) \sim d|x|^{-\mu}, \qquad x \to \infty, \tag{2.5.18}$$

with $\mu > 0$. Then there exist sequences a_n and $b_n > 0$ such that the distribution of the centered and normalized sum

$$Z_n = \frac{1}{b_n}\left(\sum_{i=1}^{n} X_i - a_n\right) \tag{2.5.19}$$

weakly converges to the stable distribution with parameters

$$\alpha = \begin{cases} \mu, & \mu \le 2, \\ 2, & \mu > 2, \end{cases}, \tag{2.5.20}$$

$$\beta = \frac{c-d}{c+d} \tag{2.5.21}$$

as $n \to \infty$:

$$F_{Z_n}(x) \Rightarrow G^A(x; \alpha, \beta). \tag{2.5.22}$$

The coefficients a_n and b_n can be taken in the form given in Table 2.1.

Table 2.1. Centering and normalizing coefficients a_n and b_n (form A).

μ	α	a_n	b_n
$0 < \mu < 1$	μ	0	$[\pi(c+d)]^{1/\alpha}[2\Gamma(\alpha)\sin(\alpha\pi/2)]^{-1/\alpha}n^{1/\alpha}$
$\mu = 1$	μ	$\beta(c+d)n\ln n$	$(\pi/2)(c+d)n$
$1 < \mu < 2$	μ	$n\mathsf{E}X$	$[\pi(c+d)]^{1/\alpha}[2\Gamma(\alpha)\sin(\alpha\pi/2)]^{-1/\alpha}n^{1/\alpha}$
$\mu = 2$	2	$n\mathsf{E}X$	$(c+d)^{1/2}[n\ln n]^{1/2}$
$\mu > 2$	2	$n\mathsf{E}X$	$\left[(1/2)\mathrm{Var}X\right]^{1/2} n^{1/2}$

Obviously, the last case ($\mu > 2$) is covered by the central limit theorem which holds true for any random variables X_i with finite variances but not for power-type distributions (2.5.17), (2.5.18) with $\mu < 2$. Recall that, as it has been said in Section 2.4, $G^A(x; 2, \beta)$ is the Gauss distribution with variance 2:

$$G^A(x; 2, \beta) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{x} e^{-z^2/4}\, dz.$$

We will not prove this theorem, but give here some heuristic reasoning following (Bouchaud & Georges, 1990) for the case of non-negative summands with the asymptotics

$$\bar{F}_X(x) \sim cx^{-\alpha}, \quad 0 < \alpha < 1, \quad x \to \infty,$$

leading to one-sided stable distributions. By (2.4.12), the density function of the maximum term M_n in the sum S_n is of the form

$$p_{M_n}(x) = dF_{M_n}(x)/dx = n \left[1 - \int_x^\infty p_X(x')\,dx'\right]^{n-1} p_X(x). \qquad (2.5.23)$$

We denote by x_n the most probable value of M_n; then

$$\left[dp_{M_n}(x)/dx\right]_{x=x_n} = 0. \qquad (2.5.24)$$

Differentiating (2.5.23) with respect to x and substituting the result in (2.5.24), we obtain

$$dp_X(x_n)/dx_n \left[1 - \int_{x_n}^\infty p_X(x')dx'\right] + (n-1)p_X^2(x_n) = 0. \qquad (2.5.25)$$

As $x \to \infty$,

$$\begin{aligned}\int_x^\infty p_X(x')\,dx' &\sim cx^{-\alpha},\\ p_X(x) &\sim c\alpha x^{-\alpha-1},\\ dp_X(x)/dx &\sim -c\alpha(\alpha+1)x^{-\alpha-2};\end{aligned} \qquad (2.5.26)$$

if $n \gg 1$, (2.5.25) yields the asymptotic relation

$$\alpha + 1 \sim nc\alpha x_n^{-\alpha};$$

therefore,

$$x_n \sim Cn^{1/\alpha}, \qquad C = \left(\frac{c\alpha}{\alpha+1}\right)^{1/\alpha}. \qquad (2.5.27)$$

Estimating the characteristic value of the sum S_n with the help of the 'truncated mean'

$$\langle X \rangle_{x_n} = \int_0^{x_n} xp_X(x)\,dx,$$

we obtain

$$S_n \sim \langle S_n \rangle_{x_n} = n \int_0^{x_n} xp_X(x)\,dx = \int_0^{Cn^{1/\alpha}} xp_X(x)\,dx.$$

Let $A > 0$ be chosen so that, as $x > A$, the density $p_X(x)$ satisfies (2.5.26). Then

$$\langle S_n \rangle_{x_n} = n \left\{ \int_0^A x p_X(x)\,dx + \int_A^{Cn^{1/\alpha}} x p_X(x)\,dx \right\} = n\langle X \rangle_A + \langle S'_n \rangle_{x_n},$$

where

$$\langle S'_n \rangle_{x_n} = \alpha c n \int_A^{Cn^{1/\alpha}} x^{-\alpha}\,dx.$$

It follows herefrom that, as $n \to \infty$,

$$S_n \sim \langle S'_n \rangle_{x_n} \sim \begin{cases} c_0 n^{1/\alpha}, & \alpha < 1; \\ c_1 n \ln n, & \alpha = 1, \end{cases} \tag{2.5.28}$$

where c_0 and c_1 are positive constants.

Estimating similarly the truncated variance of the sum S_n with $\alpha > 1$, we obtain

$$\begin{aligned} \langle (S_n - \langle S'_n \rangle)^2 \rangle_{x_n} &\sim \int_0^{x_n} (x - \langle X \rangle)^2 p_X(x) dx \\ &\sim \begin{cases} c_3 n^{1/\alpha}, & \alpha < 2, \\ c_4 n \ln n, & \alpha = 2. \end{cases} \end{aligned} \tag{2.5.29}$$

Formulae (2.5.28) and (2.5.29) conform with the generalized limit theorem.

2.6. Summary

In this chapter we outline general features of the stable laws, without using much mathematics (except, maybe, the preceding section) and illustrate some basic ideas by simple examples. Let us formulate here the basic results.

The normal law possesses the stability property: the sum of independent normally distributed r.v.'s is distributed by the normal law as well; it and only it turns out to be the limiting law for the normalized sums

$$Z_n = \left(\sum_{i=1}^n X_i - n a_1 \right) \Big/ \left(b_1 \sqrt{n} \right) \tag{2.6.1}$$

of identically distributed r.v.'s X_i with finite second moment

$$\int_{-\infty}^{\infty} x^2 p_X(x)\,dx < \infty. \tag{2.6.2}$$

(the central limit theorem).

However, the normal law is not a unique stable law. There is a whole family of stable distributions, and the normal law is just one of them. Similarly to the normal case, each of stable distributions $G(x; \alpha, \beta)$ has a domain of attraction, i.e., there exist r.v.'s X and sequences a_n, b_n such that the distribution of the normalized sums of independent summands distributed by the same law as X

$$Z_n = \left(\sum_{i=1}^{n} X_i - a_n \right) \Big/ b_n \tag{2.6.3}$$

converge to $G(x; \alpha, \beta)$ as $n \to \infty$. Condition (2.6.2) does not take place anymore, and is replaced by the conditions

$$\int_x^\infty p_X(x)\,dx \sim cx^{-\alpha}, \qquad x \to \infty, \tag{2.6.4}$$

$$\int_{-\infty}^x p_X(x)\,dx \sim d|x|^{-\alpha}, \qquad x \to -\infty, \tag{2.6.5}$$

which determine the rate of growth of the centering and normalizing coefficients a_n and b_n (generalized limiting theorem). We will use the term long (heavy) tails for such tails, and the term short (light) tails for those satisfying the relation

$$\int_x^\infty p_X(x)\,dx = o(x^{-2}), \qquad x \to \infty,$$

$$\int_{-\infty}^x p_X(x)\,dx = o(x^{-2}), \qquad x \to -\infty,$$

The stability property of and the limiting part played by stable distributions are common both for normal law and for other stable laws.

Let us highlight the distinctions of other stable distributions from the widely known normal law. First of all, we note that the stable distribution family contains not only symmetric laws satisfying the condition $p(x) = p(-x)$ and in this sense being similar to normal. Asymmetry of stable distributions is characterized by the parameter $\beta \in [-1, 1]$ which is zero for symmetrical distributions. Together with the characteristic exponent α, related to the asymptotic behaviour of densities, it determines a two-parameter set of stable densities $q(x; \alpha, \beta)$. As it was shown above, the case $\alpha = 1$, $\beta = 0$ corresponds to the symmetric Cauchy distribution, and the case $\alpha = 1/2$, $\beta = 1$, to the one-sided Lévy distribution.

With the exception of the Gauss, Cauchy and Lévy laws, explicit expressions for the densities of stable distributions in terms of elementary functions are unknown, and for their use it is necessary to refer to tables (see Appendix) or use numerical algorithms.

Considering in more detail the summation of non-negative random variables X which satisfy condition (2.6.4), we arrive at the following qualitative conclusions.

For $0 < \alpha < 1$, the 'typical' values of the sum $S_n \equiv \sum_{i=1}^n X_i$ (for example, the most probable value corresponding to the position of the maximum of the density) behave as $n^{1/\alpha}$, i.e., increase much faster than in the case where $\alpha > 1$ and the expectation hence exists. A 'dispersion' of values of the random sum S_n grows with the same rate, so its relative fluctuations (the ratio of the width to the most probable value) do not vanish.

For $1 < \alpha \le 2$, there exists $\mathsf{E}X = a_1$, and in view of the law of large numbers the typical value of the sum is proportional to the number of summands. However, the 'dissipation' of the sum characterized by the factor b_n grows as $n^{1/\alpha}$ (for $\alpha < 2$) or $(n \ln n)^{1/2}$ (for $\alpha = 2$). Thus, the relative fluctuations decrease as $n^{1/\alpha-1}$ or $\sqrt{\ln n}/\sqrt{n}$, i.e., essentially slower than in the normal case (2.6.2), where finiteness of the variance leads us to the traditional behaviour of the relative fluctuations which grow as $n^{-1/2}$.

Discussing the distinctions between the cases $\alpha = 2$ and $\alpha < 2$, it is necessary to highlight the following important fact. In the case of attraction to the normal law, only a few terms are necessary to get a satisfactory goodness-of-fit to the normal law in the central domain. Further increase of the number of summands improves only the tails of the distribution which, however, never perfectly coincide with the limiting one. In the case of summation of n uniformly distributed on $(-1/2, 1/2)$ r.v.'s, the density function for the sums outside the interval $(-n/2, n/2)$ is always equal to zero.

Another situation occurs in the case of summation of random variables satisfying conditions (2.6.4) and (2.6.5). Both tails of the density behave as the power function, which coincides with the behavior of the summands; the density of the sum S_n coincides with the limiting density at the tails for any n but differs from it in the central domain. As n grows, this distinction decreases with rate determined by the distribution of summands.

Despite a 'good' behavior of summands providing very light (normal) tails of the limit law for the sums S_n, the random sums S_{N_n} may have limit distributions with heavy tails, e.g., stable, due to a 'bad' behavior of the index N_n. By doing this we will try to shake a prejudice existing among applied specialists. According to this prejudice, the heavy-tailedness of the law which is limit for sums of independent r.v.'s (e.g., inherent in stable laws) is necessarily due to the heavy-tailedness of the distributions of summands (e.g., to the non-existence of their moments of orders less than two). This idea lies upon well-known results (see, e.g., (Gnedenko & Kolmogorov, 1954, Chapter 7, §35, Theorem 2). Of course, we do not make an attempt to call these classic results in question. We simply attract attention to that when solving applied problems (especially in financial or actuarial mathematics where stable laws are widely used) it is very important to choose an appropriate structure model. From the results given below (see (Korolev, 1997)) it follows that the heavy tails of the limit distribution for sums may occur due to the randomness of the number of summands. We show that random sums of independent summands with the properties described above are asymptotically strictly stable if and only if so are their indexes.

Consider a sequence of independent random variables $\{X_i\}_{i\ge 1}$ which are not necessarily identically distributed. Assume that $\mathsf{E}X_i = 0$, $\mathsf{E}X_i^2 = \sigma_i^2 < \infty$, $i \ge 1$. Moreover, assume that the r.v.'s $\{X_i\}_{i\ge 1}$ satisfy the Lindeberg condition: for any $\tau > 0$

$$\lim_{n\to\infty} \frac{1}{B_n^2} \sum_{i=1}^{n} \int_{|x|>\tau B_n} x^2 dF_i(x) = 0,$$

where $B_n^2 = \sigma_1^2 + \ldots + \sigma_n^2$, $F_i(x) = \mathsf{P}(X_i < x)$. Let $\{N_n\}_{n\ge 1}$ be a sequence of positive integer-valued r.v.'s such that for each n the r.v.'s N_n and $\{X_i\}_{i\ge 1}$ are independent. For a natural k denote $S_k = X_1 + \ldots + X_k$.

Let $\{D_n\}_{n\ge 1}$ be a sequence of positive numbers such that $D_n \to \infty$ as $n \to \infty$.

THEOREM 2.6.1. *Assume that $N_n \to \infty$ in probability as $n \to \infty$. Then*

$$\lim_{n\to\infty} \mathsf{P}\left(\frac{S_{N_n}}{D_n} < x\right) = G^C(x; \alpha, 0), \qquad x \in \mathbb{R}, \tag{2.6.6}$$

if and only if

$$\lim_{n\to\infty} \mathsf{P}\left(\frac{B_{N_n}^2}{D_n^2} < x\right) = G^C(x; \alpha/2, \alpha/2), \qquad x \in \mathbb{R}.$$

The proof consists in the sequential application of Theorem 16 from (Korolev, 1994) according to which under the above assumptions

$$\mathsf{P}\left(\frac{S_{N_n}}{D_n} < x\right) \to \mathsf{P}(Z < x), \qquad n \to \infty,$$

(where Z is some r.v.) if and only if there exists a r.v. $U \ge 0$ such that first,

$$\mathsf{P}(Z < x) = \int_0^\infty \Phi(x/u) d\mathsf{P}(U < u), \qquad x \in \mathbb{R},$$

and second,

$$\frac{B_{N_n}}{D_n} \to U, \qquad n \to \infty,$$

and Theorem 3.3.1 in (Zolotarev, 1986), according to which

$$G^C(x; \alpha, 0) = \int_0^\infty \Phi(x/\sqrt{u}) dG^C(u; \alpha/2, \alpha/2), \qquad x \in \mathbb{R},$$

with regard to the identifiability of scale mixtures of normal laws and the absolute continuity of stable distributions.

COROLLARY 2.6.1. *If, in addition to the conditions of the theorem, the summands* $\{X_j\}_{j\geq 1}$ *are identically distributed, then convergence* (2.6.6) *takes place if and only if*

$$\lim_{n\to\infty} \mathsf{P}\left(\frac{N_n}{D_n^2} < x\right) = G^C(\sigma_2^2 x; \alpha/2, \alpha/2), \qquad x \in \mathbb{R}.$$

These are the general properties of stable laws and limit processes related to them, which could be obtained by elementary technique used in this introductory chapter. In the following chapters, we will get acquainted with more effective mathematical methods, namely with the characteristic functions which allow us to obtain more important and accurate results.

3

Characteristic functions

3.1. Characteristic functions

The distribution function or the density function completely characterize a random variable, but they are not so convenient in problems of summation of independent r.v.'s as compared with the characteristic function, which also contains the complete information about the r.v's under consideration.

DEFINITION OF A CHARACTERISTIC FUNCTION. The complex-valued function

$$f_X(k) = \mathsf{E}e^{ikX} \tag{3.1.1}$$

is called the characteristic function (c.f.) of a real r.v. X.

Here k is some real-valued variable. If the density $p_X(x)$ exists, (3.1.1) is the Fourier transform of that density:

$$f_X(k) = \int_{-\infty}^{\infty} e^{ikx} p_X(x)\, dx. \tag{3.1.2}$$

The inverse Fourier transform

$$p_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} f_X(k)\, dk \tag{3.1.3}$$

allows us to reconstruct the density of a distribution from a known known c.f. (the uniqueness theorem).

The following properties of c.f.'s immediately follow from their definition:

(a) $f_X(0) = 1$, $|f_X(k)| \leq 1$ for all k; and $f_X(k) \to 0$ as $k \to \infty$ under the assumption that there exists a density of the r.v. X;

(b) $f_{a+bX}(k) = e^{ika} f_X(bk)$;

(c) $f_X^\star(k) = f_X(-k) = f_{-X}(k)$, where $\star$ means the complex conjugation. If X is symmetric about zero,

$$X \stackrel{d}{=} -X,$$

then its c.f. is real-valued, and vice versa.

(d) If

$$\mathsf{E}|X|^n < \infty, \qquad n \geq 1,$$

then there exists the continuous nth derivative of the c.f., and

$$f^{(n)}(0) = i^n \mathsf{E} X^n;$$

(e) if S_n is the sum of independent r.v.'s $X_1, \dots, X_n$, then

$$f_{S_n}(k) = f_{X_1}(k) \dots f_{X_n}(k).$$

(f) any c.f. $f_X(k)$ is a uniformly continuous function;

(g) if $X \geq 0$, then $f_X(k)$ is defined in the half-plane of the complex variable k, then $\Im k \geq 0$; $|f_X(k)| \leq 1$, $f_X(k)$ is thus an analytical function in the domain $\Im k > 0$, and is continuous in the domain $\Im k \geq 0$.

Definition (3.1.1) can be applied to the case of degenerate distributions as well: if $X = a$ with probability one, then

$$p_X(x) = \delta(x - a),$$
$$f_X(k) = e^{iak}, \tag{3.1.4}$$

and the inversion formula (3.1.3) leads us to the integral representation of the δ-function

$$\delta(x - a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-a)}\, dk,$$

which is commonly used in theoretical physics.

The c.f. of the uniform distribution on $(0, 1)$ is of the form

$$f_X(k) = \int_0^1 e^{ikx} dx = \frac{e^{ik} - 1}{ik}. \tag{3.1.5}$$

The c.f. of the symmetric uniform on $(-1, 1)$ distribution of Y can be obtained using this property by the transformation $Y = 2X - 1$:

$$f_Y(k) = e^{-ik} f_X(2k) = \frac{e^{ik} - e^{-ik}}{2ik} = \frac{\sin k}{k}. \tag{3.1.6}$$

This result corresponds to property (c): the c.f. of a symmetric r.v. is real-valued. Besides, in view of property (a), both functions (3.1.5) and (3.1.6) tend to zero as $k \to \infty$, whereas function (3.1.4) does not possess such a property, because its density, which is defined by the generalized δ-function, does not exist in a common sense (the distribution function $F(x)$ is not differentiable at the discontinuity point).

In the general case, the problem to find the distribution function with c.f. given is solved by the following theorem.

THEOREM 3.1.1 (inversion theorem). *Any distribution function $F(x)$ is uniquely defined by its c.f. $f(k)$. If a and b are some continuity points of $F(x)$, then the inversion formula*

$$F(b) - F(a) = \lim_{c\to\infty} \frac{1}{2\pi} \int_{-c}^{c} \frac{e^{-ikb} - e^{-ika}}{ik} f(k)\, dk$$

is true.

COROLLARY 3.1.1. *If $|f(k)/k|$ is integrable at infinity, then*

$$F(b) - F(a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ikb} - e^{-ika}}{ik} f(k)\, dk.$$

COROLLARY 3.1.2. *If the c.f. $f(k)$ is absolutely integrable on $(-\infty, \infty)$, then the corresponding $F(x)$ is an absolutely continuous distribution function with bounded continuous density $p(x) = F'(x)$ defined by the formula*

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} f(k)\, dk. \tag{3.1.7}$$

The principal advantage of the use of c.f.'s in actual practice consists in the fact that the c.f. of a sum of independent r.v.'s is equal to the mere product of the c.f.'s of the summands:

$$f_{X_1+X_2}(k) = f_{X_1}(k) f_{X_2}(k), \tag{3.1.8}$$

while the density function of the sum is given by integral (2.1.9):

$$p_{X_1+X_2}(x) = \int_{-\infty}^{\infty} p_{X_1}(x - x') p_{X_2}(x')\, dx'.$$

Property (3.1.8) can be rewritten in a more simple form via the so-called second characteristic:

$$\psi_X(k) = \ln f_X(k) \tag{3.1.9}$$

Indeed,

$$\psi_{X_1+X_2}(k) = \psi_{X_1}(k) + \psi_{X_2}(k). \tag{3.1.10}$$

Property (e) is of special importance in the problems on summation of a large number of independent random variables where the application of density functions leads us to the necessity of evaluating multiple integrals. For this reason, the c.f.'s play the central role in the theory of stable laws. In this connection, the following theorem is of particular importance.

THEOREM 3.1.2 (continuity theorem). *Let $f_n(k)$, $n = 1, 2, \dots$ be a sequence of c.f.'s, and let $F_n(x)$ be a sequence of the corresponding distribution functions. If $f_n(k) \to f(k)$ as $n \to \infty$, for all k and $f(k)$ is continuous at $k = 0$, then $f(k)$ is the c.f. of a cumulative distribution function $F(x)$, and the sequence $F_n(x)$ weakly converges to $F(x)$, $F_n \Rightarrow F$. The inverse is also true: if $F_n \Rightarrow F$ and F is a distribution function, then $f_n(k) \to f(k)$, where $f(k)$ is the c.f. of the distribution function F.*

3.2. The characteristic functions of symmetric stable distributions

Let us begin with the c.f.'s of symmetric stable distributions whose densities are expressed in terms of elementary functions.

Henceforth we use the notation $g(k; \alpha, \beta)$ for a characteristic function of a reduced stable distribution density $q(x; \alpha, \beta)$:

$$g(k; \alpha, \beta) = \int_{-\infty}^{\infty} e^{ikx} q(x; \alpha, \beta)\, dx.$$

We calculate first the c.f.'s of the normal r.v.'s with density (2.2.5)

$$q(x; 2, 0) \equiv p^{\mathrm{G}}(x; 0, \sqrt{2}) = \frac{1}{2\sqrt{\pi}} e^{-x^2/4}. \tag{3.2.1}$$

By virtue of (3.1.2),

$$g(k; 2, 0) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u(k,x)}\, dx,$$

where

$$u(k, x) = x^2/4 - ikx.$$

We rewrite the last function as

$$u(k, x) = (x - 2ikx)^2/4 + k^2,$$

and pass to the complex integration variable $z = (x - 2ikx)/2$; then we obtain

$$g(k; 2, 0) = \frac{1}{\sqrt{\pi}} e^{-k^2} I, \tag{3.2.2}$$

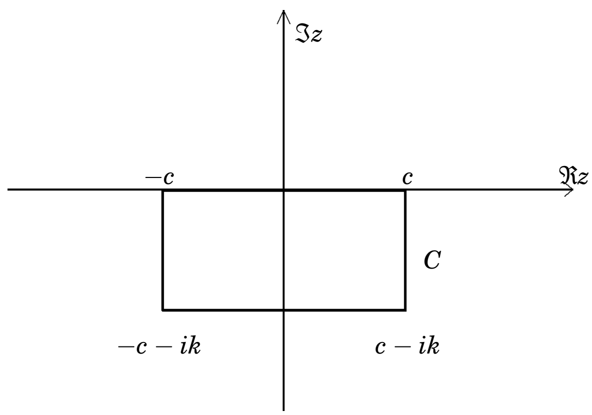


Figure 3.1. The integral of e^{-z^2} along C is zero

where

$$I = \int_{-\infty - ik}^{\infty - ik} e^{-z^2}\, dz$$

which must be understood as the limit

$$I = \lim_{c\to\infty} \int_{-c-ik}^{c-ik} e^{-z^2}\, dz. \tag{3.2.3}$$

Due to the absence of poles of the integrand inside the rectangle with corner points c, $c - ik$, $-c - ik$, and $-c$ (see Fig. 3.1), the integral along the closed contour is equal to zero by virtue of the well-known Cauchy's theorem. Since the integrals over the intervals $[c, c - ik]$ and $[-c - ik, -c]$ tend to zero sufficiently fast as $c \to \infty$, limit (3.2.3) can be rewritten as the improper integral along the real axis

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}, \tag{3.2.4}$$

which was considered in the preceding chapter. The substitution of (3.2.4) into (3.2.2) yields

$$g(k; 2, 0) = e^{-k^2}. \tag{3.2.5}$$

We note that the use of the traditional form of the normal distribution

$$p^{\mathrm{G}}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

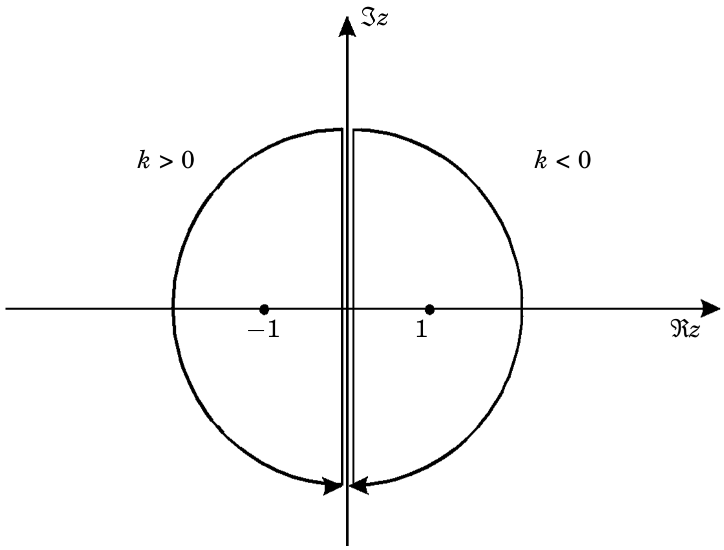


Figure 3.2.

leads us to the c.f.

$$f^{G}(k) = e^{-k^2/2}.$$

This c.f., as a function of k, is of the same form as the density p^{G} which is the function of its own argument x, but this coincidence is, in a sense, accidental: it takes place only for the normal distribution.

Let us now consider the Cauchy distribution (2.3.2) whose c.f. is

$$g(k; 1, 0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{1 + x^2} \, dx.$$

By the change of variable $z = ix$, it can be represented as

$$g(k; 1, 0) = -\frac{1}{\pi i} \int_{-i\infty}^{+i\infty} \frac{e^{kz}}{(z + 1)(z - 1)} \, dz.$$

Closing the contour of integration to the left for $k > 0$ and to the right for $k < 0$ (Fig. 3.2) and using the well-known theorem on residues, we obtain

$$f^{C}(k) = e^{-|k|}. \tag{3.2.6}$$

To derive the c.f. of a symmetric stable distribution with an arbitrary parameter α, we use formula (2.4.1) applied to the sum of independent strictly stable variables

$$\sum_{i=1}^{n} Y_i \stackrel{d}{=} b_n Y, \qquad b_n = n^{1/\alpha}. \tag{3.2.7}$$

According to Properties (b) and (e), it is possible to rewrite the equation following from (2.2.7) as

$$f_Y^n(k) = f_Y(n^{1/\alpha} k). \tag{3.2.8}$$

We recall that, by virtue of symmetry of the distribution, $f_Y(k)$ is real-valued. In a neighborhood of $k = 0$, the c.f. is positive; therefore, it has to be positive for any argument, as it follows from (3.2.8). Taking the logarithm of (3.2.8), we arrive at the equation for the second characteristic (3.1.9)

$$n\psi_Y(k) = \psi_Y(n^{1/\alpha} k). \tag{3.2.9}$$

Setting here

$$\psi_Y(k) = -ck^{\mu}, \qquad k > 0, \tag{3.2.10}$$

we see that (3.2.10) satisfies (3.2.9) with $\mu = \alpha$ and arbitrary complex-valued constant c. Presenting the latter as

$$c = \lambda[1 - ic_1],$$

with real λ and c_1, we obtain

$$\psi_Y(k) = -\lambda k^{\alpha}[1 - ic_1], \quad k > 0, \quad 0 < \alpha \leq 2. \tag{3.2.11}$$

To find $\psi_Y(k)$ in the field of negative values of the argument, we make use of property (c) of the c.f.'s, which yields

$$\psi_Y(k) = -\lambda[|k|^{\alpha} - ik\omega(k)], \quad -\infty < k < \infty, \quad 0 < \alpha \leq 2,$$

where

$$\omega(k) = c_1 |k|^{\alpha - 1}.$$

Since $|f_X(k)| \leq 1$, and therefore, $\Re\psi_X(k) \leq 0$, the real-valued constant λ should be non-negative. If $\lambda = 0$, we arrive at the confluent case (3.1.4) with $a = 0$. In the case where $\lambda > 0$, this constant is determined by the choice of a scale of r.v., and, without loss of generality, can be set equal to 1. The real-valued function $\omega(k)$ plays a more essential part; namely, it relates to the parameters α and β of a stable distribution as

$$w(k) = \omega(k; \alpha, \beta).$$

Thus, the c.f. of a strictly stable distribution with parameters α and β can be presented as

$$g(k;\alpha,\beta) = \exp\left\{-|k|^\alpha + ik\omega(k;\alpha,\beta)\right\}, \tag{3.2.12}$$

where the explicit form of dependence of function $\omega(\alpha,\beta)$ on α and β has yet to be determined. In the symmetric case where $\beta = 0$, it is very simple. In view of property (c), the c.f. is real-valued; hence

$$\omega(k;\alpha,0) = 0, \tag{3.2.13}$$

and therefore,

$$g(k;\alpha,0) = e^{-|k|^\alpha}. \tag{3.2.14}$$

The above c.f.'s (3.2.5) and (3.2.6) are mere cases of general expression (3.2.10):

$$f^{\mathrm{G}}(k) = e^{-k^2} \equiv g(k;2,0), \tag{3.2.15}$$

$$f^{\mathrm{C}}(k) = e^{-|k|} \equiv g(k;1,0). \tag{3.2.16}$$

Let us demonstrate the effectiveness of the method of the characteristic functions method in those two examples considered in the preceding chapter with the help of distribution densities:

$$q(x;2,0) * q(x;2,0) = q(x/\sqrt{2};2,0)/\sqrt{2},$$
$$q(x;1,0) * q(x;1,0) = q(x/2;1,0)/2.$$

By virtue of (3.2.7) and Properties (b) and (e), these relations are equivalent to

$$[g(k;2,0)]^2 = g(\sqrt{2}k;2,0),$$
$$[g(k;1,0)]^2 = g(2k;1,0).$$

They immediately follow from (3.2.15) and (3.2.16) by simple raising them to power two.

Let us look at Fig. 3.3, where the second characteristic $\psi(k;\alpha,0)$ in the neighborhood of zero is drawn. For $\alpha > 1$, the slope of curves changes continuously, but for $\alpha \le 1$ it breaks at the point $k = 0$, and the first derivative does not exist. For this reason, the first moment of distributions with $\alpha \le 1$ exists only in the Cauchy sense:

$$\lim_{A\to\infty} \int_{-A}^{A} q(x;\alpha,\beta)x\,dx = 0.$$

A comparison of the graphs of the corresponding densities reveals the following generality: the faster the tails of distribution decrease, the smoother the central part of characteristic function is, and vice versa (Feller, 1966).

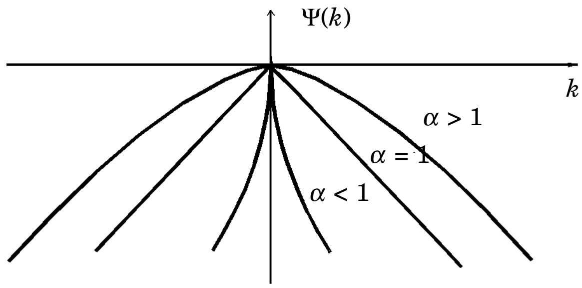


Figure 3.3. The second characteristics of symmetric stable laws

3.3. Skew stable distributions with $\alpha < 1$

The only asymmetric stable density for which an explicit representation via elementary functions is known is the Lévy density ($\alpha = 1/2$, $\beta = 1$):

$$q(x; 1/2, 1) = \frac{1}{\sqrt{2\pi}} e^{-1/(2x)} x^{-3/2}, \qquad x > 0. \tag{3.3.1}$$

It is concentrated on the positive semi-axis, and its c.f. is

$$g(k; 1/2, 1) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{ikx-1/(2x)} x^{-3/2}\, dx. \tag{3.3.2}$$

We consider the real and imaginary parts of c.f. (3.3.2)

$$\Re g(k; 1/2, 1) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-1/(2x)} x^{-3/2} \cos kx\, dx,$$
$$\Im g(k; 1/2, 1) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-1/(2x)} x^{-3/2} \sin kx\, dx.$$

Omitting bulky evaluations, we give the final results from (Oberhettinger, 1973):

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-1/(2x)} x^{-3/2} \cos kx\, dx = \sqrt{2\pi} e^{-\sqrt{|k|}} \cos \sqrt{|k|},$$
$$\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-1/(2x)} x^{-3/2} \sin kx\, dx = \sqrt{2\pi} e^{-\sqrt{|k|}} \sin \sqrt{|k|} \operatorname{sign} k,$$

where

$$\operatorname{sign} k = k/|k|.$$

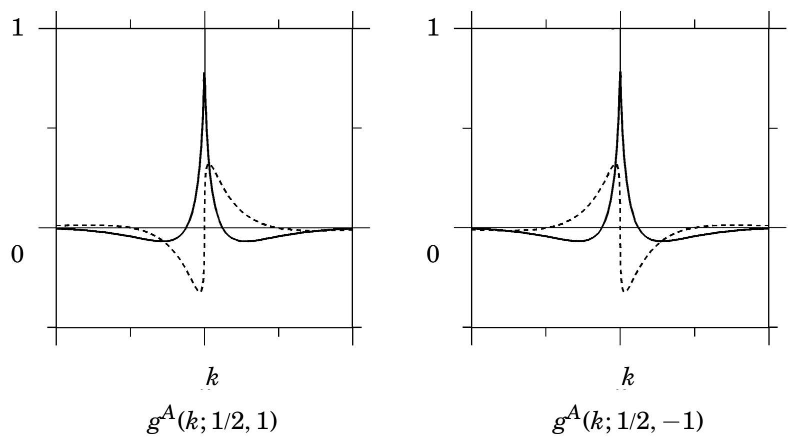


Figure 3.4. The c.f.'s $g(k; 1/2, 1)$ and $g(k; 1/2, -1)$ (— stands for $\Re g$, and - - -, for $\Im g$

Thus, c.f. (3.3.2) takes the form

$$g(k; 1/2, 1) = e^{-\sqrt{|k|}(1-i\,\mathrm{sign}\,k)}. \tag{3.3.3}$$

This function is complex-valued, but its modulus

$$|g(k; 1/2, 1)| = e^{-\sqrt{|k|}}$$

coincides with the c.f. of the symmetric stable distribution with the same parameter $\alpha = 1/2$.

The Lévy distribution (3.3.1) and the 'reflected' one (i.e., concentrated on the negative semi-axis) with the c.f.

$$g(k; 1/2, -1) = e^{-\sqrt{|k|}(1+i\,\mathrm{sign}\,k)} \tag{3.3.4}$$

are extreme in the set of stable distributions with parameter $\alpha = 1/2$. The graphs of these c.f.'s are shown in Fig. 3.4. Comparing (3.3.3) and (3.3.4) with (3.2.12), we obtain

$$\omega(k; 1/2, 1) = |k|^{-1/2},$$
$$\omega(k; 1/2, -1) = -|k|^{-1/2}.$$

The relation

$$g(k; 1/2, 1)g(k; 1/2, -1) = e^{-\sqrt{|k|}} \tag{3.3.5}$$

following from (3.3.3) and (3.3.4) demonstrates that the difference of two independent random variables $Y(1/2, 1)$ is the symmetric stable variable with the same characteristic exponent:

$$Y_1(1/2,1) - Y_2(1/2,1) \stackrel{d}{=} Y_1(1/2,1) + Y_2(1/2,-1) \stackrel{d}{=} 4Y(1/2,0). \tag{3.3.6}$$

In what follows, separate symbols of a random variable $Y(\alpha, \beta)$ stand for independent realizations of this random variable:

$$Y_1(\alpha,\beta) + Y_2(\alpha,\beta) \equiv Y(\alpha,\beta) + Y(\alpha,\beta). \tag{3.3.7}$$

Then property (3.3.6) takes the form

$$Y(1/2,1) - Y(1/2,1) \stackrel{d}{=} Y(1/2,1) + Y(1/2,-1) \stackrel{d}{=} 4Y(1/2,0). \tag{3.3.8}$$

Let us now pose the problem to find the c.f. $g(k; \alpha, \beta)$ for all $\beta \in [-1, 1]$, $\alpha \in (0, 1)$. Since explicit expressions for these densities are absent, we should refer to the definition of the stable laws. In view of the definition, we search for the c.f.'s as the limits of the c.f.'s of normalized sums (2.6.3), as $n \to \infty$:

$$g(k;\alpha,\beta) = \lim_{n\to\infty} f_{Z_n}(k), \tag{3.3.9}$$

where

$$Z_n = \frac{1}{b_n}\left(\sum_{i=1}^{n} X_i - a_n\right).$$

As concerns the distribution of individual terms which should satisfy conditions (2.5.18), we choose it in the simplest (Zipf–Pareto) form (Fig. 3.5)

$$p_X(x) = \begin{cases} \alpha c x^{-\alpha-1}, & x > \varepsilon, \\ 0, & -\varepsilon < x < \varepsilon, \\ \alpha d|x|^{-\alpha-1}, & x < -\varepsilon. \end{cases} \tag{3.3.10}$$

We have already dealt with a particular symmetric case of such a distribution in Section 2.5. We managed to calculate the one-fold convolution of such distributions with $\alpha = 1$, but the evaluation of multiple convolutions of these distributions is rather difficult. Therefore, we will solve it by means of c.f.'s.

The c.f. of an individual summand distributed by (3.3.10) is of the form

$$\begin{aligned} f_X(k) &= \int_\varepsilon^\infty e^{ikx} p_X(x)\,dx + \int_{-\infty}^{-\varepsilon} e^{ikx} p_X(x)\,dx \\ &= \alpha a \int_\varepsilon^\infty e^{ikx} x^{-\alpha-1}\,dx + \alpha b \int_\varepsilon^\infty e^{-ikx} x^{-\alpha-1}\,dx. \end{aligned} \tag{3.3.11}$$

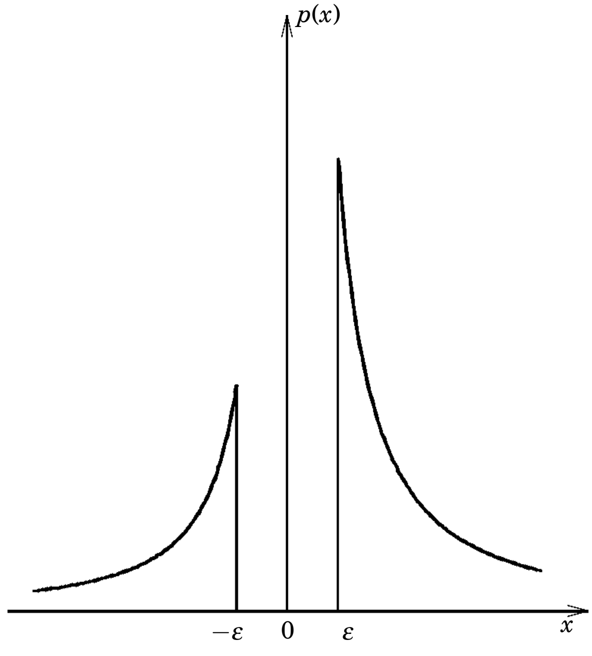


Figure 3.5. Density (3.3.10)

By virtue of property (a),

$$f_X(0) = (c+d)\varepsilon^{-\alpha} = 1. \tag{3.3.12}$$

Let $k > 0$; then, by the change of variable $z = kx$, c.f. (3.3.11) is reduced to

$$f_X(k) = \alpha(c+d)k^\alpha \left[I_c^{(-\alpha)}(\varepsilon k) + i\beta I_s^{(-\alpha)}(\varepsilon k)\right], \tag{3.3.13}$$

where

$$I_s^{(\mu)}(k) = \int_k^\infty z^{\mu-1} \sin z \, dz, \tag{3.3.14}$$

$$I_c^{(\mu)}(k) = \int_k^\infty z^{\mu-1} \cos z \, dz, \tag{3.3.15}$$

$$\beta = (c-d)/(c+d).$$

By virtue of Properties (b) and (e) of a characteristic function,

$$f_{Z_n}(k) \equiv f_{S_n/b_n}(k) = [f_X(k/b_n)]^n. \tag{3.3.16}$$

We make use of expression (2.6.3) for a normalized sum, and obtain $a_n = 0$, since the distributions under consideration are strictly stable ($\alpha < 1$). The positive coefficients b_n infinitely increase as $n \to \infty$, so the c.f. f_X enters into (3.3.16) for small values of the argument k/b_n.

In the domain of small k, c.f. (3.3.13) can be simplified. To this end, we present the limiting values of integrals (3.3.14) and (3.3.15) which enter into (3.3.13) as

$$\lim_{k\to+0} I_s^{(\mu)}(k) = \Gamma(\mu)\sin(\mu\pi/2), \qquad 0 < |\mu| < 1, \tag{3.3.17}$$

$$\lim_{k\to+0} I_c^{(\mu)}(k) = \Gamma(\mu)\cos(\mu\pi/2), \qquad 0 < \mu < 1. \tag{3.3.18}$$

If $\mu = -\alpha$ and $0 < \alpha < 1$, then only the first limit exists:

$$\lim_{k\to+0} I_s^{(-\alpha)}(k) = -\Gamma(-\alpha)\sin(\alpha\pi/2). \tag{3.3.19}$$

By integration by parts the remaining integral is transformed to

$$I_c^{(-\alpha)}(k) = \alpha^{-1}k^{-\alpha}\cos k - \alpha^{-1}I_s^{(1-\alpha)}(k). \tag{3.3.20}$$

The second term tends to the finite limit (3.3.17) as $k \to +0$, and we obtain

$$I_c^{(-\alpha)}(k) \sim \alpha^{-1}k^{-\alpha} - \alpha^{-1}\Gamma(1-\alpha)\cos(\alpha\pi/2), \quad k \to +0. \tag{3.3.21}$$

Substituting (3.3.19) and (3.3.21) into (3.3.13) and taking (3.3.12) into account, we arrive at expression

$$f_X(k) \sim 1 - (c+d)k^\alpha\left\{\Gamma(1-\alpha)\cos(\alpha\pi/2) + i\alpha\beta\Gamma(-\alpha)\sin(\alpha\pi/2)\right\},$$

which, after simple transformations, takes the form

$$f_X(k) \sim 1 - C(\alpha)(c+d)k^\alpha[1 - i\beta\tan(\alpha\pi/2)], \qquad k \to +0,$$

where

$$C(\alpha) = \Gamma(1-\alpha)\cos(\alpha\pi/2).$$

Using property (c) to determine the c.f. in the domain of negative k, we obtain the general expression for the c.f. in neighborhood of zero

$$f_X(k) \sim 1 - C(\alpha)(c+d)|k|^\alpha[1 - i\beta\tan(\alpha\pi/2)\operatorname{sign} k], \quad k \to 0. \tag{3.3.22}$$

Substituting (3.3.22) into (3.3.16), we obtain

$$f_{Z_n}(k) \sim \left\{1 - C(\alpha)(c+d)b_n^{-\alpha}|k|^\alpha[1 - i\beta\tan(\alpha\pi/2)\operatorname{sign} k]\right\}^n, \quad n \to \infty. \tag{3.3.23}$$

and, after introducing

$$y_n = -C(\alpha)(c+d)b_n^{-\alpha}|k|^\alpha[1 - i\beta\tan(\alpha\pi/2)\operatorname{sign}k],$$

we represent (3.3.23) as

$$f_{Z_n}(k) \sim \left[(1+y_n)^{1/y_n}\right]^{ny_n}, \qquad n \to \infty. \tag{3.3.24}$$

Since $y_n \to 0$ as $n \to \infty$, the expression in square brackets tends to e. For the limit of (3.3.24) to be a c.f., it is necessary that $nb_n^{-\alpha}$ tends to a finite limit different from zero. We set

$$b_n = b_1 n^{1/\alpha}, \qquad b_1 = [C(\alpha)(c+d)]^{1/\alpha}; \tag{3.3.25}$$

then

$$\begin{aligned} g(k;\alpha,\beta) &= \lim_{n\to\infty} f_{Z_n}(k) \\ &= \exp\left\{-|k|^\alpha[1 - i\beta\tan(\alpha\pi/2)\operatorname{sign}k]\right\}, \quad 0 < \alpha < 1. \end{aligned} \tag{3.3.26}$$

This is the c.f. sought for. For $\beta = 0$, we obtain the c.f. of the symmetric distribution (3.2.10); for $\alpha = 1/2$ and $\beta = \pm 1$, we have the c.f. of one-sided distributions (3.3.3) and (3.3.4); and the general expression (3.3.26) for normalizing coefficients b_n coincides with (2.4.8), differing from it only by the factor $b_1 \neq 1$, which appears due to the difference of (3.3.10) from the stable distribution.

3.4. The general form of stable characteristic functions

In the case where $\alpha > 1$, the mathematical expectation of the r.v. X given by (3.3.10) exists:

$$\mathsf{E}X = \alpha(c-d)\int_\varepsilon^\infty x^{-\alpha}dx = \frac{\alpha(c-d)}{(\alpha-1)\varepsilon^{\alpha-1}} \equiv a, \tag{3.4.1}$$

and, as a consequence, the expectation of the sum exists:

$$\mathsf{E}S_n = na.$$

The latter can be chosen as the centering constant, namely $a_n = na$ in (2.6.3), which yields

$$Z_n = (S_n - na)/b_n. \tag{3.4.2}$$

Formula (3.3.13) remains valid for $\alpha > 1$ as well, but the problem consists now in that both integrals contained there diverge as $k \to 0$. Therefore, alongside with the use of (3.3.20), we need to transform the first integral (3.3.14) in an appropriate way:

$$I_s^{(-\alpha)}(k) = \alpha^{-1}k^{-\alpha}\sin k + \alpha^{-1}I_c^{(1-\alpha)}(k)$$
$$= \alpha^{-1}k^{-\alpha}\sin k + [\alpha(\alpha-1)]^{-1}k^{-\alpha+1}\cos k - [\alpha(\alpha-1)]^{-1}I_s^{(2-\alpha)}(k). \quad (3.4.3)$$

We recall that $k > 0$. Here the latter integral possesses a finite limit equal to (3.3.17) as $k \to 0$. Substituting (3.3.20) and (3.4.3) into (3.3.13), and collecting the leading terms as $k \to 0$, we obtain

$$f_X(k) \sim 1 + ika - k^\alpha C(\alpha)(c+d)[1 - i\beta\tan(\alpha\pi/2)], \quad k \to 0, \quad k > 0, \quad (3.4.4)$$

or, by extending it to the whole real axis,

$$f_X(k) \sim 1 + ika - C(\alpha)(c+d)|k^\alpha|[1 - i\beta\tan(\alpha\pi/2)\,\mathrm{sign}\,k], \quad k \to 0. \quad (3.4.5)$$

Expression (3.4.5) differs from (3.3.22) with $\alpha < 1$ only by the presence of ika_1 occurred due to the finiteness of expectation 3.4.1.

According to property (c), the characteristic function of r.v. (3.4.2) is

$$f_{Z_n}(k) = e^{-ikna/b_n} f_{S_n}(k/b_n) = e^{-ika/b_n}[f_X(k/b_n)]^n. \quad (3.4.6)$$

By virtue of the smallness of the terms containing k and $|k|^\alpha$, c.f. (3.4.5) can be represented in the asymptotically equivalent form

$$f_X(k) \sim [1 + ika]\{1 - C(\alpha)(c+d)|k|^\alpha[1 - i\beta\tan(\alpha\pi/2)\,\mathrm{sign}\,k]\}, \quad k \to 0. \quad (3.4.7)$$

Substituting (3.4.7) into (3.4.6) and, as before, setting b_n according to (3.3.25), we arrive at the limiting expression

$$g(k;\alpha,\beta) = \exp\left\{-|k|^\alpha[1 - i\beta\tan(\alpha\pi/2)\,\mathrm{sign}\,k]\right\}, \quad 1 < \alpha < 2,$$

which coincides with (3.3.26).

Thus, as we foretold in Section 2.2, the c.f.'s of strictly stable laws look as

$$g(k;\alpha,\beta) = \exp\{-|k|^\alpha + ik\omega(k;\alpha,\beta)\}, \quad (3.4.8)$$

where

$$\omega(k;\alpha,\beta) = |k|^{\alpha-1}\beta\tan(\alpha\pi/2), \quad \alpha \neq 1. \quad (3.4.9)$$

The peculiarity of the case where $\alpha = 1$, $\beta \neq 0$, which is not covered by the strictly stable laws, is that the mathematical expectation does not exist, but it is impossible to drop the centering constants a_n here. Rather than (3.4.2) we should consider

$$Z_n = (S_n - a_n)/b_n, \tag{3.4.10}$$

where a_n are to be chosen later.

Let us turn back to the expression for the c.f.'s of summands (3.3.13) which, in the case under consideration, takes the form

$$f_X(k) = (c+d)k\left[I_c^{(-1)}(\varepsilon k) + i\beta I_s^{(-1)}(\varepsilon k)\right]. \tag{3.4.11}$$

Transforming both the integrals $I_c^{(-1)}$ and $I_s^{(-1)}$ by means of integration by parts, we reduce them to

$$I_c^{(-1)}(k) = k^{-1}\cos k - \pi/2 + \mathrm{Si}(k) \sim k^{-1} - \pi/2, \qquad k \to +0, \tag{3.4.12}$$

$$I_s^{(-1)}(k) = k^{-1}\sin k - \pi/2 + \mathrm{Ci}(k) \sim -\ln k, \qquad k \to +0, \tag{3.4.13}$$

where $\mathrm{Si}(k)$ and $\mathrm{Ci}(k)$ are the integral sine and cosine respectively. Substituting (3.4.12) and (3.4.13) into (3.4.11) under normalization (3.3.12), and extending the results to the domain $k < 0$, we obtain

$$\begin{aligned} f_X(k) &= 1 - (\pi/2)(c+d)|k| - i(c+d)\beta k \ln[(c+d)|k|] \\ &\sim [1 - (\pi/2)(c+d)|k|]\{1 - i(c+d)\beta k \ln[(c+d)|k|]\}, \qquad k \to 0. \end{aligned} \tag{3.4.14}$$

Characteristic function (3.4.10) is related to (3.4.14) as follows:

$$f_{Z_n}(k) = e^{-i(a_n/b_n)k}[f_X(k/b_n)]^n \sim e^{-i(a_n/b_n)k}[(1+y_n)^{1/y_n}]^{ny_n}[(1+z_n)^{1/z_n}]^{nz_n},$$

where

$$\begin{aligned} y_n &= -(\pi/2)(c+d)b_n^{-1}|k| \to 0, \qquad n \to \infty, \\ z_n &= -i\beta(c+d)b_n^{-1}k\ln[(c+d)b_n^{-1}|k|] \to 0, \qquad n \to \infty. \end{aligned}$$

Hence

$$\begin{aligned} f_{Z_n}(k) \sim\ & \exp\{-(\pi/2)(c+d)b_n^{-1}n|k|\} \\ & \times \exp\{-i(a_n/b_n)k - i\beta(c+d)b_n^{-1}nk\ln[(c+d)b_n^{-1}|k|]\}. \end{aligned}$$

The first factor shows that the sequence b_n must be chosen so that

$$b_n = b_1 n, \qquad b_1 = (\pi/2)(c+d), \tag{3.4.15}$$

and from the second one, we see that the sequence

$$c_n = a_n/n + \beta(c+d)\ln|(2/\pi)k/n| \tag{3.4.16}$$

should possess a finite limit. Rewriting (3.4.16) as

$$c_n = a_n/n - \beta(c+d)\ln(2n/\pi) + \beta(c+d)\ln|k|,$$

we see that it is sufficient to set

$$a_n = \beta(c+d)n\ln(2n/\pi) \sim \beta(c+d)n\ln n, \qquad n\to\infty. \tag{3.4.17}$$

Thus, the c.f.'s of stable laws with parameter $\alpha = 1$ can be presented as

$$g(k;1,\beta) = \exp\left\{-|k| - i(2/\pi)\beta k\ln|k|\right\}. \tag{3.4.18}$$

Setting

$$\omega(k;1,\beta) = -(2/\pi)\beta\ln|k| \tag{3.4.19}$$

we gather the results obtained above as follows:

$$g(k;\alpha,\beta) = \exp\{-|k|^\alpha + ik\omega(k;\alpha,\beta)\}, \qquad 0<\alpha\le 2, \tag{3.4.20}$$

where $\omega(k;\alpha,\beta)$ is determined by (3.4.9) and (3.4.10)[1].

According to the definition of the stable distributions and Properties (b) and (e) of the c.f.'s, for the stable c.f. $g(k)$ the relation

$$g(b_1k)g(b_2k) = g(bk)e^{iak} \tag{3.4.21}$$

holds. It can easily be seen that the above c.f.'s satisfy this condition. Indeed, for $\alpha \ne 1$ we have

$$g(k;\alpha,\beta) = \exp\{-b^\alpha|k|^\alpha[1 - i\beta\tan(\alpha\pi/2)\,\mathrm{sign}\,k]\},$$

Substituting this into (3.4.21), we obtain

$$\exp\{-(b_1^\alpha + b_2^\alpha)|k|^\alpha[1 - i\beta\tan(\alpha\pi/2)\,\mathrm{sign}\,k]\}$$
$$= e^{iak}\exp\{-b^\alpha|k|^\alpha[1 - i\beta\tan(\alpha\pi/2)\,\mathrm{sign}\,k]\}.$$

It follows herefrom that

$$a = 0, \qquad b = (b_1^\alpha + b_2^\alpha)^{1/\alpha}. \tag{3.4.22}$$

For $a = 0$, (3.4.21) takes the form

$$g(b_1k)g(b_2k) = g(bk), \tag{3.4.23}$$

[1]The calculations performed here and in the preceding section can be considered as the proof of a somewhat simplified version of the generalized limit theorem in Section 2.5, where conditions (2.5.17)–(2.5.18) are replaced with simpler constraints (3.3.10).

and hence corresponds to strictly stable c.f.'s, so the above c.f.'s for $\alpha \neq 1$ are strictly stable indeed. The symmetric ($\beta = 0$) distribution with parameter $\alpha = 1$ also belongs to the family of strictly stable laws, and its c.f. can be derived as the limit of c.f. (3.4.8):

$$g(k; 1, 0) = \lim_{\alpha \to 1} g(k; \alpha, 0).$$

The reader should keep in mind that, while deriving c.f.'s, we set $\lambda = 1$ in (3.2.11). Assigning an arbitrary positive value to it and taking the shift parameter in the form $\gamma\lambda$, we obtain the second characteristic of a stable law in the general form

$$\ln g(k; \alpha, \beta, \gamma, \lambda) = \lambda[ik\gamma - |k|^\alpha + ik\omega(k; \alpha, \beta)]. \tag{3.4.24}$$

Thus, we have a four-parameter family of stable c.f.'s whose parameters are listed in their arguments. We extend the appropriate notation to the density of strictly stable distributions as well:

$$q(x; \alpha, \beta, \gamma, \lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} g(k; \alpha, \beta, \gamma, \lambda)\, dk. \tag{3.4.25}$$

Representation (3.4.24) rewritten in the form

$$\ln g(k; \alpha, \beta, \gamma', \lambda) = ik\gamma' + \lambda[-|k|^\alpha + ik\omega(k; \alpha, \beta)],$$

used in many works (see (3.5.24) and (3.5.25)), provides us with the opportunity to supplement the family with the degenerate distribution with second characteristic $ik\gamma'$.

3.5. Stable laws as infinitely divisible laws

As we have seen in Section 2.2, stable laws belong to the set of infinitely divisible laws. Now we can easily verify this, having the explicit form for c.f.'s at our disposal. To this end, we represent definition (3.4.24) of infinitely divisible distributions in terms of c.f.'s as

$$f(k) = [f_n(k)]^n.$$

The functions $f_n(k)$ are uniquely determined via the function $f(k)$ as

$$f_n(k) = [f(k)]^{1/n}, \tag{3.5.1}$$

where the principal branch of the radical is taken.

It is clear that c.f. (4.23) is infinitely divisible, because the radical $f^{1/n}$ does not change its form if we replace λ by λ/n.

Nevertheless, the set of infinitely divisible distributions is not exhausted by the stable laws. Indeed, the Poisson distribution

$$p_n = \mathsf{P}\{N = n\} = \frac{a^n}{n!}e^{-a}, \qquad a > 0, \tag{3.5.2}$$

possesses the characteristic function

$$\begin{aligned} f(k) = \mathsf{E}e^{ikN} &= e^{-a}\sum_{n=0}^{\infty}\frac{(ae^{ik})^n}{n!} \\ &= \exp\{a(e^{ik} - 1)\}. \end{aligned}$$

Taking the root

$$[f(k)]^{1/n} = \exp\left\{\frac{a}{n}(e^{ik} - 1)\right\}$$

merely modifies the parameter a that has the sense of the expectation of N.

Let us now consider the sum of a random number of identically distributed independent summands

$$S_N = \sum_{i=1}^{N} X_i. \tag{3.5.3}$$

Let $F_X(x)$ and $f_X(k)$ be the distribution function and the c.f. of X_i, respectively, and let N be independent of $X_1, X_2, \dots$ and distributed by the Poisson law (3.5.2). Then for r.v. (3.5.3) we have

$$F_{S_N}(x) = e^{-a}\sum_{n=0}^{\infty}\frac{a^n}{n!}\underbrace{F_X * F_X * \dots * F_X}_{n}, \tag{3.5.4}$$

$$f_{S_N}(x) = \exp\{a[f_X(k) - 1]\}. \tag{3.5.5}$$

Distribution (3.5.4) is called the generalized Poisson distribution, in the case of a degenerate r.v. ($X = 1$), it coincides with (3.5.2). From (3.5.5), we can immediately see that this distribution belongs also to the class of infinitely divisible distributions. It is clear that neither (3.5.2) nor (3.5.4) are stable distributions (f_X in (3.5.5) is an arbitrary c.f.). On the other hand, the uniform on $(-1, 1)$ distribution is not infinitely divisible, because we cannot take the nth root of its c.f.

$$f(k) = \frac{\sin k}{k} \tag{3.5.6}$$

in such a way that the result satisfies the conditions imposed on c.f.'s. For a c.f. to belong to a class of infinitely divisible distributions, it is necessary, but not sufficient, that there are no zeros on the real axis. Stable distributions and the generalized Poisson distribution satisfy this condition, whereas the uniform distribution (3.5.6) does not.

The generalized Poisson distribution (3.5.4) plays an important part in the theory of infinitely divisible distributions: any infinitely divisible distribution is a limit of a sequence of generalized Poisson distributions (Feller, 1966). Introducing, for the sake of generality, an arbitrary centering of sums S_ν, we present the c.f. of an infinitely divisible distribution as

$$
\begin{aligned}
f(k) &= e^{\psi(k)}, \\
\psi(k) &= \lim_{n\to\infty} c_n[f_n(k) - 1 - ia_n k],
\end{aligned}
\tag{3.5.7}
$$

where c_n are positive constants, a_n are real-valued centering constants. The centering by expectations is a rather natural way for those distributions that possess mathematical expectations. In a general case, the centering can be performed by different ways. It seems likely that the most elementary way is to require that

$$\psi_n(k) = c_n[f_n(k) - 1 - ia_n k] \tag{3.5.8}$$

is real-valued for $k = 1$ (Feller, 1966). Then

$$
\begin{aligned}
\Im\psi_n(1) &= 0, \\
\Im f_n(1) &= \int_{-\infty}^{\infty} \sin x\, dF_n(x) = a_n.
\end{aligned}
\tag{3.5.9}
$$

Here we take account for

$$f_n(k) = \int_{-\infty}^{\infty} e^{ikx}\, dF_n(x). \tag{3.5.10}$$

Substituting (3.5.9) and (3.5.10) into (3.5.8), we obtain

$$\psi_n(k) = c_n \int_{-\infty}^{\infty} (e^{ikx} - 1 - ik\sin x)\, dF_n(x). \tag{3.5.11}$$

Passing to the limit as $n \to \infty$, we represent this expression as

$$\psi(k) = \int_{x\neq 0} (e^{ikx} - 1 - ik\sin x)\, dH(x), \tag{3.5.12}$$

where the function $H(x)$, defined on the whole x-axis with the only point $x = 0$ excluded, does not decrease on the semi-axes $x < 0$, $x > 0$, tends to zero as $|x| \to \infty$, and satisfies the condition

$$\int_{0<|x|<1} x^2\, dH(x) < \infty.$$

The representation

$$f(k) = \exp\left\{ika - bk^2 + \int_{x\neq 0}^{\infty} \left(e^{ikx} - 1 - ik\sin x\right)\, dH(x)\right\}, \quad (3.5.13)$$

where a and $b \geq 0$ are real numbers, is called the canonical form of an infinitely divisible c.f., and the function $H(x)$ is called the spectral function of an infinitely divisible distribution. There are also other forms of infinitely divisible c.f.'s, for example, the Lévy–Khinchin form

$$f(k) = \exp\left\{ika - bk^2 + \int_{-\infty}^{\infty} \left(e^{ikx} - 1 - \frac{ikx}{1+x^2}\right) \frac{1+x^2}{x^2}\, dH(x)\right\}. \quad (3.5.14)$$

The following theorem allows us to express the c.f.'s of stable distributions in terms of the spectral functions.

THEOREM 3.5.1 (canonical form of a stable c.f.). *The c.f. of any non-degenerate stable distribution is of canonical form* (3.5.14)*, where either*

$$b \neq 0, \qquad H(x) = 0, \quad (3.5.15)$$

or

$$b = 0, \qquad H(x) = \begin{cases} -cx^{-\alpha}, & x > 0 \\ d|x|^{-\alpha}, & x < 0 \end{cases} \quad (3.5.16)$$

with parameters $\alpha \in (0,2)$*,* $c \geq 0$*,* $d \geq 0$*,* $c + d > 0$.

The reverse is true as well: c.f. (3.5.15) *under conditions* (3.5.15)–(3.5.16) *is stable.*

We prove here only the second part of this theorem following (Lukacs, 1960). Condition (3.5.15) immediately leads us to the c.f. of the normal distribution. Substitution of (3.5.16) into (3.5.14) yields

$$\psi(k) = ika - bk^2 + \alpha c I_+(k) + \alpha d I_-(k), \quad (3.5.17)$$

where

$$I_+(k) = \int_0^{\infty} \left(e^{ikx} - 1 - \frac{ikx}{1+x^2}\right) \frac{dx}{x^{1+\alpha}}, \quad (3.5.18)$$

$$I_-(k) = \int_{-\infty}^{0} \left(e^{ikx} - 1 - \frac{ikx}{1+x^2}\right) \frac{dx}{|x|^{1+\alpha}}. \quad (3.5.19)$$

We first consider the case where $\alpha \in (0,1)$. Then, as one can easily see, the integrals

$$\int_{-\infty}^{0} \frac{x}{1+x^2} \frac{dx}{|x|^{1+\alpha}} \quad \text{and} \quad \int_0^{\infty} \frac{x}{1+x^2} \frac{dx}{x^{1+\alpha}}$$

are finite. Therefore, formula (3.5.17) can be rewritten as

$$\ln f(k) = ika' + \alpha d \int_{-\infty}^{0} (e^{ikx} - 1)\frac{dx}{|x|^{1+\alpha}} + \alpha c \int_{0}^{\infty} (e^{ikx} - 1)\frac{dx}{x^{1+\alpha}}. \quad (3.5.20)$$

We assume that $k > 0$; the change of variables in (3.5.20) gives us

$$\psi(k) = ika' + \alpha k^{\alpha} \left[d \int_{0}^{\infty} (e^{-iv} - 1)\frac{dv}{v^{1+\alpha}} + c \int_{0}^{\infty} (e^{iv} - 1)\frac{dv}{v^{1+\alpha}} \right]. \quad (3.5.21)$$

Let Γ be some contour which consists of the segment $[r, R]$, the real axis, the arc $z = Re^{i\varphi}$, $0 \le \varphi \le \pi/2$, of a circle of radius R with center at zero, the segment $[iR, ir]$ of the imaginary axis, and the arc $z = re^{i\varphi}$, $\pi/2 \ge \varphi \ge 0$, of the circle of radius r and center at zero. It follows from the Cauchy theorem that

$$\int_{\Gamma} (e^{iz} - 1)\frac{dz}{z^{1+\alpha}} = 0.$$

Besides, we easily see that the integrals over arcs of circles tend to zero as $r \to 0$ and $R \to \infty$. Thus,

$$\int_{0}^{\infty} (e^{iv} - 1)\frac{dv}{v^{1+\alpha}} = e^{-i\alpha\pi/2} L_1(\alpha)$$

where

$$L_1(\alpha) = \int_{0}^{\infty} (e^{-y} - 1)\frac{dy}{y^{1+\alpha}} = \Gamma(-\alpha) < 0.$$

Similar reasoning yields

$$\int_{0}^{\infty} (e^{-iv} - 1)\frac{dv}{v^{1+\alpha}} = e^{i\alpha\pi/2} L_1(\alpha).$$

It follows from (3.5.21) that

$$\psi(k) = ika' + k^{\alpha} \alpha L_1(\alpha)(c + d)\cos(\alpha\pi/2) \left[1 - i\frac{c - d}{c + d}\tan(\alpha\pi/2) \right].$$

Taking property (c) of the c.f.'s into account, and introducing

$$\lambda = -\alpha L_1(\alpha)(c + d)\cos(\alpha\pi/2) > 0,$$
$$\beta = \frac{c - d}{c + d},$$

we see that for $0 < \alpha < 1$ and for all k

$$\ln f(k) = ika' - \lambda|k|^{\alpha} \left(1 - i\beta\frac{k}{|k|}\tan(\alpha\pi/2) \right), \quad (3.5.22)$$

where $\lambda > 0$ and $|\beta| \le 1$.

Consider now the case where $1 < \alpha < 2$. It can easily be seen that

$$\int_0^\infty \frac{x^2}{1+x^2}\frac{dx}{x^\alpha} = \int_{-\infty}^0 \frac{x^2}{1+x^2}\frac{dx}{|x|^\alpha} < \infty.$$

We can rewrite (3.5.17) as

$$\ln f(k) = ika'' + \alpha d\int_{-\infty}^0 (e^{ikx} - 1 - ikx)\frac{dx}{|x|^{1+\alpha}} + \alpha c\int_0^\infty (e^{ikx} - 1 - ikx)\frac{dx}{x^{1+\alpha}},$$

or, for $k > 0$,

$$\ln f(k) = ika'' + t^\alpha \alpha \left\{ d\int_0^\infty (e^{-iv} - 1 + iv)\frac{dv}{v^{1+\alpha}} + c\int_0^\infty (e^{iv} - 1 - iv)\frac{dv}{v^{1+\alpha}} \right\}. \quad (3.5.23)$$

Integrating the function $(e^{-iz} - 1 + iz)/z^{1+\alpha}$ over the contour Γ and following the reasoning above, we obtain

$$\int_0^\infty (e^{-iv} - 1 + iv)\frac{dv}{v^{1+\alpha}} = e^{i\alpha\pi/2}L_2(\alpha),$$
$$\int_0^\infty (e^{iv} - 1 - iv)\frac{dv}{v^{1+\alpha}} = e^{-i\alpha\pi/2}L_2(\alpha),$$

where

$$L_2(\alpha) = \int_0^\infty (e^{-v} - 1 + v)\frac{dv}{v^{1+\alpha}} = \Gamma(-\alpha) > 0.$$

From (3.5.23) we obtain for $k > 0$

$$\ln f(k) = ika'' - \lambda k^\alpha \left(1 - \beta\tan(\alpha\pi/2)\right),$$

where

$$\lambda = -\alpha(c + d)L_2(\alpha)\cos(\alpha\pi/2) > 0,$$
$$\beta = (c - d)/(c + d).$$

For $k < 0$, the function $f(k)$ can be defined using property (c) of the c.f.'s; namely, for $1 < \alpha < 2$

$$\ln f(k) = ika'' - \lambda|k|^\alpha \left(1 - i\beta\frac{k}{|k|}\tan(\alpha\pi/2)\right) \quad (3.5.24)$$

where $c > 0$, $|\beta| \le 1$.

Now we consider the case where $\alpha = 1$. The following equality is used below for the calculation of the integrals in (3.5.17):

$$\int_0^\infty \frac{1-\cos y}{y^2} dy = \frac{\pi}{2}.$$

If $k > 0$, then

$$\begin{aligned}
&\int_0^\infty \left(e^{ikx} - 1 - \frac{ikx}{1+x^2}\right)\frac{dx}{x^2} \\
&\quad = \int_0^\infty \frac{\cos kx - 1}{x^2} dx + i\int_0^\infty \left(\sin kx - \frac{kx}{1+x^2}\right)\frac{dx}{x^2} \\
&\quad = -\frac{\pi}{2}k + i\lim_{\varepsilon\to 0}\left[\int_\varepsilon^\infty \frac{\sin kx}{x^2}dx - k\int_\varepsilon^\infty \frac{dx}{x(1+x^2)}\right] \\
&\quad = -\frac{\pi}{2}k + i\lim_{\varepsilon\to 0}\left\{-k\int_\varepsilon^{\varepsilon k}\frac{\sin v}{v^2}dv + k\left[\int_\varepsilon^\infty \left(\frac{\sin v}{v^2} - \frac{1}{v(1+v^2)}\right)dv\right]\right\}.
\end{aligned}$$

It is easy to see that as $\varepsilon \to 0$

$$\int_\varepsilon^\infty \left(\frac{\sin v}{v^2} - \frac{1}{v(1+v^2)}\right) dv \to A < \infty;$$

moreover,

$$\lim_{\varepsilon\to 0}\int_\varepsilon^{\varepsilon k}\frac{\sin v}{v^2}dv = \lim_{\varepsilon\to 0}\int_\varepsilon^{\varepsilon k}\frac{dv}{v} = \ln k;$$

therefore,

$$\int_0^\infty \left(e^{ikx} - 1 - \frac{ikx}{1+x^2}\right)\frac{dx}{x^2} = -\frac{\pi}{2}k - ik\ln k + Aik.$$

Since integrals (3.5.18) and (3.5.19) are complex conjugate integrals,

$$\begin{aligned}
\int_{-\infty}^0 \left(e^{ikx} - 1 - \frac{ikx}{1+x^2}\right)\frac{dx}{x^2} &= \int_0^\infty \left(e^{-ikx} - 1 + \frac{ikx}{1+x^2}\right)\frac{dx}{x^2} \\
&= -\frac{\pi}{2}k + ik\ln k - iAk;
\end{aligned}$$

therefore, for $k > 0$

$$\ln f(k) = ika'' - (c+d)(\pi/2)k - (c-d)ik\ln k.$$

From property (c) for $\alpha = 1$ and all real k we obtain

$$\ln f(k) = ika'' - \lambda|k|\left\{1 + i\beta\frac{2}{\pi}\frac{k}{|k|}\ln|k|\right\}. \tag{3.5.25}$$

Here

$$\lambda = (c+d)\pi/2, \qquad \beta = (c-d)/(c+d).$$

Combining (3.5.22), (3.5.24) and (3.5.25), we arrive at (3.4.24) with $a'' = \lambda\gamma$.

3.6. Various forms of stable characteristic functions

So far as there are different forms of representation of stable characteristic functions, we refer to the representation used above as form A:

$$\ln g^A(k;\alpha,\beta,\gamma,\lambda) = \lambda(ik\gamma - |k|^\alpha + ik\omega^A(k;\alpha,\beta)), \tag{3.6.1}$$

where

$$\omega^A(k;\alpha,\beta) = \begin{cases} |k|^{\alpha-1}\beta\tan(\alpha\pi/2), & \alpha \neq 1, \\ -\beta(2/\pi)\ln|k|, & \alpha = 1, \end{cases} \tag{3.6.2}$$

$0 < \alpha < 2$, $-1 \le \beta \le 1$, $-\infty < \gamma < \infty$, $\lambda > 0$. It is labeled by index A for the purposes of distinguishing this representation from the others which we consider in this section. As the necessity appears, we assign the corresponding indices to the parameters, too.

While investigating the analytical properties of the family of stable laws, form B appears to be more convenient:

$$\ln g^B(k;\alpha,\beta,\gamma,\lambda) = \lambda|i|k\gamma - |k|^\alpha\omega^B(k;\alpha,\beta), \tag{3.6.3}$$

where

$$\omega^B(k;\alpha,\beta) = \begin{cases} \exp[-i\beta\Phi(\alpha)\operatorname{sign}k], & \alpha \neq 1, \\ \pi/2 + i\beta\ln|k|\operatorname{sign}k, & \alpha = 1, \end{cases}$$

and the parameters are in the same domain as in form A (3.6.2). In this case, only the characteristic exponents coincide:

$$\alpha^A = \alpha^B.$$

The connection between other parameters in representations A and B and the form of the function $\Phi(\alpha)$ can be determined by comparing (3.6.1) and (3.6.3). We equate the real and imaginary parts of the second characteristics of those two forms:

$$\Re\psi^A = \Re\psi^B, \qquad \Im\psi^A = \Im\psi^B;$$

we thus obtain

$$-\lambda^A|k|^\alpha = -\lambda^B|k|^\alpha\cos(\beta^B\Phi(\alpha))$$

and

$$\lambda^A\gamma^A k + \lambda^A|k|^\alpha\beta^A\tan(\alpha\pi/2)\operatorname{sign}k = \lambda^B\gamma^B k + \lambda^B|k|^\alpha\beta^A\sin(\beta^B\Phi(\alpha))\operatorname{sign}k,$$

which yields (for $\alpha \neq 1$):

$$\lambda^A = \lambda^B\cos(\beta^B\Phi(\alpha)), \tag{3.6.4}$$

$$\gamma^A = \gamma^B[\cos(\beta^B\Phi(\alpha))]^{-1}, \tag{3.6.5}$$

$$\beta^A = \tan(\beta^B\Phi(\alpha))/\tan(\alpha\pi/2). \tag{3.6.6}$$

The constant λ^B, as well as λ^A, should be positive; hence

$$-\pi/2 < \beta^B \Phi(\alpha) < \pi/2.$$

Let us find $\Phi(\alpha)$ such that $\beta^B = 1$ if $\beta^A = 1$ irrespective of α. The following two equations must be satisfied:

$$-\pi/2 < \Phi(\alpha) < \pi/2,$$
$$\tan \Phi(\alpha) = \tan(\alpha\pi/2).$$

For $\alpha < 1$, $0 < \alpha\pi/2 < \pi/2$,

$$\Phi(\alpha) = \alpha\pi/2, \qquad \alpha < 1. \tag{3.6.7}$$

For $\alpha > 1$ we use the trigonometric transformation

$$\tan(\alpha\pi/2) = \tan(\pi + (\alpha/2 - 1)\pi) = \tan((\alpha - 2)\pi/2).$$

Now the argument of the tangent obeys the condition

$$-\pi/2 < (\alpha - 2)\pi/2 < 0,$$

and we obtain

$$\Phi(\alpha) = (\alpha - 2)\pi/2, \qquad \alpha > 1. \tag{3.6.8}$$

Uniting (3.6.7) and (3.6.8), we obtain

$$\Phi(\alpha) = [\alpha - 1 - \operatorname{sign}(\alpha - 1)]\pi/2. \tag{3.6.9}$$

In the case $\alpha = 1$,

$$\beta^A = \beta^B, \quad \gamma^A = (2/\pi)\gamma^B, \quad \lambda^A = (\pi/2)\lambda^B.$$

We put emphasis on the fact that the function has a discontinuity at the point $\alpha = 1$. Nevertheless, we can find a continuous solution of the form

$$\Phi'(\alpha) = [1 - |1 - \alpha|]\pi/2. \tag{3.6.10}$$

But this results in the difference in signs of parameters β for $\alpha > 1$: the positive values of β^A correspond to negative values of β^B, and vice versa. Such representation, up to the factor $\pi/2$, was used in (Lukacs, 1960; Zolotarev, 1966).

In form B, as well as in form A, the stable laws (as functions of parameters) are not continuous at the point $\alpha = 1$. The passage to form M removes this peculiarity:

$$\ln g^M(k; \alpha, \beta, \gamma, \lambda) = \lambda(ik\gamma - |k|^\alpha + ik\omega^M(k; \alpha, \beta)), \tag{3.6.11}$$

where

$$\omega^M(k;\alpha,\beta) = \begin{cases} (|k|^{\alpha-1} - 1)\beta\tan(\alpha\pi/2), & \alpha \neq 1, \\ -\beta(2/\pi)\ln|k|, & \alpha = 1. \end{cases}$$

The domain of parameter variations is the same as in the two preceding forms, and they relate to the parameters of form A as

$$\alpha^A = \alpha^M, \quad \beta^A = \beta^M, \quad \gamma^A = \gamma^M - \beta^M\tan(\alpha\pi/2), \quad \lambda^A = \lambda^M.$$

The class of strictly stable laws is determined by the equality

$$\ln g(b_1k) + \ln g(b_2k) = \ln g(bk)$$

following from (3.4.23). In this case, the family of c.f.'s becomes three-parameter, and its second characteristics can be presented in form C:

$$\ln g^C(k;\alpha,\delta,\lambda) = -\lambda|k|^\alpha \exp(-i\alpha\delta(\pi/2)\operatorname{sign} k), \tag{3.6.12}$$

where

$$0 < \alpha \leq 2, \quad |\delta| \leq \delta_\alpha = \min\{\alpha, 2-\alpha\}, \quad \lambda > 0.$$

Sometimes, it turns out to be more convenient to use the parameter

$$\rho = \frac{\alpha+\delta}{2}.$$

In this parametric system, the case $\alpha = 1$ plays a special part, because the values of parameters $\alpha = 1$, δ, λ for $|\delta| = \alpha$ correspond to the degenerate distribution. If this case is excluded from consideration, then for parameters in forms B and C the equalities

$$\begin{aligned} & & \alpha^C &= \alpha^B, & & \\ \delta &= 2\beta^B\Phi(\alpha)/\pi, & \lambda^C &= \lambda^B, & & \alpha \neq 1, \\ \delta &= (2\alpha/\pi)\arctan(2\gamma^B/\pi), & \lambda^C &= \lambda^B(\pi^2/4 + \gamma^{B2})^{1/2}, & & \alpha = 1 \end{aligned}$$

hold. An essential difference between the classes of stable and strictly stable laws arises as soon as $\alpha = 1$. In this case, only symmetric ($\beta = 0$) distribution (the Cauchy distribution) appears to be strictly stable, whereas the asymmetric stable distributions ($\beta \neq 0$) do not belong to this class.

Finally, we give here one more form of c.f. for strictly stable law, namely form E:

$$\ln g^E(k;\nu,\theta,\tau) = -\exp\{\nu^{-1/2}(\ln|k| + \tau - i\theta(\pi/2)\operatorname{sign} k) + C(\nu^{-1/2} - 1)\}, \tag{3.6.13}$$

where $C = 0.577\ldots$ is the Euler constant, and the parameters vary within the bounds

$$\nu \geq 1/4, \quad |\theta| \leq \min\{1, 2\sqrt{\nu} - 1\}, \quad |\tau| < \infty$$

and are related to the parameters in form C as

$$\nu = \alpha^{-2}, \quad \theta = \delta/\alpha, \quad \tau = \alpha^{-1}\ln\lambda^C + C(\alpha^{-1} - 1).$$

The c.f.'s and the distributions corresponding to them with particular values of scale and shift parameters are called reduced, and denoted by

$$\begin{aligned} g^{A,B,M}(k;\alpha,\beta) &= g^{A,B,M}(k;\alpha,\beta,0,1);\\ g^C(k;\alpha,\delta) &= g_C(k;\alpha,\delta,1), \quad \delta = 0 \text{ if } \alpha = 1,\\ g^E(k;\nu,\theta) &= g^E(k;\nu,\theta,C(\sqrt{\nu}-1)), \quad \theta = 0 \text{ if } \nu = 1. \end{aligned}$$

The c.f. $g^B(k;\alpha,\beta)$ and the corresponding distribution are considered standard, and the subscript B is omitted:

$$g(k;\alpha,\beta) = g^B(k;\alpha,\beta).$$

For the sake of convenience we give explicit expressions of the second characteristics $\psi(k;\dots) \equiv \ln g(k,\dots)$:

$$\begin{aligned} \psi^{A,M}(k;\alpha,\beta) &= -|k|^\alpha + ik\omega^{A,M}(k;\alpha,\beta),\\ \psi^B(k;\alpha,\beta) &= -|k|^\alpha\omega^B(k;\alpha,\beta),\\ \psi^C(k;\alpha,\delta) &= -|k|^\alpha \exp\{-i\delta(\pi/2)\operatorname{sign} k\},\\ \psi^E(k;\nu,\theta) &= -\exp\{\nu^{-1/2}(\ln|k| - i\theta(\pi/2)\operatorname{sign} k)\}. \end{aligned}$$

Some particular cases of the expressions are given in Table 3.1.

In each of the forms of parametrization, there is an indicated domain of variation of the parameters corresponding to this form, which we call the domain of admissible values of the parameters.

Of course, one may puzzle over the question: why such an abundance of different forms for expressing the characteristic functions of stable laws exists? While studying the analytic properties of the distributions of stable laws, we encounter groups of properties with their own diverse features. The expression of analytic relations connected with stable distributions can be simpler or more complicated depending on how felicitous the choice of the parameters determining the distributions for our problem turns out to be. By associating with a particular group of properties the parametrization form most natural for it, we thereby minimize the complexity involved in expressing these properties. In this approach, the extraneous complexity is, as it was, isolated from the problem and relegated to the formulae for passing from one form of expressing the characteristic functions g to another.

Table 3.1. Reduced second characteristics of the Lévy, Cauchy and Gauss laws in different forms

Form	Lévy	Cauchy	Gauss
A	$\psi^A(k;1/2,1) = -\sqrt{\lvert k\rvert}(1 - i \operatorname{sign} k)$	$\psi^A(k;1,0) = -\lvert k\rvert$	$\psi^A(k;2,\beta) = -k^2$
B	$\psi^B(k;1/2,1) = -\sqrt{\lvert k\rvert/2}(1 - i \operatorname{sign} k)$	$\psi^B(k;1,0) = -\lvert\pi k/2\rvert$	$\psi^B(k;2,\beta) = -k^2$
C	$\psi^C(k;1/2,1/2) = -\sqrt{\lvert k\rvert/2}(1 - i \operatorname{sign} k)$	$\psi^C(k;1,0) = -\lvert k\rvert$	$\psi^C(k;2,\delta) = -k^2$
M	$\psi^M(k;1/2,1) = -ik - \sqrt{\lvert k\rvert}(1 - \operatorname{sign} k)$	$\psi^M(k;1,0) = -\lvert k\rvert$	$\psi^M(k;2,\beta) = -k^2$
E	$\psi^E(k;4,1) = -\sqrt{\lvert k\rvert/2}(1 - \operatorname{sign} k)$	$\psi^E(k;1,0) = -\lvert k\rvert$	$\psi^E(k;1/4,0) = -k^2$

Table 3.2. Densities of the reduced Lévy, Cauchy and Gauss laws in different forms

Form	Lévy	Cauchy	Gauss
A	$q^A(x;1/2,1) = (2\pi x^3)^{-1/2}\exp\{-(2x)^{-1}\}$	$q^A(x;1,0) = [\pi(1+x^2)]^{-1}$	$q^A(x;2,0) = (4\pi)^{-1/2}\exp\{-x^2/4\}$
B	$q^B(x;1/2,1) = (4\pi x^3)^{-1/2}\exp\{-(4x)^{-1}\}$	$q^B(x;1,0) = [2(\pi^2/4+x^2)]^{-1}$	$q^B(x;2,0) = (4\pi)^{-1/2}\exp\{-x^2/4\}$
C	$q^C(x;1/2,1/2) = (4\pi x^3)^{-1/2}\exp\{-(4x)^{-1}\}$	$q^C(x;1,0) = [\pi(1+x^2)]^{-1}$	$q^C(x;2,0) = (4\pi)^{-1/2}\exp\{-x^2/4\}$
M	$q^M(x;1/2,1) = (2\pi(x+1)^3)^{-1/2}\exp\{-(2x+2)^{-1}\}$	$q^M(x;1,0) = [\pi(1+x^2)]^{-1}$	$q^M(x;2,0) = (4\pi)^{-1/2}\exp\{-x^2/4\}$
E	$q^E(x;4,1) = (4\pi x^3)^{-1/2}\exp\{-(4x)^{-1}\}$	$q^E(x;1,0) = [\pi(1+x^2)]^{-1}$	$q^E(x;1/4,0) = (4\pi)^{-1/2}\exp\{-x^2/4\}$

3.7. Some properties of stable random variables

The explicit expressions for the characteristic functions of stable laws obtained above allow us to reveal a series of interrelations between them. In many cases, it is convenient and rather intuitive to treat relations between stable distributions as relations between random variables having these distributions. The reverse passage is quite simple and does not require additional explanations.

We denote by $Y_D(\Gamma)$ the random variable having the c.f. $g^D(k;\Gamma)$, where D is one of the symbols A, B, M, C, E, and let Γ stand for the set of parameters corresponding to the form chosen, so the r.v. $Y_A(\alpha,\beta,\gamma,\lambda)$ has the c.f. $g^A(k;\alpha,\beta,\gamma,\lambda)$, and so on. For the reduced forms, we use the shortened sets of parameters $Y_A(\alpha,\beta) \equiv Y_A(\alpha,\beta,0,1)$, etc.

Using this notation, we can obtain relations between stable r.v.'s presented in different forms. In view of property (c) (Section 3.1), the second characteristics of the r.v. $a + bX$ is related to the corresponding characteristics of X as

$$\psi_{a+bX}(k) = iak + \psi_X(bk).$$

Applying this relation to the r.v.

$$X = a + bY_A(\alpha,\beta),$$

we obtain ($b > 0$)

$$\psi_X(k) = -|bk|^\alpha + i\{ak + \beta^A|bk|^\alpha \tan(\alpha\pi/2)\,\mathrm{sign}\,k\}, \qquad \alpha \neq 1,$$
$$\psi_X(k) = -|bk|^\alpha + i\{ak - (2/\pi)\beta^A bk[\ln|k| + \ln b]\}, \qquad \alpha = 1.$$

The second characteristics of $Y_B(\alpha,\beta)$ is

$$\ln g^B(k;\alpha,\beta) = -|k|^\alpha \cos[\beta^B\Phi(\alpha)] + i|k|^\alpha \sin[\beta^B\Phi(\alpha)]\,\mathrm{sign}\,k, \qquad \alpha \neq 1,$$
$$\ln g^B(k;1,\beta) = -(\pi/2)|k| - i\beta^B k \ln|k|.$$

Equating the real and imaginary parts of ψ_X and $\ln g^B$, we obtain

$$a = 0, \qquad b = [\cos(\beta^B\Phi(\alpha))]^{1/\alpha}, \quad \alpha \neq 1,$$
$$a = \beta^B \ln(\pi/2), \quad b = \pi/2, \qquad \alpha = 1.$$

Thus,

$$Y_B(\alpha,\beta) = [\cos(\beta\Phi(\alpha))]^{1/\alpha} Y_A(\alpha,\beta^A), \qquad \alpha \neq 1,$$
$$Y_B(1,\beta) = \beta\ln(\pi/2) + (\pi/2)Y_A(1,\beta).$$

We note that both forms of the symmetric r.v.'s $Y(\alpha, 0)$ coincide (for $\alpha \neq 1$).

Similar reasoning yields

$$
\begin{aligned}
Y_M(\alpha,\beta) &= Y_A(\alpha,\beta) - \beta\tan(\alpha\pi/2), && \alpha \neq 1,\\
Y_M(1,\beta) &= Y_A(1,\beta);\\
Y_C(\alpha,\delta) &= [\cos(\delta\pi/2)]^{1/\alpha} Y_A(\alpha,\beta), && \alpha \neq 1,\\
Y_C(1,0) &= Y_A(1,0);\\
Y_E(\nu,\theta) &= [\cos(\nu^{-1/2}\theta\pi/2)]^{\nu^{1/2}} Y_A(\alpha,\beta), && \nu \neq 1,\\
Y_E(1,0) &= Y_A(1,0).
\end{aligned}
$$

Now we present some properties which are true for both forms A and B, so we omit the subscripts.

(1) Any two admissible parameter quadruples $(\alpha,\beta,\gamma,\lambda)$ and $(\alpha',\beta',\gamma',\lambda')$ uniquely determine real $a > 0$ and b such that

$$Y(\alpha,\beta,\gamma,\lambda) \stackrel{d}{=} aY(\alpha,\beta,\gamma',\lambda') + \lambda b. \tag{3.7.1}$$

In form A, a and b relate on the parameters as

$$
\begin{aligned}
a &= (\lambda/\lambda')^{1/\alpha},\\
b &= \begin{cases} \gamma - \gamma'(\lambda/\lambda')^{1/\alpha}, & \alpha \neq 1,\\ \gamma - \gamma' + (2/\pi)\beta\ln(\lambda/\lambda'), & \alpha = 1. \end{cases}
\end{aligned}
$$

There exists an important particular case of (3.7.1). Let $\gamma' = 0$ and $\lambda' = 1$; then

$$Y(\alpha,\beta,\gamma,\lambda) \stackrel{d}{=} \begin{cases} \lambda^{1/\alpha} Y(\alpha,\beta) + \lambda\gamma, & \alpha \neq 1,\\ \lambda^{1/\alpha} Y(\alpha,\beta) + \lambda[\gamma + (2/\pi)\beta\ln\lambda], & \alpha = 1. \end{cases} \tag{3.7.2}$$

Equality (3.7.2) shows that λ stands for the scale parameter, while γ corresponds to the translation parameter (rigorously speaking, the pure shift of the distribution is a linear function of γ).

(2) For any admissible parameter quadruple $(\alpha,\beta,\gamma,\lambda)$,

$$Y(\alpha,-\beta,-\gamma,\lambda) \stackrel{d}{=} -Y(\alpha,\beta,\gamma,\lambda). \tag{3.7.3}$$

The useful content of this property is, in particular, that it allows us, without loss of generality, to consider the distribution functions $G(x;\alpha,\beta,\gamma,\lambda)$ with the only (according to our choice) additional condition that the signs of the argument x, the parameter β, or the parameter γ are preserved.

(3) Any admissible parameter quadruples $(\alpha, \beta_k, \gamma_k, \lambda_k)$ and any real h, c_k, $k = 1, \dots, m$, uniquely determine a parameter quadruple $(\alpha, \beta, \gamma, \lambda)$ such that[2]

$$Y(\alpha, \beta, \gamma, \lambda) \stackrel{d}{=} c_1 Y(\alpha, \beta_1, \gamma_1, \lambda_1) + \dots + c_m Y(\alpha, \beta_m, \gamma_m, \lambda_m) + h. \quad (3.7.4)$$

In form A, the quadruple $(\alpha, \beta, \gamma, \lambda)$ depends on the parameters and numbers chosen as follows:

$$\lambda = \sum_{n=1}^{m} \lambda_n |c_n|^\alpha,$$
$$\lambda\beta = \sum_{n=1}^{m} \lambda_n \beta_n |c_n|^\alpha \operatorname{sign} c_n,$$
$$\lambda\gamma = \sum_{n=1}^{m} \lambda_n \gamma_n c_n + h',$$
$$h' = \begin{cases} h, & \alpha \neq 1, \\ h - (2/\pi) \sum_{n=1}^{m} \lambda_n \beta_n c_n \ln |c_n|, & \alpha = 1. \end{cases}$$

We mention some special cases that are of independent interest.

(a) An arbitrary admissible parameter quadruple $(\alpha, \beta, \gamma, \lambda)$ and any $\beta_1 \leq \beta$ and $\beta_2 \geq \beta$ uniquely determine positive c_1, c_2 and real h such that

$$Y(\alpha, \beta, \gamma, \lambda) \stackrel{d}{=} c_Y(\alpha, \beta_1) + c_2 Y(\alpha, \beta_2) + h. \quad (3.7.5)$$

In form A, the parameters are inter-related as follows:

$$c_1 = [\lambda(\beta_2 - \beta)/(\beta_2 - \beta_1)]^{1/\alpha},$$
$$c_2 = [\lambda(\beta - \beta_1)/(\beta_2 - \beta_1)]^{1/\alpha},$$
$$h = \begin{cases} \lambda\gamma, & \alpha \neq 1, \\ \lambda\gamma - (2/\pi)(\beta_1 c_1 \ln c_1 + \beta_2 c_2 \ln c_2), & \alpha = 1. \end{cases}$$

Choosing $\beta_1 = -1$, $\beta_2 = 1$, and using the equality

$$Y(\alpha, -1) = -Y(\alpha, 1),$$

we obtain

$$Y(\alpha, \beta, \gamma, \lambda) \stackrel{d}{=} (\lambda/2)^{1/\alpha} \left[(1 + \beta)^{1/\alpha} Y(\alpha, 1) - (1 - \beta)^{1/\alpha} Y(\alpha, 1) \right] + h. \quad (3.7.6)$$

[2]Here and in what follows we use rule (3.3.7).

We hence conclude that any r.v. $Y(\alpha, \beta, \gamma, \lambda)$ can be expressed as a linear combination of two independent r.v.s $Y(\alpha, 1)$ (in the sense of the equality $\stackrel{d}{=}$). For standardized strictly stable r.v.'s, (3.7.6) takes the form

$$Y(\alpha, \beta) \stackrel{d}{=} (1/2)^{1/\alpha} \left\{(1+\beta)^{1/\alpha} Y(\alpha, 1) - (1-\beta)^{1/\alpha} Y(\alpha, 1)\right\}. \tag{3.7.7}$$

(b) For any admissible parameter quadruple $(\alpha, \beta, \gamma, \lambda)$,

$$Y(\alpha, \beta, \gamma, \lambda) - Y(\alpha, \beta, \gamma, \lambda) \stackrel{d}{=} Y(\alpha, 0, 0, 2\lambda). \tag{3.7.8}$$

(c) Any admissible parameter quadruple $(\alpha, \beta, \gamma, \lambda)$ uniquely determines an admissible parameter quadruple $(\alpha, \beta^*, \gamma^*, \lambda)$ such that

$$Y(\alpha, \beta, \gamma, \lambda) - (1/2)Y(\alpha, \beta, \gamma, \lambda) - (1/2)Y(\alpha, \beta, \gamma, \lambda) \stackrel{d}{=} Y(\alpha, \beta^*, \gamma^*, \lambda^*). \tag{3.7.9}$$

Thus,

$$\beta^* = [(1 - 2^{1-\alpha})/(1 + 2^{1-\alpha})]\beta,$$
$$\gamma^* = \begin{cases} 0, & \alpha \neq 1, \\ -(\beta/\pi)\ln 2, & \alpha = 1, \end{cases}$$
$$\lambda^* = (1 + 2^{1-\alpha})\lambda.$$

It is not hard to see that the r.v.'s on the right-hand sides of equalities (3.7.8)–(3.7.9) possess strictly stable distributions. This feature of the transformations of independent r.v.'s with arbitrary stable distribution on the left-hand sides of equalities appears to be very useful in the problem of statistical estimation of the parameters of stable laws.

All the properties mentioned above are deduced from explicit expressions for the c.f.'s of stable r.v.'s As an example, let us prove Property 3 in the case where $\alpha \neq 1$. In terms of c.f.'s, relation (3.7.4) takes the form

$$g(k; \alpha, \beta, \gamma, \lambda) = e^{ikh} \prod_{n=1}^{m} g(c_n k; \alpha, \beta_n, \gamma_n, \lambda_n). \tag{3.7.10}$$

Using the explicit expression for c.f. (3.5.23), we equate the logarithms of both parts of equality (3.7.10), and obtain

$$\lambda(ik\gamma - |k|^\alpha + ik|k|^{\alpha-1}\beta \tan(\alpha\pi/2))$$
$$= ikh + \sum_{n=1}^{m} \lambda_n (ik\gamma_n c_n - c_n^\alpha |k|^\alpha + ik|k|^{\alpha-1} c_k^\alpha \beta_k \tan(\alpha\pi/2)).$$

A comparison of the coefficients of the functions ik, $|k|^\alpha$, and $ik|k|^{\alpha-1}$ gives the relations determining the parameters β, γ, λ. It is clear that γ and λ take admissible values. Writing out the parameter β in the final form

$$\beta = \sum_{n=1}^{m} \lambda_n |c_n|^\alpha \beta_n \operatorname{sign} c_n \bigg/ \sum_{n=1}^{m} \lambda_n |c_n|^\alpha,$$

and taking the inequality $|\beta_n| \le 1$ into account, we see that the condition $|\beta| \le 1$ is satisfied indeed.

Additional information about the properties of stable r.v.'s can be found in (Zolotarev, 1986).

3.8. Conclusion

There is one particular error which is repeated in fairly many papers (connected in one way or another with stable laws). Namely, to describe the family of stable laws they use form A for $g(k)$ with the sign in front of $it\omega^A(k;\alpha,\beta)$ chosen to be 'minus' in the case $\alpha \neq 1$. Along with this, it is commonly assumed that the value $\beta = 1$ corresponds to the stable laws appearing as limit distributions of the normalized sums Z_n with positive terms. But this assumption contradicts the choice of 'minus' in front of ω^A.

The error evidently become widespread because it found its way into the well-known monograph (Gnedenko & Kolmogorov, 1954). Hall (Hall, 1981) devoted a special note to a discussion of this error, calling it (with what seems unnecessary pretentiousness) a 'comedy of errors'.

Though he is undoubtedly right on the whole, in our view he exaggerates unnecessarily, presenting the matter as if the mistake he observed is almost universal. In reality, this defect in (Gnedenko & Kolmogorov, 1954) was noticed long ago. For example, special remarks on this were made in (Zolotarev, 1961a; Skorokhod, 1954). And in general, there are more than a few papers and books whose authors were sufficiently attentive and did not lapse into this 'sin'. For example, we can mention (Linnik, 1954; Feller, 1966).

4

Probability densities

4.1. Symmetric distributions

The characteristic functions uniquely determine the corresponding densities of stable laws; nevertheless, it is hard to calculate the densities by the direct application of the inversion theorem, because we have to operate with improper integrals of oscillating functions. Therefore, all numerical calculations of densities are based on other representations emerging from somewhat transformed inversion formula. In actual practice, three basic representations are used: convergent series, asymptotic series, and integrals of non-oscillating functions.

We begin with expansions of symmetric distributions. For strictly stable laws (form C),

$$
\begin{aligned}
q(x;\alpha,\delta) &= (2\pi)^{-1}\int_{-\infty}^{\infty} e^{-ikx} g(k;\alpha,\delta)\,dk \\
&= \pi^{-1}\Re\int_0^{\infty} \exp\{-ikx\}\exp\{-k^\alpha e^{-i\delta\pi/2}\}\,dk. \qquad (4.1.1)
\end{aligned}
$$

Substituting $\delta = 0$, we obtain

$$
q(x;\alpha,0) = \pi^{-1}\int_0^{\infty} e^{-k^\alpha}\cos kx\,dk. \qquad (4.1.2)
$$

We expand the cosine entering the integrand into a series; after the change

$t = k^\alpha$ we obtain

$$q(x;\alpha,0) = \pi^{-1}\int_0^\infty \sum_{m=0}^\infty \frac{(-1)^m}{(2m)!}(kx)^{2m}e^{-k^\alpha}dk$$
$$= (\pi\alpha)^{-1}\sum_{m=0}^\infty \frac{(-1)^m}{(2m)!}x^{2m}\int_0^\infty e^{-t}t^{(2m+1)/\alpha-1}\,dt$$
$$= (\pi\alpha)^{-1}\sum_{m=0}^\infty \frac{(-1)^m}{(2m)!}\Gamma\left(\frac{2m+1}{\alpha}\right)x^{2m}. \qquad (4.1.3)$$

Recalling the well-known convergence criteria, we easily see that this series converges for $\alpha \ge 1$. In particular, for $\alpha = 2$

$$q(x;2,0) = (2\pi)^{-1}\sum_{m=0}^\infty \frac{(-1)^m}{(2m)!}\Gamma(m+1/2)x^{2m} = (2\sqrt{\pi})^{-1}\sum_{m=0}^\infty \frac{(2m-1)!!}{(2m)!}(-x^2/2)^m$$
$$= (2\sqrt{\pi})^{-1}\sum_{m=0}^\infty \frac{1}{m!}(-x^2/4)^m = (2\sqrt{\pi})^{-1}\exp\{-x^2/4\},$$

and we arrive at the normal distribution, whereas for $\alpha = 1$

$$q(x;1,0) = \pi^{-1}\sum_{m=0}^\infty (-x^2)^m = \frac{1}{\pi(1+x^2)}$$

the series stands for the Cauchy distribution.

In order to get a convergent series for $\alpha < 1$, we have to transform the integration path; we will dwell upon this in following section, while here we make use of formula 82 of (Oberhettinger, 1973)

$$\int_0^\infty k^\mu \exp(-ak^c)\cos(kx)dk$$
$$= -\sum_{n=0}^\infty \{(-a)^n(n!)^{-1}\Gamma(\mu+1+nc)\sin[\pi(\mu+nc)/2]\}x^{-\mu-1-nc},$$

which holds for $\mu > -1$ and $0 < c \le 1$. Setting $\mu = 0$, $a = 1$, and $c = \alpha$, we arrive at the series

$$q(x;\alpha,0) = \pi^{-1}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n!}\Gamma(n\alpha+1)\sin(n\alpha\pi/2)x^{-n\alpha-1}, \qquad (4.1.4)$$

which converges for $\alpha < 1$.

By integrating (4.1.3) from 0 to x and (4.1.4) from x to ∞, we immediately obtain the corresponding expansions of the distribution functions $G(x;\alpha,0)$. Since, in view of symmetry,

$$G(0;\alpha,0) = 1/2,$$

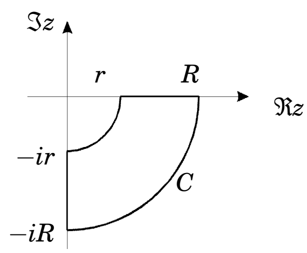


Figure 4.1. Integration contour C for $\alpha < 1$

we obtain

$$G(x;\alpha,0) = \begin{cases} 1 - \pi^{-1}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\Gamma(n\alpha)\sin(n\alpha\pi/2)x^{-n\alpha}, & \alpha < 1, \\ 1/2 + (\pi\alpha)^{-1}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}\Gamma\left(\frac{2m+1}{\alpha}\right)x^{2m+1}, & \alpha \geq 1. \end{cases} \quad (4.1.5)$$

4.2. Convergent series for asymmetric distributions

We turn back to the universal formula (4.1.1). Because

$$q(x;\alpha,\delta) = q(-x;\alpha,-\delta), \quad (4.2.1)$$

it suffices to evaluate the integral

$$q(x;\alpha,\delta) = \pi^{-1}\Re\int_0^\infty \exp\{-ikx\}\exp\{-k^\alpha e^{-i\delta\pi/2}\}\,dk \quad (4.2.2)$$

for all δ and and x belonging to either positive or negative semiaxis.

Let $\alpha < 1$, $-\alpha < \delta \leq \alpha$, and $x > 0$. By virtue of the Cauchy theorem, the integral

$$\oint_C \exp\{-izx - z^\alpha e^{-i\delta\pi/2}\}\,dz$$

along the closed contour given in Fig. 4.1 vanishes. Setting $r \to 0$ and $R \to \infty$, by making use of the Jordan lemma we obtain

$$\int_0^\infty \exp\{-ikx - k^\alpha e^{-i\delta\pi/2}\}\,dk = -i\int_0^\infty \exp\{-xk - k^\alpha e^{-i\rho\pi}\}\,dk, \quad (4.2.3)$$

where $\rho = (\delta + \alpha)/2$. Changing xk for t in the right-hand side and substituting the result into (4.2.2), we obtain

$$q(x;\alpha,\delta) = \frac{1}{\pi x}\Re\left\{-i\int_0^\infty \exp\{-t - (t/x)^\alpha e^{-i\rho\pi}\}\,dt\right\}.$$

Expanding $\exp[-(t/x)^\alpha e^{-i(\rho\pi)}]$ into a series and interchanging integration and summation signs, we arrive at the representation of a stable density (for $\alpha < 1$ and any $x > 0$) in the form of a convergent series

$$q(x; \alpha, \delta) = \pi^{-1} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\Gamma(\alpha n + 1)}{n!} \sin(n\rho\pi)\, x^{-\alpha n - 1}. \tag{4.2.4}$$

Using the known relation

$$\sin(\pi x) = \frac{\pi}{\Gamma(x)\Gamma(1-x)}, \tag{4.2.5}$$

we are able to rewrite (4.2.4) in the equivalent form containing only gamma functions

$$q(x; \alpha, \delta) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \frac{\Gamma(\alpha n + 1)}{\Gamma(\rho n)\Gamma(1 - \rho n)} x^{-\alpha n - 1}, \qquad \rho = (\delta + \alpha)/2. \tag{4.2.6}$$

We recall that in the domain $\alpha < 1$ the asymmetry parameter δ ranges from $-\alpha$ to α, which corresponds to β^A varying between -1 and 1. We set $\delta = \alpha$ and use formula (4.2.1) for negative x:

$$q(x; \alpha, \alpha) = q(|x|; \alpha, -\alpha).$$

Substituting series (4.2.4) into the right-hand side of this equality with $\rho = 0$, we obtain

$$q(x; \alpha, \alpha) = 0, \quad x < 0, \quad \alpha < 1.$$

Therefore, a stable distribution with parameters $\alpha < 1$ and $\delta = \alpha$ ($\beta^A = \beta^B = 1$) is concentrated in the positive semi-axis. Similarly, a distribution with parameters $\alpha < 1$ and $\delta = -\alpha$ ($\beta^A = \beta^B = -1$) is concentrated in the negative semiaxis.

Let us consider the special case of a distribution concentrated in the positive semiaxis with characteristic parameter $\alpha = 1/2$ ($\delta = \rho = 1/2$). Because the factor

$$\frac{1}{\Gamma(1 - \rho n)} = \frac{1}{\Gamma(1 - n/2)}$$

vanishes at all even natural n, sum (4.2.6) contains only summands with odd indices $n = 2m + 1$, $m = 0, 1, 2\dots$ Passing to summation over m, we obtain

$$\begin{aligned} q(x; 1/2, 1/2) &= \sum_{m=0}^{\infty} \frac{\Gamma(m + 3/2)}{(2m+1)!\,\Gamma(m + 1/2)\Gamma(1/2 - m)} x^{-m-3/2} \\ &= \frac{1}{2} \sum_{m=0}^{\infty} \frac{1}{(2m)!\,\Gamma(1/2 - m)} x^{-m-3/2}. \end{aligned} \tag{4.2.7}$$

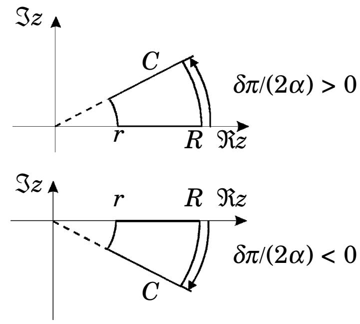


Figure 4.2. Integration contour C for $\alpha > 1$

Taking advantage of the well-known relations

$$(2m)! = \Gamma(2(m + 1/2)) = \frac{2^{2m}}{\sqrt{\pi}}\Gamma(m + 1/2)\Gamma(m + 1)$$

and

$$\Gamma(1/2 - m)\Gamma(1/2 + m) = \frac{\pi}{\cos \pi m} = \frac{\pi}{(-1)^m},$$

we rewrite (4.2.7) as

$$\begin{aligned} q(x; 1/2, 1/2) &= \frac{x^{-3/2}}{2\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}(4x)^{-m} \\ &= \frac{1}{2\sqrt{\pi}} x^{-3/2} \exp\{-1/(4x)\}, \qquad x \geq 0 \end{aligned}$$

which corresponds to the Lévy law. It might be well to point out that there exists no power expansions of one-sided distributions with respect to the point $x = 0$: derivatives of all orders at this point are equal to zero.

For $\alpha > 1$ and $x > 0$ the form of the contour C depends on the sign of δ: (see Fig. 4.2). In either case the integrals along the arcs tend to zero as $r \to 0$ and $R \to \infty$. By the Cauchy theorem,

$$\int_0^\infty \exp\{-ikx - k^\alpha e^{-i\delta\pi/2}\}dk = e^{i\delta\pi/(2\alpha)} \int_0^\infty \exp\{-ikxe^{i\delta\pi/(2\alpha)} - k^\alpha\}\, dk.$$

Changing k^α for t, we obtain

$$\int_0^\infty \exp\{-ikx - k^\alpha e^{-i\delta\pi/2}\}\, dk = \alpha^{-1} e^{i\delta\pi/(2\alpha)} \int_0^\infty \exp\{-ixt^{1/\alpha} e^{i\delta\pi/(2\alpha)}\} e^{-t} t^{1/\alpha - 1}\, dt.$$

Substituting this expression into (4.2.2), we obtain

$$q(x;\alpha,\delta) = (\alpha\pi)^{-1}\Re\left\{e^{i\delta\pi/(2\alpha)}\int_0^\infty \exp\{-ixt^{1/\alpha}e^{i\delta\pi/(2\alpha)}\}e^{-t}t^{1/\alpha-1}\,dt\right\}; \tag{4.2.8}$$

expanding the first exponential in the integrand into a series, we arrive at the series

$$q(x;\alpha,\delta) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\frac{\Gamma(n/\alpha+1)}{n!}\sin(n\rho\pi/\alpha)x^{n-1}, \qquad x > 0, \tag{4.2.9}$$

which converges for $\alpha > 1$ and all admissible δ. Once more using (4.2.5), we arrive at an analogue of (4.2.6) for $\alpha > 1$

$$q(x;\alpha,\delta) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n!}\frac{\Gamma(n/\alpha+1)}{\Gamma(n\rho/\alpha)\Gamma(1-n\rho/\alpha)}x^{n-1}. \tag{4.2.10}$$

Now the extreme (that is, with $\delta = \pm(2-\alpha)$) distributions are not one-sided but are dispersed along the whole axis from $-\infty$ to ∞.

In the symmetric case $\delta = 0$, $\rho = \alpha/2$, the series (4.2.6) and (4.2.10) take the forms (4.1.4) and (4.1.3) respectively.

The contributions of the leading terms of the expansions for some distributions are given in Figures 4.3–4.6 (the numbers near the curves show how many leading terms of the expansion are used).

Beyond the scope of strictly stable laws, we are able to arrive at the expansion of $q^B(x;1,\beta)$, $\beta \neq 0$ (Zolotarev, 1986):

$$q^B(x;1,\beta) = \pi^{-1}\sum_{n=1}^\infty (-1)^{n-1}nb_n(\beta)x^{n-1}, \tag{4.2.11}$$

where

$$b_n(\beta) = \frac{1}{\Gamma(n+1)}\int_0^\infty \exp\{-\beta t\ln t\}t^{n-1}\sin\left[(1+\beta)t\pi/2\right]dt. \tag{4.2.12}$$

We integrate (4.2.4) along the semiaxis (x,∞) and (4.2.9), (4.2.11) along the interval $(0,x)$, and thus obtain the corresponding expansions for the distribution functions. For $\alpha < 1$ and $x > 0$,

$$1 - G(x;\alpha,\delta) = (\pi\alpha)^{-1}\sum_{n=1}^\infty (-1)^{n-1}\frac{\Gamma(\alpha n+1)}{n!n}\sin(n\rho\pi)x^{-\alpha n}; \tag{4.2.13}$$

for $\alpha > 1$ and $x > 0$,

$$G(x;\alpha,\delta) - G(0;\alpha,\delta) = \pi^{-1}\sum_{n=1}^\infty (-1)^{n-1}\frac{\Gamma(n/\alpha+1)}{n!n}\sin(n\rho\pi/\alpha)x^n; \tag{4.2.14}$$

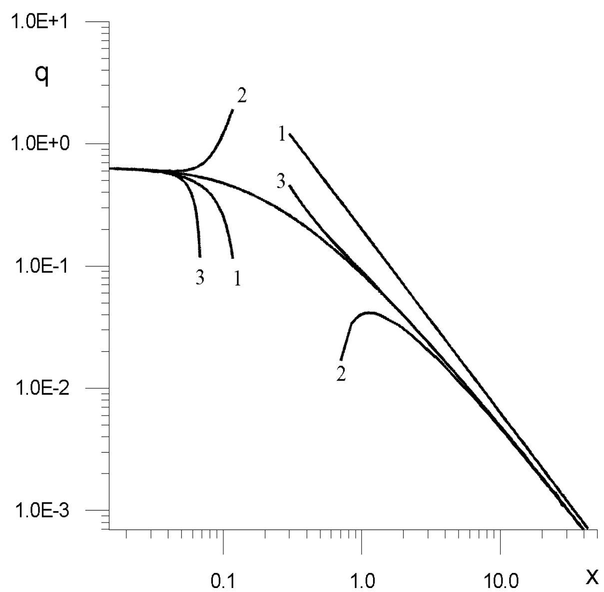


Figure 4.3. Representation of $q^C(x; 1/2, 0)$ by (4.2.6) (for large x) and by (4.3.3) for small x (the numbers near the curves show how many leading terms of the expansion are used)

for $\alpha = 1$, $\beta > 0$,

$$G^B(x; 1, \beta) = 1 - \pi^1 b_0(\beta) + \pi^{-1} \sum_{n=1}^{\infty} (-1)^{n-1} b_n(\beta) x^n, \qquad (4.2.15)$$

where the coefficients $b_n(\beta)$ are determined by formula (4.2.12), too. The value of the distribution function $G(0; \alpha, \beta)$ entering into (4.2.14) will be determined in Section 4.5.

As concerns one-sided stable distributions $q^B(x; \alpha, 1)$, $\alpha < 1$, there exists an expansion into a convergent series in the generalized Laguerre polynomials

$$L_n^{(s)}(x) = \left(\frac{\Gamma(n+1)}{\Gamma(n+1+s)} \right)^{1/2} \sum_{k=0}^{n} (-1)^k \frac{\Gamma(n+1+s) x^k}{\Gamma(k+1)\Gamma(n-k+1)\Gamma(1+s+k)},$$

$s > -1$, $n = 0, 1, \ldots$, that form a complete orthonormalized system in the Hilbert space of real-valued functions on $[0, \infty)$ which are square integrable with respect to the measure $\mu(dx) = x^s \exp(-x)\, dx$. According to (Brockwell & Brown, 1978), for any $\alpha \in (0, 1)$ and $x > 0$, the density $q^B(x; \alpha, 1)$ can be

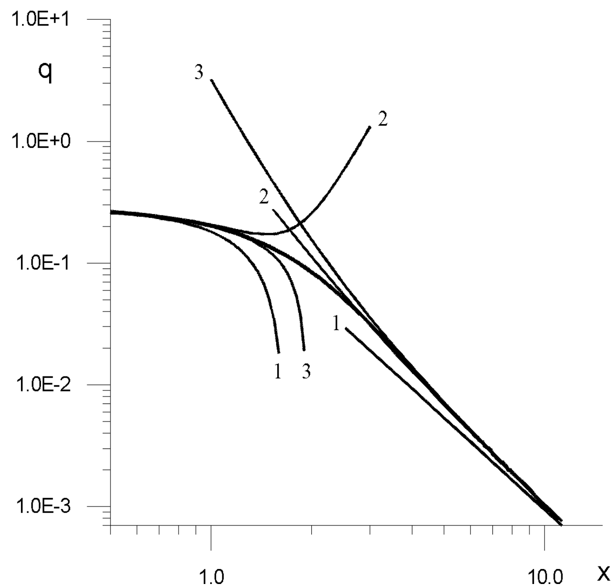


Figure 4.4. Representation of $q^C(x; 3/2, 0)$ by (4.2.10) (for small x) and by (4.3.2) for large x (the numbers near the curves show how many leading terms of the expansion are used)

represented in the form of a convergent series

$$q^B(x; \alpha, 1) = x \exp(-x) \sum_{n=0}^{\infty} k_n^{(s)}(\alpha) L_n^{(s)}(x),$$

where s is an arbitrary fixed number exceeding -1, and

$$k_n^{(s)}(\alpha) = \alpha \left(\frac{\Gamma(n+1)}{\Gamma(n+1+s)} \right)^{1/2} \sum_{m=0}^{n} \frac{(-1)^m \Gamma(1+s+n)}{\Gamma(m+1)\Gamma(n-m+1)\Gamma(1+\alpha(s+m))}.$$

For the sake of brevity, we use various asymmetry parameters. All of them, except $\rho = \alpha/2$, are equal to zero for symmetric laws. The extreme values of the parameters used are given in Table 4.1.

4.3. Long tails

As we have seen in the preceding section, the support of $G(x; \alpha, \delta)$ is the semiaxis $(0, \infty)$ in the case where $\alpha < 1$, $\delta = \alpha$; the semiaxis $(-\infty, 0)$ in the case

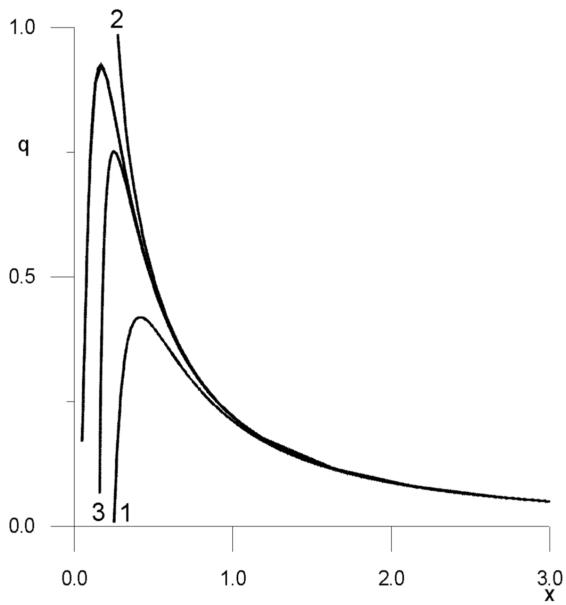
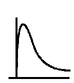
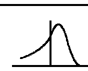
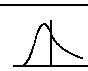


Figure 4.5. Representation of one-sided $q^C(x; 1/2, 1/2)$ by (4.2.6) (the numbers near the curves show how many leading terms of the expansion are used)

Table 4.1. Extreme values of asymmetry parameters

Graphs			
$\alpha < 1$	β^A	-1	-1
	β^B	-1	1
	δ	$-\alpha$	α
	ρ	0	α
	θ	-1	1
Graphs			
$\alpha > 1$	β^A	-1	1
	β^B	-1	1
	δ	$2-\alpha$	$\alpha-2$
	ρ	1	$\alpha-1$
	θ	$2/\alpha-1$	$1-2/\alpha$

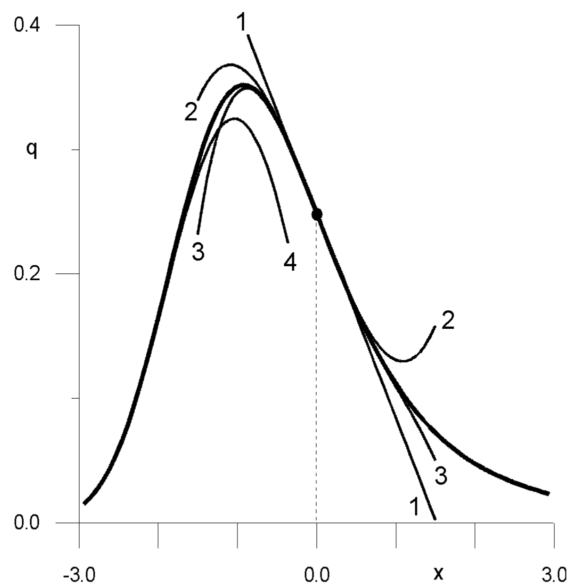


Figure 4.6. Representation of extreme $q^C(x; 3/2, -1/2)$ by (4.2.10) in a neighborhood of $x = 0$; the numbers near the curves show how many leading terms of the expansion are used, label 4 stands for representation by the first leading term of (4.7.12) in the domain of left short tail

where $\alpha < 1$, $\delta = -\alpha$, and the whole real axis otherwise. But this does not allow us to judge the behavior of the functions $G(x; \alpha, \delta)$ and $q(x; \alpha, \delta)$ at $x = 0$ for $\alpha < 1$ and at $x = \infty$ for $1 < \alpha < 2$ (except for the case where $\alpha = 1$, $\delta = 0$), because these points of the complex plane are singular for the functions under consideration: in the first case, it is the branch point, whereas in the second case it is the essential singularity, which follows from the asymptotic expansions of the functions G and q in neighborhoods of the corresponding singularities. Furthermore, the expansions for short tails ($x \to \infty$ for $\alpha > 1$ and $\delta = 2 - \alpha$; $x \to -\infty$ for $\alpha > 1$ and $\delta = \alpha - 2$; $x \to 0$ for $\alpha < 1$ and $\delta = \pm\alpha$) essentially differ from the expansions for long tails. We begin with the analysis of long tails, and postpone the study of short tails to Section 4.7.

Let $\alpha > 1$. We turn back to relation (4.2.8). Bringing the argument of the first exponential in the integrand into the form

$$
\begin{aligned}
-ixt^{1/\alpha}e^{i\delta\pi/(2\alpha)} &= -xt^{1/\alpha}e^{i(\delta+\alpha)\pi/(2\alpha)} \\
&= -xt^{1/\alpha}e^{(i\delta+\alpha-1)\pi/(2\alpha)}e^{i\pi/(2\alpha)},
\end{aligned}
$$

we turn to the integration variable

$$\lambda = xt^{1/\alpha}e^{i\pi/(2\alpha)}$$

thus obtaining

$$q(x;\alpha,\delta) = (\pi x)^{-1}\Re\left\{e^{i(\delta-1)\pi/(2\alpha)}\int_0^\infty \exp\left[-\lambda e^{i(\delta+\alpha-1)\pi/(2\alpha)}\right]\right.$$
$$\left.\times\ \exp\left[-(\lambda/x)^\alpha e^{-i\pi/2}\right]d\lambda\right\}.$$

Applying the Taylor formula to the second exponential of the integrand, we obtain

$$\exp\left[-(\lambda/x)^\alpha e^{-i\pi/2}\right] = \sum_{n=0}^{m}\frac{(-1)^n}{n!}e^{-i\pi n/2}(\lambda/x)^{\alpha n} + \frac{(\lambda/x)^{\alpha(m+1)}}{(m+1)!}\theta, \qquad |\theta| \le 1.$$

Then

$$q(x;\alpha,\delta) = \frac{1}{\pi x}\Re\left\{e^{i(\delta-1)\pi/(2\alpha)}\left[\sum_{n=0}^{m}\frac{(-1)^n}{n!}J_n e^{-i\pi n/2}x^{-\alpha n}\right.\right.$$
$$\left.\left.+\theta\frac{J_{m+1}}{(m+1)!}x^{-\alpha(m+1)}\right]\right\}, \quad (4.3.1)$$

where

$$J_n = \int_0^\infty \lambda^{\alpha n}\exp\{-\lambda e^{i\varphi}\}\,d\lambda,$$
$$\varphi = (\delta+\alpha-1)\pi/(2\alpha) \in [\pi - 3\pi/(2\alpha), \pi/(2\alpha)].$$

To evaluate the integrals J_k, we consider the Cauchy integral in the complex plane

$$\oint_C z^{\alpha n}\exp\{-ze^{i\varphi}\}\,dz = 0$$

along the contour given in Fig. 4.7.

As $r \to 0$ and $R \to \infty$, the integrals along the arcs vanish, and

$$J_n = \exp(-i\varphi\alpha n - i\varphi)\Gamma(\alpha n + 1).$$

Substituting this into (4.3.1) and introducing $\rho = (\delta+\alpha)/2$, we obtain

$$q(x;\alpha,\delta) = (\pi x)^{-1}\sum_{n=1}^{m}\frac{(-1)^{n-1}\Gamma(\alpha n+1)}{n!}\sin(n\rho\pi)x^{-\alpha n} + O(x^{-\alpha(m+1)})$$
$$\sim (\pi x)^{-1}\sum_{n=1}^{\infty}\frac{(-1)^{n}\Gamma(\alpha n+1)}{n!}\sin(n\rho\pi)x^{-\alpha n}, \quad (4.3.2)$$

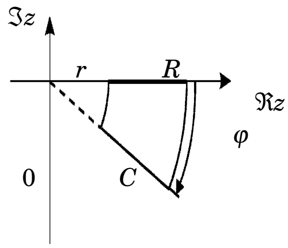


Figure 4.7.

which is the asymptotic expansion of a strictly stable density as $x \to \infty$ for $\alpha > 1$ and $\delta \neq -\alpha$.

For $\alpha < 1$ and $x > 0$ we use formula (4.1.1) applying the Taylor theorem to the first exponential of the integrand, thus obtaining

$$e^{-ikx} = \sum_{n=0}^{m} \frac{(-ix)^n}{n!} k^n + \theta \frac{x^{m+1} k^{m+1}}{(m+1)!}, \qquad |\theta| \leq 1.$$

Setting

$$J_n = \int_0^\infty k^n \exp\{-k^\alpha e^{-i\delta\pi/2}\}\, dk,$$

instead of (4.1.1) we obtain

$$q(x;\alpha,\delta) = \pi^{-1} \Re \left\{ \sum_{n=0}^{m} \frac{(-ix)^n}{n!} J_n + \theta \frac{x^{m+1}}{(m+1)!} J_{m+1} \right\}.$$

It is easy to evaluate the integrals J_n if we use the Cauchy theorem. Taking the line $z = u \exp\{i\delta\pi/(2\alpha)\}$, $0 \leq u < \infty$, as the integration path, we obtain

$$J_n = \alpha^{-1} \Gamma((n+1)/\alpha) \exp\{i(n+1)\delta\pi/(2\alpha)\}.$$

Therefore, for $\alpha < 1$ and $x \to 0$

$$q(x;\alpha,\delta) \sim \pi^{-1} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(n/\alpha + 1)}{n!} \sin(n\rho\pi/\alpha) x^{n-1}, \qquad (4.3.3)$$

where, as before, $\rho = (\delta + \alpha)/2$.

Following the same way, we obtain the corresponding expansions of the distribution functions: for $\alpha < 1$, $x \to 0$,

$$G(x;\alpha,\delta) - G(0;\alpha,\delta) \sim \pi^{-1} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(n/\alpha + 1)}{n!n} \sin(n\rho\pi/\alpha) x^n; \quad (4.3.4)$$

for $\alpha > 1, x \to \infty$,

$$1 - G(x; \alpha, \delta) = (\pi\alpha)^{-1} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(\alpha n + 1)}{n!n} \sin(n\rho\pi) x^{-\alpha n}; \qquad (4.3.5)$$

It is easily seen that series (4.2.9) and (4.2.14) which converge for $\alpha > 1$ turn into asymptotic expansions (4.3.3) and (4.3.4) as $\alpha < 1$, $x \to 0$, whereas the series (4.2.4) and (4.2.13) which converge for $\alpha < 1$, into expansions (4.3.2) and (4.3.5) as $\alpha > 1$, $x \to \infty$. The contributions of the leading terms of the asymptotic expansions are given in Figures 4.3–4.5.

4.4. Integral representation of stable densities

Both convergent and asymptotic expansions are convenient tool of numerical analysis in the cases where the number of terms required to guarantee a reasonable accuracy is not very large. Otherwise one should prefer the integral representation of the density. From the computational viewpoint, the definite integral can be treated as the limit of a sequence of integral sums, i.e., as a series, too; the existence of various schemes of numerical integration offers considerable scope for further improvements.

It is clear that the presence of oscillating integrand plagues the computation. In this section, we transform the inversion formula for stable density into an integral of non-oscillating function.

Let

$$q(x; \alpha, \beta) = q(-x; \alpha, -\beta) = \pi^{-1} \Re \int_0^\infty e^{ikx} g(k; \alpha, -\beta)\, dk. \qquad (4.4.1)$$

Without loss of generality, we assume that $x > 0$ if $\alpha \neq 1$ and $\beta > 0$ if $\alpha = 1$, and make use of form B with

$$\ln g(k; \alpha, -\beta) = \begin{cases} -|k|^\alpha \exp\{i\beta\Phi(\alpha)\operatorname{sign} k\}, & \alpha \neq 1, \\ -|k|(\pi/2 - i\beta \ln|k|), & \alpha = 1, \end{cases}$$

$$\Phi(\alpha) = \begin{cases} \alpha\pi/2, & \alpha < 1, \\ (\alpha - 2)\pi/2, & \alpha > 1. \end{cases}$$

The function $g(k; \alpha, \beta)$ allows the analytic continuation from the positive semiaxis to the complex plane with the cut along the ray $\arg z = -3\pi/4$. We denote this continuation by $g^+(z; \alpha, \beta)$. It is not hard to see that

$$\ln g^+(z; \alpha, \beta) = \begin{cases} -z^\alpha \exp\{-i\beta\Phi(\alpha)\}, & \alpha \neq 1, \\ -z(\pi/2 + i\beta \ \ln\ z), & \alpha = 1, \end{cases}$$

where z^α and $\ln z$ stand for the principal branches of these functions.

We consider the integral

$$J = \int_L e^{ixz} g^+(z; \alpha, -\beta)\, dz \equiv \int_L e^{-W(z,x)}\, dz$$

along the contour L which starts from zero and goes to infinity so that the $W(z, x)$ takes only real values. In view of the abovesaid,

$$W(z,x) = \begin{cases} -ixz + z^\alpha \exp\{i\beta\Phi(\alpha)\}, & \alpha \neq 1, \\ -ixz + z(\pi/2 - i\beta \ \ln\ z), & \alpha = 1. \end{cases}$$

Substituting

$$z = \rho e^{i\varphi} = \rho\cos\varphi + i\rho\sin\varphi$$

and keeping in mind that

$$\begin{aligned} z^\alpha \exp\{i\beta\Phi\} &= \rho^\alpha \exp\{i(\alpha\varphi + \beta\Phi)\} \\ &= \rho^\alpha \cos(\alpha\varphi + \beta\Phi) + i\rho^\alpha \sin(\alpha\varphi + \beta\Phi), \end{aligned}$$

we obtain

$$\begin{aligned} W(z,x) &= \rho^\alpha \cos(\alpha\varphi + \beta\Phi) + x\rho\sin\varphi \\ &\quad - i\left[x\rho\cos\varphi - \rho^\alpha \sin(\alpha\varphi + \beta\Phi)\right], \qquad \alpha \neq 1, \\ W(z,x) &= \rho\left[x\sin\varphi + (\pi/2 + \beta\varphi)\cos\varphi + \beta\sin\varphi\ln\rho\right] \\ &\quad - i\rho\left[x\cos\varphi + \beta\cos\varphi\ln\rho - (\pi/2 + \beta\varphi)\sin\varphi\right], \qquad \alpha = 1. \end{aligned}$$

Setting

$$\Im W(z,x) = \begin{cases} x\rho\cos\varphi - \rho^\alpha \sin(\alpha\varphi + \beta\Phi) = 0, & \alpha \neq 1, \\ x\cos\varphi + \beta\cos\varphi\ln\rho - (\pi/2 + \beta\varphi)\sin\varphi = 0, & \alpha = 1, \end{cases} \tag{4.4.2}$$

we arrive at the equation for the contour L in the polar coordinates:

$$\rho(\varphi) = \left(\frac{\sin(\alpha\varphi + \beta\Phi)}{x\cos\varphi}\right)^{1/(1-\alpha)}, \qquad \alpha \neq 1, \tag{4.4.3}$$

$$\rho(\varphi) = \exp\left\{-x/\beta + \left(\varphi + \pi/(2\beta)\tan\varphi\right)\right\}, \qquad \alpha = 1. \tag{4.4.4}$$

The following lemma allows us to change integration along the real semi-axis in (4.4.1) for integration along the contour L whose points satisfy (4.4.2).

LEMMA 4.4.1. *In the complex plane z with the cut along the ray* $\arg z = -3\pi/4$, *let $\{\Gamma\}$ be a family of contours possessing the following properties each:*

(1) *it begins at $z = 0$;*

(2) *it does not cross the cut;*

(3) *it goes to infinity so that, beginning with some place, the argument of all its points lies in the domain*

$$0 \le \arg z \le \pi - \varepsilon,$$
$$\varepsilon - [\pi/2 - \beta\Phi(\alpha)]/\alpha \le \arg z \le [\pi/2 + \beta\Phi(\alpha)]/\alpha,$$

where $0 < \alpha < 2$, $|\beta| \le 1$, *and* $\varepsilon > 0$ *is as small as desired.*

Then for any contour Γ *and any pair of admissible parameters* α, β *we have*

$$\int_0^\infty e^{izx} g^+(z; \alpha, -\beta)\, dz = \int_\Gamma e^{izx} g^+(z; \alpha, -\beta)\, dz.$$

The proof of this lemma was given in (Zolotarev, 1986), where, in addition, it was established that the contour L determined by (4.4.2) belongs to the family $\{\Gamma\}$. Therefore,

$$q(x; \alpha, \beta) = \pi^{-1} \Re \int_0^\infty e^{-W(z,x)}\, dz = \pi^{-1} \int_L e^{-V(\varphi,x)}\, d(\rho \cos\varphi), \qquad (4.4.5)$$

where

$$V(\varphi, x) = \rho(\varphi) x \sin\varphi + \rho^\alpha(\varphi) \cos(\alpha\varphi + \beta\Phi), \qquad \alpha \ne 1, \quad (4.4.6)$$
$$V(\varphi, x) = \rho(\varphi)[x \sin\varphi + (\pi/2 + \beta\varphi)\cos\varphi + \beta \sin\varphi \ln \rho(\varphi)], \quad \alpha = 1. \quad (4.4.7)$$

We arrive at the final form of $V(\varphi, x)$ after substituting $\rho = \rho(\varphi)$: for $\alpha \ne 1$

$$\begin{aligned} V(\varphi, x) &= \rho^\alpha \frac{\sin(\alpha\varphi + \beta\Phi)\sin\varphi}{\cos\varphi} + \rho^\alpha \cos(\alpha\varphi + \beta\Phi) \\ &= \rho^\alpha \frac{\cos((\alpha - 1)\varphi + \beta\Phi)}{\cos\varphi}; \end{aligned} \qquad (4.4.8)$$

for $\alpha = 1$

$$V(\varphi, x) = \rho(\beta\varphi + \pi/2)(\cos\varphi + \sin^2\varphi / \cos\varphi). \qquad (4.4.9)$$

In order to give the final form of the integral representation, we have to clear up the structure of the differential $d(\rho \cos\varphi)$. We begin with the case $\alpha \ne 1$. By virtue of (4.4.2)

$$x\rho \cos\varphi = \rho^\alpha \sin(\alpha\varphi + \beta\Phi).$$

Therefore,

$$\begin{aligned} x\, d(\rho\cos\varphi) &= \alpha\rho^\alpha \cos(\alpha\varphi + \beta\Phi)\, d\varphi + \alpha\rho^{\alpha-1} \sin(\alpha\varphi + \beta\Phi)\, d\rho \\ &= \alpha\rho^\alpha \cos(\alpha\varphi + \beta\Phi)\, d\varphi + \alpha x \cos\varphi\, d\rho \\ &= \alpha x\, d(\rho\cos\varphi) + \alpha[x\rho\sin\varphi + \rho^\alpha \cos(\alpha\varphi + \beta\Phi)]\, d\varphi. \end{aligned}$$

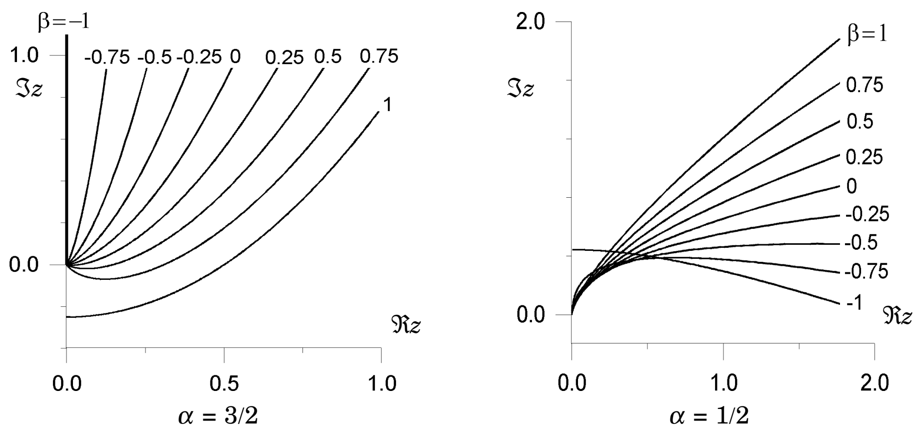


Figure 4.8. The family of contours L for $\alpha = 3/2$, $\alpha = 1/2$, $x = 1$, and various β

The second term is $\alpha V(\varphi, x)\, d\varphi$, thus

$$x\, d(\rho \cos\varphi) = \alpha x\, d(\rho\cos\varphi) + \alpha V(\varphi, x)\, d\varphi,$$

hence

$$d\,(\rho\cos\varphi) = \frac{\alpha}{(1-\alpha)x} V(\varphi, x)\, d\varphi, \qquad \alpha \neq 1. \tag{4.4.10}$$

The integration limits in φ are determined by

$$\rho(\varphi_1) = 0, \qquad \rho(\varphi_2) = \infty.$$

For $\alpha < 1$

$$\varphi_1 = -\beta\pi/2, \qquad \varphi_2 = \pi/2,$$

whereas for $\alpha > 1$

$$\varphi_1 = \pi/2, \qquad \varphi_2 = \beta(\pi/\alpha - \pi/2).$$

The corresponding families of contours are given in Fig. 4.8.

Substituting (4.4.10) into (4.4.5), we obtain (for $x > 0$)

$$q(x; \alpha, \beta) = \frac{\alpha}{\pi(1-\alpha)x} \int_{\varphi_1}^{\varphi_2} V(\varphi, x)\, d\varphi. \tag{4.4.11}$$

In view of (4.4.3) and (4.4.8),

$$V(\varphi, x) = \left[\frac{\sin(\alpha\varphi + \beta\Phi)}{x\cos\varphi}\right]^{\alpha/(1-\alpha)} \frac{\cos((\alpha-1)\varphi + \beta\Phi)}{\cos\varphi}.$$

We set

$$U(\varphi;\alpha,\beta) = \left[\frac{\sin(\alpha\varphi + \beta\Phi(\alpha))}{\cos\varphi}\right]^{\alpha/(1-\alpha)} \frac{\cos((\alpha-1)\varphi + \beta\Phi(\alpha))}{\cos\varphi}, \quad (4.4.12)$$

and obtain

$$V(\varphi, x) = x^{\alpha/(\alpha-1)} U(\varphi;\alpha,\beta). \quad (4.4.13)$$

Returning to (4.4.11) and using the relation

$$q(-x;\alpha,\beta) = q(x;\alpha,-\beta),$$

we arrive at the formula

$$q(x;\alpha,\beta) = \frac{\alpha|x|^{1/(\alpha-1)}}{\pi|1-\alpha|} \int_{-\beta^*\Phi/\alpha}^{\pi/2} \exp\{-|x|^{\alpha/(\alpha-1)} U(\varphi;\alpha,\beta^*)\} U(\varphi;\alpha,\beta^*)\, d\varphi \quad (4.4.14)$$

which is true for all x, where

$$\beta^* = \begin{cases} \beta, & x > 0, \\ -\beta, & x < 0. \end{cases}$$

For $\alpha = 1$, the way described above leads us to

$$q(x;1,\beta) = \frac{1}{\pi|\beta|} e^{-x/\beta} \int_{-\pi/2}^{\pi/2} \exp\{-e^{-x/\beta} U(\varphi;1,\beta)\} U(\varphi;1,\beta)\, d\varphi, \quad (4.4.15)$$

where

$$U(\varphi;1,\beta) = \frac{\pi/2 + \beta\varphi}{\cos\varphi} \exp\{(\varphi + \pi/(2\beta))\tan\varphi\}.$$

Expressions (4.4.14) and (4.4.15) are exactly the representations of stable densities as integrals of non-oscillating functions sought for. Differentiating them, we arrive at integral representations of derivatives of densities (Lukacs, 1960; Zolotarev, 1986), while integrating them we obtain representations of distribution functions. Let us dwell on the latter topic.

4.5. Integral representation of stable distribution functions

Integrating (4.2.1) we see that the relation

$$G(-x;\alpha,\beta) + G(x;\alpha,-\beta) = 1 \quad (4.5.1)$$

is valid for all real x and all admissible (in forms A and B) parameters α, β.

Let $\alpha < 1$, $x > 0$. Since

$$dx^{\alpha/(\alpha-1)} = \frac{\alpha}{\alpha-1} x^{1/(\alpha-1)}\,dx,$$

by integrating (4.4.14) (form B) we obtain

$$\begin{aligned}
G(x;\alpha,\beta) &= 1 - \int_x^\infty q(x';\alpha,\beta)\,dx' \\
&= 1 + \pi^{-1}\int_{-\beta\pi/2}^{\pi/2} d\varphi \\
&\quad \times \int_x^\infty \exp\{-(x')^{\alpha/(\alpha-1)}U(\varphi;\alpha,\beta)\}\,d\{(x')^{\alpha/(\alpha-1)}U(\varphi;\alpha,\beta)\} \\
&= 1 - \pi^{-1}\int_{-\beta\pi/2}^{\pi/2} d\varphi \int_0^{V(\varphi,x)} e^{-y}\,dy \\
&= (1-\beta)/2 + \pi^{-1}\int_{-\beta\pi/2}^{\pi/2} e^{-V(\varphi,x)}\,d\varphi, \qquad (4.5.2)
\end{aligned}$$

where the function $V(\varphi, x)$ is determined by formulae (4.4.12) and (4.4.13).

For $\alpha > 1$ and $x > 0$,

$$\begin{aligned}
G(x;\alpha,\beta) &= 1 - \pi^{-1}\int_{-\beta\Phi/\alpha}^{\pi/2} d\varphi \int_{V(\varphi,x)}^\infty e^{-y}\,dy \\
&= 1 - \pi^{-1}\int_{-\beta\Phi/\alpha}^{\pi/2} e^{-V(\varphi,x)}\,d\varphi. \qquad (4.5.3)
\end{aligned}$$

Finally, for $\alpha = 1$ and $\beta > 0$,

$$G(x;1,\beta) = \pi^{-1}\int_{-\pi/2}^{\pi/2} \exp\{e^{-x/\beta}U(\varphi;1,\beta)\}\,d\varphi. \qquad (4.5.4)$$

The cases where $x < 0$ for $\alpha \neq 1$ and where $\beta < 0$ for $\alpha = 1$ are reduced to the just analyzed by (4.5.1).

Let us find the value of the distribution function at the point $x = 0$. We recall that

$$V(\varphi, x) = x^{\alpha/(\alpha-1)}U(\varphi;\alpha,\beta);$$

therefore, for $\alpha < 1$

$$V(\varphi, x) \to \infty, \qquad x \to 0,$$

whereas for $\alpha > 1$

$$V(\varphi, x) \to 0, \qquad x \to 0.$$

In the former case, (4.5.2) yields

$$G(0;\alpha,\beta) = (1-\beta)/2, \qquad (4.5.5)$$

and in the latter case formula (4.5.3) yields

$$G(0;\alpha,\beta) = 1/2 - \beta\Phi(\alpha)/(\alpha\pi). \tag{4.5.6}$$

Since

$$\Phi(\alpha) = \begin{cases} \alpha\pi/2, & \alpha < 1, \\ (\alpha-2)\pi/2, & \alpha > 1, \end{cases}$$

formula (4.5.6) unifies the two cases $\alpha < 1$ and $\alpha > 1$. The property

$$G(0;\alpha,1) = 0$$

following from (4.5.5) has been discussed above (see Section 4.2).

4.6. Duality law

This law relates, in the framework of the class of strictly stable distributions, the distributions with parameter $\alpha \geq 1$ to the distributions with parameter $\alpha' = 1/\alpha$. In what follows, we use form C.

DUALITY LAW. *For any pair of admissible parameters $\alpha \geq 1$, δ, and any $x > 0$ the equalities*

$$\alpha[1 - G(x;\alpha,\delta)] = G(x^{-\alpha};\alpha',\delta') - G(0;\alpha',\delta'), \tag{4.6.1}$$

$$q(x;\alpha,\delta) = x^{-1-\alpha} q(x^{-\alpha};\alpha',\delta') \tag{4.6.2}$$

are true, where the parameters α', δ' relate to α, δ as follows:

$$\alpha' = 1/\alpha, \qquad \delta' + \alpha' = (\delta + \alpha)/\alpha. \tag{4.6.3}$$

Relation (4.6.2) is the result of differentiation of (4.6.1), so it suffices to establish the validity of (4.6.1). By virtue of (4.5.3), the left-hand side of equality (4.6.1) can be represented as

$$\alpha[1 - G(x;\alpha,\delta)] = (\alpha/\pi) \int_{-\delta\pi/(2\alpha)}^{\pi/2} \exp\{-x^{\alpha/(\alpha-1)} U(\varphi;\alpha,\delta)\}\, d\varphi, \tag{4.6.4}$$

where we utilize the formula $\delta = 2\beta\Phi(\alpha)/\pi$ which relates the asymmetry parameters δ and β in forms C and B respectively (see Section 3.6). Since $\alpha' = 1/\alpha$, we obtain

$$x^{\alpha/(\alpha-1)} = (x^{-\alpha})^{1/(1-\alpha)} = (x^{-\alpha})^{\alpha'/(\alpha'-1)}. \tag{4.6.5}$$

Moreover,

$$U(\varphi;\alpha,\delta) = \left[\frac{\sin(\alpha\varphi + \delta\pi/2)}{\cos\varphi}\right]^{\alpha/(1-\alpha)} \frac{\cos((\alpha-1)\varphi + \delta\pi/2)}{\cos\varphi}$$
$$= \left[\frac{\sin(\alpha\varphi + \delta\pi/2)}{\cos\varphi}\right]^{1/(1-\alpha)} \frac{\cos((\alpha-1)\varphi + \delta\pi/2)}{\sin(\alpha\varphi + \delta\pi/2)}$$
$$= \left[\frac{\cos\varphi}{\sin\alpha\varphi + \delta\pi/2}\right]^{\alpha'/(1-\alpha')} \frac{\cos((\alpha-1)\varphi + \delta\pi/2)}{\sin(\alpha\varphi + \delta\pi/2)}.$$

Introducing

$$\varphi' = \pi/2 - (\alpha\varphi + \delta\pi/2),$$

we obtain

$$\alpha\varphi + \delta\pi/2 = \pi/2 - \varphi',$$
$$(\alpha-1)\varphi + \delta\pi/2 = (\alpha'-1)\varphi' + \delta'\pi/2;$$

hence

$$U(\varphi;\alpha,\delta) = \left[\frac{\sin(\alpha'\varphi' + \delta'\pi/2)}{\cos\varphi'}\right]^{\alpha'/(1-\alpha')} \frac{\cos((\alpha'-1)\varphi' + \delta'\pi/2)}{\cos\varphi'}$$
$$= U(\varphi';\alpha',\delta'). \tag{4.6.6}$$

The integration limits $\varphi_1 = -\delta\pi/(2\alpha)$, $\varphi_2 = \pi/2$ are transformed into $\varphi_1' = \pi/2$, $\varphi_2' = -\delta'\pi/(2\alpha')$. Since

$$d\varphi = -\alpha' d\varphi', \tag{4.6.7}$$

substituting (4.6.5)–(4.6.7) into (4.6.4) we obtain

$$\alpha[1 - G(x;\alpha,\delta)] = \pi^{-1} \int_{-\delta'\pi/(2\alpha')}^{\pi/2} \exp\{-(x^{-\alpha})^{\alpha'/(\alpha'-1)} U(\varphi';\alpha',\delta')\}\, d\varphi'$$

In view of (4.5.2) and (4.5.5), the right-hand side of the above equality is exactly the right-hand side of (4.6.1), as we wished to prove.

We do not need to consider the case where $\alpha = 1$ in its own right, because the validity of (4.6.1) follows from the continuity of the distributions belonging to the class of strictly stable laws in α, δ in the whole domain of variation of these parameters.

In terms of random variables, the duality law can be expressed as follows:

$$\alpha \mathsf{P}\{Y(\alpha,\delta) \geq x\} = \mathsf{P}\{1/Y(\alpha',\delta') > x^{\alpha}\}.$$

In particular, if $\alpha = 1$ and $\delta = 0$, then $\alpha' = 1$ and $\delta' = 0$; therefore,

$$Y(1,0) \stackrel{d}{=} [Y(1,0)]^{-1}.$$

If $\alpha = 1/2$ and $\delta = 1/2$, then $\alpha' = 2$ and $\delta' = 0$, which yields

$$Y(1/2,1/2) = [Y(2,0)]^{-2}.$$

We have discussed this in Section 2.3.

4.7. Short tails

The formulae of the preceding sections allow us to conclude the asymptotic analysis of stable densities with examination of the behaviour of the short tails (that is, $x \to 0$ for $\alpha < 1$, $\beta = 1$, and $x \to \infty$ for $\alpha \ge 1$, $\beta = -1$ in form B). This study is based on the well-known method of asymptotic representation of integrals due to Laplace. We utilize one of the simplest versions of this method given in (Zolotarev, 1986).

We consider even, analytic on the interval $(-\pi, \pi)$ functions $s(\psi)$ and $\omega(\psi)$, and assume that

(1) $w(\psi)$ is strictly monotone in the interval $(0, \pi)$;

(2) $\mu = s(0) > 0$, $\tau = w(0) > 0$, $\sigma^2 = w''(0) > 0$;

(3) $s(\psi) = O(w(\psi))$ as $\psi \to \pi$.

With the use of these functions, we consider the integrals

$$I_N = (2\pi)^{-1} \int_{-\pi}^{\pi} s(\psi) \exp\{-Nw(\psi)\}\, d\psi, \tag{4.7.1}$$

which exist for any $N > 0$.

LEMMA 4.7.1. *As $N \to \infty$, the representation*

$$I_N \sim \frac{\mu}{\sqrt{2\pi}\sigma} N^{-1/2} \exp(-\tau N) \left[1 + \sum_{n=1}^{\infty} Q_n N^{-n}\right] \tag{4.7.2}$$

holds, where

$$Q_n = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p_n(\psi) e^{-\psi^2/2} d\psi, \tag{4.7.3}$$

and $p_n(\psi)$ are polynomials which are the coefficients of the expansion into series in powers of $h^2 = (\sigma^2 N)^{-1}$ of the even function

$$\begin{aligned}\omega(\psi, h) &= \frac{1}{\mu} s(\psi h) \exp\left\{-\frac{1}{\sigma^2 h^2}\left[w(\psi h) - \tau - (\sigma\psi h)^2/2\right]\right\}\\ &= 1 + \sum_{n=1}^{\infty} p_n(\psi) h^{2n}.\end{aligned} \tag{4.7.4}$$

We set $\beta = 1$ in (4.4.14) and consider the case $\alpha < 1$ where $\Phi(\alpha) = \alpha\pi/2$. For $x > 0$ we obtain

$$q(x; \alpha, 1) = \frac{\alpha x^{1/(\alpha-1)}}{\pi(1-\alpha)} \int_{-\pi/2}^{\pi/2} \exp\left\{-x^{\alpha/(\alpha-1)} U(\varphi; \alpha, 1)\right\} U(\varphi; \alpha, 1)\, d\varphi,$$

where

$$U(\varphi;\alpha,1) = \left[\frac{\sin(\alpha(\varphi+\pi/2))}{\cos\varphi}\right]^{\alpha/(\alpha-1)} \frac{\cos((\alpha-1)\varphi+\alpha\pi/2)}{\cos\varphi}.$$

Changing $\varphi + \pi/2$ for ψ, we transform the function $x^{\alpha/(\alpha-1)}U(\varphi;\alpha,1)$ into $\xi(x,\alpha)w(\psi,\alpha)$, where

$$\xi(x,\alpha) = (1-\alpha)(\alpha/x)^{\alpha/(1-\alpha)},$$
$$w(\psi,\alpha) = \left[\frac{\sin(\alpha\psi)}{\alpha\sin\psi}\right]^{\alpha/(\alpha-1)} \frac{\sin((1-\alpha)\psi)}{(1-\alpha)\sin\psi}.$$

The density $q(x;\alpha,1)$ thus becomes

$$q(x;\alpha,1) = \frac{[\xi(x,\alpha)]^{1/\alpha}}{2\pi(1-\alpha)^{1/\alpha}} \int_{-\pi}^{\pi} w(\psi,\alpha)\exp\{-\xi(x,\alpha)w(\psi,\alpha)\}\,d\psi. \quad (4.7.5)$$

If $\alpha = 1$, $\beta = -1$, we obtain

$$q(x;1,-1) = \frac{1}{2\pi}\xi(x,1)\int_{-\pi}^{\pi} w(\psi,1)\exp\{-\xi(x,1)w(\psi,1)\}\,d\psi, \quad (4.7.6)$$

where

$$\xi(x,1) = \exp(x-1),$$
$$w(\psi,1) = \frac{\psi}{\sin\psi}\exp(1-\psi\cot\psi).$$

For $\alpha > 1$ the function $w = w(\psi,\alpha)$ is defined via the function $w(\psi,\alpha)$ with $\alpha < 1$ by changing α for $1/\alpha$.

Setting

$$\nu = \nu(\alpha) = \begin{cases} |1-\alpha|^{-1/\alpha}, & \alpha \neq 1 \\ 1, & \alpha = 1, \end{cases}$$

we reduce (4.7.5) and (4.7.6) to

$$q(x;\alpha,\beta) = \frac{\nu(\alpha)}{2\pi}[\xi(x,\alpha)]^{1/\alpha}\int_{-\pi}^{\pi} w(\psi,\alpha)\exp\{-\xi(x,\alpha)w(\psi,\alpha)\}\,d\psi, \quad (4.7.7)$$

This formula remains valid for $\alpha > 1$ as well, which immediately follows from the duality law and the following properties of the function $\xi(x,\alpha)$: if $\alpha > 1$, then for any $x > 0$

$$\xi(x^{-\alpha}, 1/\alpha) = \xi(x,\alpha);$$

and

$$\nu(1/\alpha)[\xi(x^{-\alpha},1/\alpha)]^{\alpha}x^{-(1+\alpha)} = \nu(\alpha)[\xi(x,\alpha)]^{1/\alpha}.$$

Similarly we can rewrite distribution functions in a unified form with the use of the duality law and relations (4.5.3), (4.5.4) in the cases $\alpha < 1$, $\beta = 1$, and $\alpha \geq 1$, $\beta = -1$ as follows:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\{-\xi(x,\alpha)w(\psi,\alpha)\}d\psi = \begin{cases} G(x;\alpha,1), & \alpha < 1, \\ 1 - G(x;\alpha,-1), & \alpha \geq 1. \end{cases} \tag{4.7.8}$$

For the sake of convenience we set

$$\alpha^* = \begin{cases} \alpha, & \text{if } \alpha < 1, \\ 1/\alpha, & \text{if } \alpha \geq 1. \end{cases} \tag{4.7.9}$$

The function $\xi(x,\alpha)$ increases beyond all bounds as $x \to 0$ and as $x \to \infty$, provided that $\alpha \geq 1$, $\beta = -1$, whereas the function $w(\psi,\alpha)$, as one can see from its definition, is an even analytic function in the interval $(-\pi,\pi)$ such that

$$\tau = w(0,\alpha) = 1, \qquad \sigma^2 = w''(0,\alpha) = \alpha^* > 0.$$

Thus, condition (2) is satisfied.

To see whether or not condition (1) is fulfilled, we consider the function

$$h(\alpha,\psi) = \alpha\cot(\alpha\psi) - \cot\psi$$

in the domain $0 < \psi < \pi$. Since

$$\partial h/\partial\alpha = [2\sin^2(\alpha\psi)]^{-1}[\sin(2\alpha\psi) - 2\alpha\psi] < 0,$$

the function $h(\alpha,\psi)$ with fixed ψ decreases while α grows. Because $h(1,\psi) = 0$, we see that

$$h(\alpha,\psi) > 0 \quad \text{if} \quad 0 < \alpha < 1.$$

Using this property, we conclude that

$$\frac{\partial w(\psi,\alpha)/\partial\psi}{w(\psi,\alpha)} = [\alpha h(\alpha,\psi) + (1-\alpha)h(1-\alpha,\psi)]/(1-\alpha) > 0.$$

It is clear that $w(\psi,\alpha)$ is positive for $0 < \psi < \pi$; therefore,

$$\partial w(\psi,\alpha)/\partial\psi > 0,$$

and condition (1) is hence satisfied. The third condition is also fulfilled ($s(\psi)$ is $w(\psi,\alpha)$ itself, or $s(\psi) \equiv 1 \leq w(\psi,\alpha)$), and we are able to apply Lemma 4.7.1 to the case under consideration.

To refine the form of the asymptotic expansion we are interested in, we have to study series (4.7.4) governing the coefficients Q_n in (4.7.3) in great depth.

First we write the expansion into power series of the function $\ln w(\psi, \alpha)$, provided that $\alpha \le 1$, making use of the known expansions of the functions $\ln(\psi^{-1}\sin\psi)$ and $\psi\cot\psi$ (Gradshtein & Ryzhik, 1963, 1.411, 1.518). We obtain

$$\ln w(\psi, \alpha) = \sum_{n=1}^{\infty} a_n(\alpha)\psi^{2n},$$

where

$$a_n(\alpha) = \frac{2^{2n}|B_{2n}|}{(2n)!\,2n}\left[\frac{\alpha(1-\alpha^{2n})}{1-\alpha} + 1 - (1-\alpha)^{2n}\right]$$

are polynomials of degree $2n - 1$ with zero constant term, and B_{2n} are the Bernoulli numbers. From the known Bruno formula we obtain

$$w(\psi, \alpha) = 1 + \alpha\psi^2/2 + \sum_{n=2}^{\infty} b_n(\alpha)\psi^{2n}, \tag{4.7.10}$$

where

$$b_n(\alpha) = (n!)^{-1}C_n(1!\,a_1, 2!\,a_2, \dots, n!\,a_n),$$

$$C_n(y_1, y_2, \dots y_n) = \sum_{\substack{k_1,k_2,\dots,k_n \ge 0 \\ k_1+2k_2+\dots+nk_n=n}} \frac{n!}{k_1!\,k_2!\dots k_n!}\left(\frac{y_1}{1!}\right)^{k_1}\left(\frac{y_2}{2!}\right)^{k_2}\cdots\left(\frac{y_n}{n!}\right)^{k_n}.$$

One can readily see that $b_n(\alpha)$ are polynomials of degree $2n - 1$ with zero constant terms.

From (4.7.10) it follows that

$$w(\psi, \alpha) = 1 + \alpha^*\psi^2/2 + \sum_{n=2}^{\infty} b_n(\alpha^*)\psi^{2n},$$

where $b_n(\alpha^*)$ is a rational function of α, provided that $\alpha \ge 1$.

Thus, if $s(\psi) = w(\psi) = w(\psi, \alpha)$, then expansion (4.7.4) takes the form

$$\begin{aligned} \omega(\psi, h) &= \exp\left\{\ln w(\psi h, \alpha) - \frac{1}{\alpha^* h^2}\left[w(\psi h, \alpha) - 1 - -\alpha^*(\psi h)^2/2\right]\right\} \\ &= \exp\left(\sum_{n=1}^{\infty} d_n(\psi, \alpha)h^{2n}\right) = 1 + \sum_{n=1}^{\infty} p_n(\psi, \alpha)h^{2n}, \end{aligned} \tag{4.7.11}$$

where

$$d_n(\psi, \alpha) = \left(a_n(\alpha^*) - (\psi^2/\alpha^*)b_{n+1}(\alpha^*)\right)\psi^{2n},$$

$$p_n(\psi, \alpha) = \frac{1}{n!}C_n(1!\,d_1, 2!\,d_2, \dots, n!\,d_n)$$

are polynomials of degree $2(n + 1)$ in ψ and of degree $2n$ in α^*.

Applying Lemma 4.7.1 to the integral representation (4.7.7), and taking expansion (4.7.10) into account, we arrive at the following assertion.

THEOREM 4.7.1 (on asymptotic behavior of short tails). *Let $\alpha < 1$, $\beta = 1$, and $x \to 0$, or $\alpha \geq 1$, $\beta = -1$, and $x \to \infty$. Then*

$$q(x;\alpha,\beta) \sim \frac{\nu}{\sqrt{2\pi\alpha}}\xi^{(2-\alpha)/(2\alpha)}\exp(-\xi)\left[1+\sum_{n=1}^{\infty}Q_n(\alpha^*)(\alpha^*\xi)^{-n}\right]. \quad (4.7.12)$$

The coefficients Q_n, determined by (4.7.3) *with p_n taken from expansion* (4.7.11)*, are polynomials of degree $2n$ in α^*.*

The leading term of expansion (4.7.12)

$$q^{(0)}(x;\alpha) = \frac{(x/\alpha)^{(1-\alpha/2)/(\alpha-1)}}{\sqrt{2\pi\alpha|1-\alpha|}}\exp\{-|1-\alpha|(x/\alpha)^{\alpha/(\alpha-1)}\}, \quad \alpha \neq 1, \quad (4.7.13)$$

$$q^{(0)}(x;1) = \frac{1}{\sqrt{2\pi}}\exp\{(x-1)/2 - e^{x-1}\} \quad (4.7.14)$$

does not depend on the parameter β. Its contribution in the case $\alpha = 3/2$, $\beta = 1$ is given in Fig. 4.6.

Curiously, it was found that for $\alpha = 1/2$ and $\alpha = 2$ the density $q^{(0)}(x;\alpha)$ is exactly the Lévy density and the Gauss density, respectively:

$$q^{(0)}(x;1/2) = q(x;1/2,1), \quad (4.7.15)$$

$$q^{(0)}(x;2) = q(x;2,0). \quad (4.7.16)$$

In view of (4.7.16), formula (4.7.13) would be expected to be a good approximation to $q(x;\alpha,-1)$ for α close to two in a wide domain of variation of x. Fig. 4.9 lends credence to this conjecture, and demonstrates that, due to (4.7.15), the asymptotic term $q^{(0)}(x;\alpha)$ provides us with a rather perfect approximation to the short tail even for $\alpha \in (1/2, 1)$.

4.8. Stable distributions with α close to extreme values

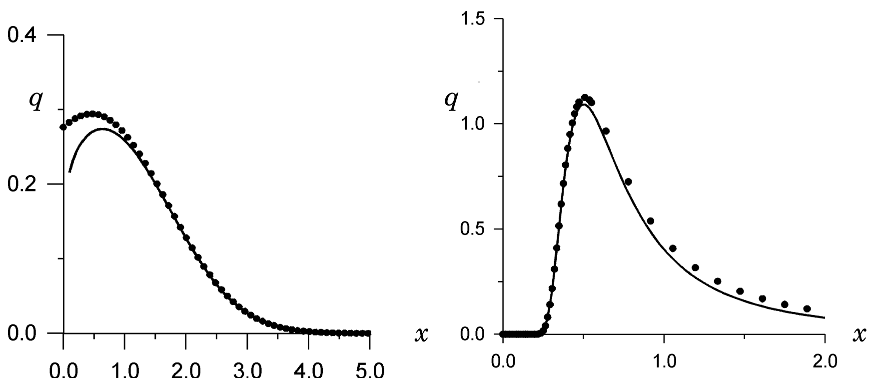
Using the infinite series expansion obtained above, one can show that if E is a random variable distributed by the exponential law with mean one, then

$$|Y_B(\alpha,\beta)|^\alpha \stackrel{d}{\to} E^{-1}, \qquad \alpha \to +0. \quad (4.8.1)$$

This result was obtained by Zolotarev (Zolotarev, 1957). We follow (Cressie, 1975) who offered a simpler way for the proof of (4.8.1).

We set

$$Z(\alpha,\beta) = |Y_B(\alpha,\beta)|^\alpha.$$



Approximation of $q^B(x; 7/4, -1)$ (bullets) by the leading term of (4.7.13) (solid line)

Approximation of $q^B(x; 1/4, 1)$ and $q^B(x; 1/2, 1)$ (bullets) by (4.8.7) (solid line)

Figure 4.9.

Then $Z(\alpha, \beta)$ becomes a positive random variable, and its density $p(z; \alpha, \beta)$ has two contributions from the density $q(x; \alpha, \beta)$ of $Y(\alpha, \beta)$: one from the point $z = x^{1/\alpha}$, and the other, from the point $z = -x^{1/\alpha}$. Denoting them by $p_R(z; \alpha, \beta)$ and $p_L(z; \alpha, \beta)$, respectively, and taking series expansion (4.2.4) into account, we represent them as

$$p_R(z; \alpha, \beta) = \frac{1}{\pi \alpha z} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \Gamma(\alpha n + 1) \sin[n(1 + \beta)\alpha\pi/2] z^{-n}, \tag{4.8.2}$$

$$p_L(z; \alpha, \beta) = \frac{1}{\pi \alpha z} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \Gamma(\alpha n + 1) \sin[n(1 - \beta)\alpha\pi/2] z^{-n}. \tag{4.8.3}$$

Thus

$$p(z; \alpha, \beta) = p_R(z; \alpha, \beta) + p_L(z; \alpha, \beta). \tag{4.8.4}$$

Using the elementary trigonometric formula

$$\sin(A + B) + \sin(A - B) = 2 \sin A \cos B,$$

and substituting (4.8.2), (4.8.3) into (4.8.4), we obtain

$$p(z; \alpha, \beta) = z^{-2} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(n\alpha + 1) \sin(n\alpha\pi/2) \cos(n\alpha\beta\pi/2)}{(n-1)!\, n\alpha\pi/2} z^{-n+1}. \tag{4.8.5}$$

If we let $\alpha \to +0$ and formally take the limit under the summation sign, then we obtain

$$\lim_{\alpha\to+0} p(z;\alpha,\beta) = z^{-2}e^{-1/z}, \qquad z > 0. \tag{4.8.6}$$

The limiting distribution does not depend on the parameter β.

This can be justified by rewriting the summand of (4.8.5) as $f_n(\alpha)$—a function of α—and observing that

$$|f_n(\alpha)| \le 2e^2n^2e^n \exp[-(1-\alpha_1)n\ln n]z^{-n+1} \equiv a_n.$$

on $[0,\alpha_1]$, $0 < \alpha_1 < 1$. The bound is obtained by using the Stirling formula (as was done in (Ibragimov & Linnik, 1971, §2.4). Thus,

$$\sum_{n=1}^{\infty} a_n < \infty,$$

and (4.8.5) is a uniformly convergent series of continuous on $[0,\alpha_1]$ functions. Therefore, from (Titchmarsh, 1939), we conclude that (4.8.5) is a continuous function of α on $[0,\alpha_1]$, and we do not lose the rigor while deducing (4.8.2).

But (4.8.6) is exactly the distribution of the random variable E^{-1}, because

$$\mathsf{P}\left\{E^{-1} < z\right\} = \mathsf{P}\left\{E > z^{-1}\right\} = e^{-1/z}$$

and

$$p_{E^{-1}}(z) = z^{-2}e^{-1/z}, \qquad z > 0.$$

Thus, (4.8.1) is true.

The asymptotic expression

$$q^{as}(x;\alpha) = \alpha e^{-x^{-\alpha}}x^{-\alpha-1}, \qquad x > 0, \tag{4.8.7}$$

following from (4.8.1) can be used to approximate the densities $q(x;\alpha,1)$ even for α close to 1/2 (see Fig. 4.10).

Let us turn to the consideration of a stable law with α close to another extreme value 2. It is known that for any fixed x, as $\alpha \to 2$,

$$q(x;\alpha,0) = q(x;2,0)(1+o(1)). \tag{4.8.8}$$

On the other hand, if α is fixed and $\alpha \neq 2$, then, as $x \to \infty$,

$$q(x;\alpha,0) = k(\alpha)x^{-1-\alpha}(1+o(1)), \tag{4.8.9}$$

where

$$k(\alpha) = \pi^{-1}\Gamma(\alpha+1)\sin(\alpha\pi/2)$$

is a positive constant depending only on α (see (4.3.2)).

Let $x = \xi(\alpha)$, moreover, $\xi(\alpha) \to \infty$ as $\alpha \to 2$. Intuition suggests that if $\xi(\alpha)$ grows fast enough, then representation (4.8.9) is valid; if $\xi(\alpha)$ grows rather slowly, then (4.8.8) remains good. The following theorem (Nagaev & Shkolnik, 1988) restates this suggestion in a rigorous way.

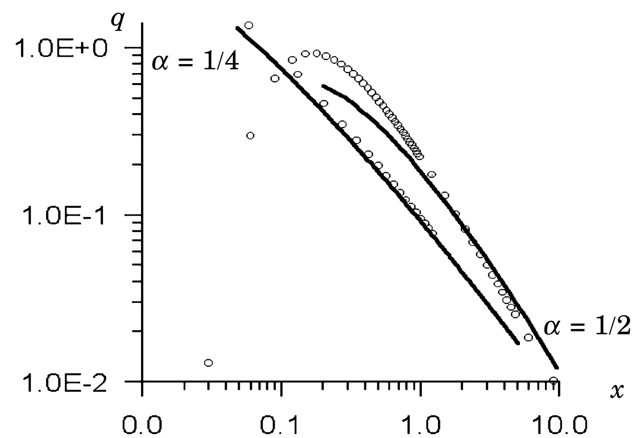


Figure 4.10. Approximation of $q^B(x; 1/4, 1)$ and $q^B(x; 1/2, 1)$ by (4.8.7) (solid line)

THEOREM 4.8.1 (on symmetric stable laws close to the normal law). *Let* $\Delta = 2-\alpha \to 0$, $x \to \infty$. *Then*

$$q(x; \alpha, 0) = q(x; 2, 0)(1 + o(1)) + \Delta x^{\Delta-3}(1 + o(1)). \qquad (4.8.10)$$

From (4.8.10), it follows, in particular, that representations (4.8.8) and (4.8.9) hold true in the domains $x \le (2-\varepsilon)|\ln\Delta|^{1/2}$ and $x \ge (2+\varepsilon)|\ln\Delta|^{1/2}$, respectively, where $\varepsilon > 0$ is as small as desired.

4.9. Summary

Summarizing the above facts, we can issue the following idea about the densities of stable laws.

The set of stable laws is determined by a four-parameter family of densities, while the set of strictly stable laws, by a three-parameter one.

The formulae describing the interrelations between parameters of various forms of representations of the densities are given in Section 3.6. The property

$$q(x; \alpha, \beta, \gamma, \lambda) = \lambda^{-1/\alpha} q(\lambda^{-1/\alpha}(x-h); \alpha, \beta, 0, 1),$$

where $h = h(\alpha, \beta, \gamma, \lambda)$ is a known function (in particular, $h = \gamma$ if $\gamma \ne 1$), allows us to exclude the shift and scale parameters from consideration and direct our attention to the study of

$$q^{A,B,M}(x;\alpha,\beta) = q^{A,B,M}(x;\alpha,\beta,0,1),$$
$$q^{C}(x;\alpha,\delta) = q^{C}(x;\alpha,\delta,1),$$
$$q^{E}(x;\nu,\theta) = q^{E}(x;\alpha,\nu,\theta,\tau_0),$$

where

$$\tau_0 = \begin{cases} C(\sqrt{\nu}-1), & \alpha \neq 1, \\ 2\ln(\pi/2), & \alpha = 1. \end{cases}$$

The common properties of stable densities are

- continuity;
- invariance with respect to the simultaneous change of the sign of the argument x and the asymmetry parameter β:

$$q(x;\alpha,-\beta) = q(-x;\alpha,\beta).$$

- for $\alpha \neq 2$, at least one of the tails appears to be long:

$$q(x;\alpha,\beta) = O(x^{-\alpha-1}), \quad x \to \infty, \quad \beta \neq -1;$$
$$q(x;\alpha,\beta) = O(|x|^{-\alpha-1}), \quad x \to -\infty, \quad \beta \neq 1;$$

- as concerns strictly stable densities, the duality law

$$q(x;\alpha,\delta) = x^{-1-\alpha}q(x^{-\alpha};1/\alpha,1+(\delta-1)/\alpha)$$

  holds.

The following particular cases are useful (form B):

$$\begin{aligned} G(0;\alpha,\beta) &= (1/2)[1-(2/\pi)\beta\Phi(\alpha)/\alpha] \\ &= \begin{cases} (1-\beta)/2, & \alpha < 1, \\ (1-\beta(1-2/\alpha))/2, & \alpha > 1, \end{cases} \\ q(0;\alpha,\beta) &= \pi^{-1}\Gamma(1+1/\alpha)\cos[\beta\Phi(\alpha)/\alpha], \\ q'(0;\alpha,\beta) &= (2\pi)^{-1}\Gamma(1+2/\alpha)\sin[2\beta\Phi(\alpha)/\alpha], \end{aligned} \tag{4.9.1}$$

The graphs of the densities $q^A(x;\alpha,\beta)$ obtained by means of the integral representation are given in Figures 4.11–4.15.

The characteristic α is common for all forms and does not change during transition from one form to another. The maximum value $\alpha = 2$ corresponds to the normal law, and the further α is from this extreme value, the greater is

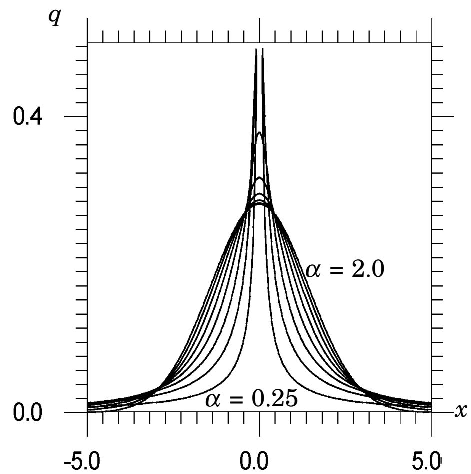


Figure 4.11. Symmetric stable distribution densities $q^A(x;\alpha,0)$ for $\alpha = 0.25$, 0.50, 0.75, 1.00 (the Cauchy law), 1.25, 1.50, 1.75, and 2 (the Gauss law)

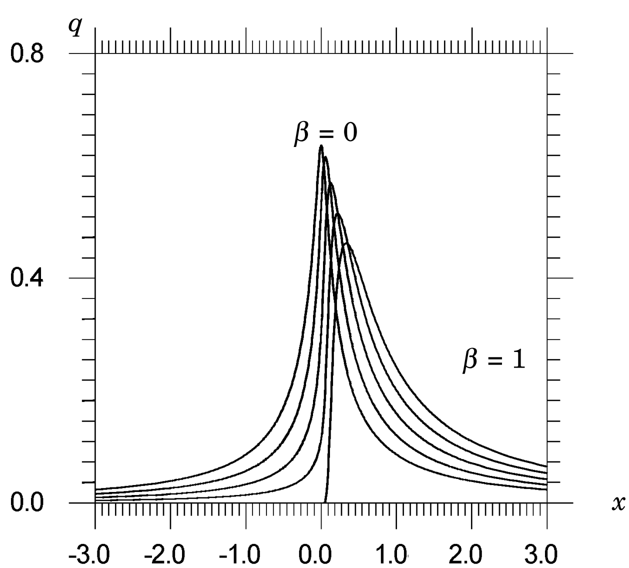


Figure 4.12. Stable distribution densities $q^A(x;1/2,\beta)$ for $\beta = 0$, 0.25, 0.5, 0.75, and 1 (the Lévy law)

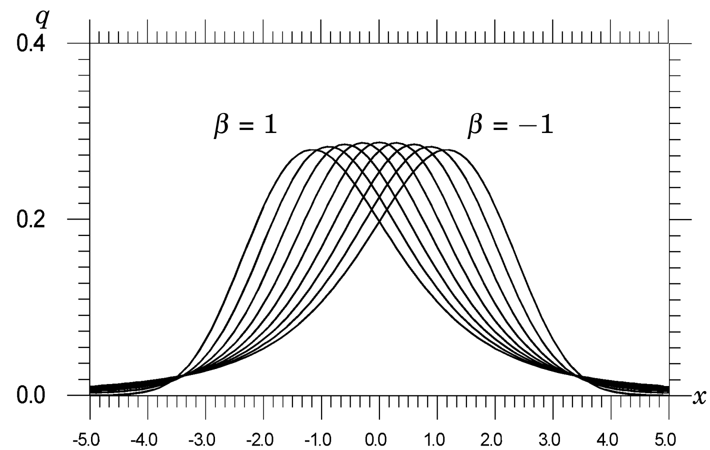


Figure 4.13. Stable distribution densities $q^A(x; 3/2, \beta)$ for $\beta = -1, -0.75, -0.50, -0.25, 0, 0.25, 0.5, 0.75$, and 1

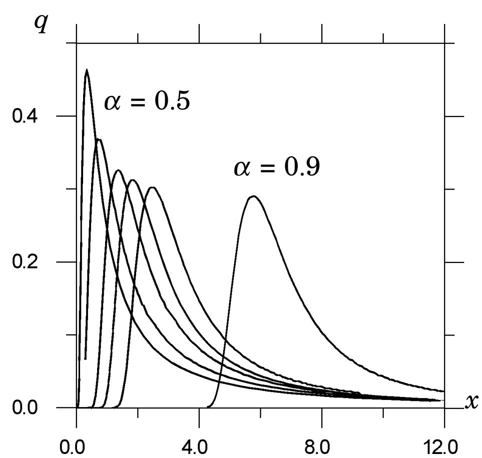


Figure 4.14. One-sided stable distribution densities $q^A(x, \alpha, 1)$ for $\alpha = 0.5$ (the Lévy law), 0.6, 0.7, 0.75, 0.8, and 0.9

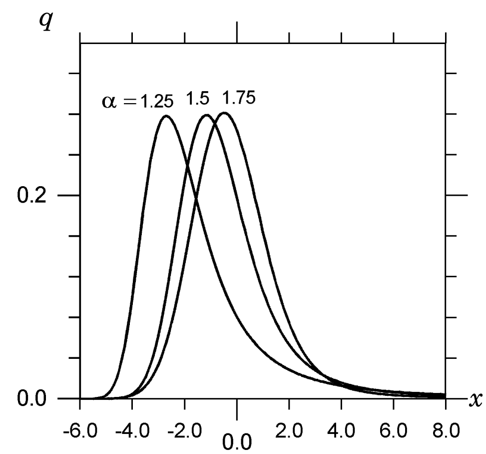


Figure 4.15. Extreme stable distribution densities $q^A(x; \alpha, 1)$ for α = 1.25, 1.50, and 1.75

the difference between the distribution and the normal law; the parameter β and related ones δ and θ characterize the asymmetry of the stable law: they are zero for symmetric distributions.

Fig. 4.11 illustrates the evolution of a symmetric stable distribution as α decreases. It is seen that the density around the maximum grows and the decrease rate of the tails lowers, so more probability becomes concentrated at small and large distances from the mean, while the intermediate domain becomes somewhat sparse as compared with the normal law.

As $\beta \neq 0$, the distributions become asymmetric about $x = 0$ (Figures 4.12 and 4.13). If $\alpha < 1$, then, as β grows from 0 to 1, the maximum of the density becomes shifted into positive x, and the positive semiaxis thus holds greater probability, which attains one for $\beta = 1$. In this extreme case, the right-hand tail of the distribution remains long, while the left-hand one becomes short. If $\alpha > 1$, then this change of β is associated with an inverse process: the maximum of the density is shifted into negative x and the probability concentrated on the negative semiaxis grows, attaining $G(0; \alpha, 1) = 1/\alpha < 1$ for $1 < \alpha < 2$, Even in the extreme case $\beta = 1$, some probability remains on the positive semiaxis whose distribution is described by a long tail opposing the short tail on the negative semiaxis. If $\beta \neq \pm 1$, both tails appear to be long.

The behavior of extreme stable densities is illustrated in Figures 4.14–4.15. For $\alpha < 1$, the extreme distributions turn out to be one-sided.

Let us make some comments concerning the evolution of the asymmetric density $q(x; \alpha, \beta)$ while α grows and passes through 1. As we have seen, a part of the probability (and all probability for extremal β) is instantly transferred from one semi-axis to another. On the face of it, we can circumvent this weird behavior by taking $\Phi'(\alpha)$ determined by formula (3.6.10) instead of $\Phi(\alpha)$ determined by (3.6.9) in representation B, which is equivalent to the change of sign of β for $\alpha > 1$, though. But this does not solve the problem.

To clear up the essence of the problem, we consider a stable distribution as the result of summation. As we know, the extreme values 1 and -1 of the parameter β^A correspond to the well-known situation of the summation of random variables with identical signs (positive or negative). Although this relation is not reversible (the density $q^A(x; \alpha, 1)$ can appear as a limit while one sums random variables X_i of different signs), it can play the role of a simple 'reference point' in properties of stable laws. This remains true in other forms of representation of stable densities, namely in forms A, M, and B (with the use of $\Phi(\alpha)$), because $\beta^A = 1$ corresponds to $\beta^M = \beta^B = 1$ and

$$\delta = \begin{cases} \alpha > 0 & \alpha < 1, \\ \alpha - 2 < 0 & \alpha > 1, \end{cases}$$

$$\theta = \begin{cases} 1 & \alpha < 1, \\ 1 - 2/\alpha < 0 & \alpha > 1. \end{cases}$$

Of course, one can re-define the parameters in the domain $\alpha > 1$, but it is more important to preserve the continuity of strictly stable distributions in the whole domain of the admissible values of the characteristic parameters.

5

Integral transformations

5.1. Laplace transformation

The characteristic functions

$$\varphi(k) = \int_{-\infty}^{\infty} p(x)e^{ikx}dx,$$

applied in Chapter 3 to the investigation of stable laws are only particular integral transforms, namely the Fourier transforms. The methodology of integral transformations as a whole is acknowledged to be among the most powerful and efficient tools of the analysis. Its essence consists in that the investigation of a function $f(x)$ is replaced by that of its integral transform

$$\psi(z) = \int_D K(z,x)f(x)\,dx, \qquad z \in I,$$

where the sets D, I, and the function $K(z,x)$, referred to as the kernel, determine the type of the transform. As concerns the Fourier transforms, D and I coincide with the real axis $\mathbb{R}$, whereas

$$K(z,x) = e^{izx}.$$

The theory of integral transforms with various kernel types was presented in (Titchmarsh, 1937). We give here some facts needed in what follows.

We begin with the Laplace transform.

DEFINITION OF THE ONE-SIDED LAPLACE TRANSFORM. Let a function $f(x)$ be defined in the positive semi-axis $x > 0$, and let the integral

$$\tilde{f}(\lambda) = \int_0^{\infty} e^{-\lambda x} f(x)\,dx \tag{5.1.1}$$

converge in some half-plane $\Re\lambda > c$; then the function $\tilde{f}(\lambda)$ is called the (one-sided) Laplace transform.

Laplace transform (5.1.1) exists for $\Re\lambda \equiv \sigma > c$, the improper integral converges absolutely and uniformly, and the transform $\tilde{f}(\lambda)$ is thus an analytic function of $\lambda > c$, provided that the integral

$$\int_0^\infty e^{-\sigma x}|f(x)|\,dx = \lim_{\substack{A\to 0\\ B\to\infty}} \int_A^B e^{-\sigma x}|f(x)|\,dx$$

converges for $\sigma = c$. The infimum σ_a of those real c for which this condition is fulfilled is referred to as the absolute convergence abscissa of Laplace transform (5.1.1).

The domain of definition of analytic function (5.1.1) is usually analytically continued to the whole complex plane, except the singularities placed to the left of the absolute convergence abscissa, and in what follows we do this with no additional stipulation.

THEOREM 5.1.1 (uniqueness theorem). *Laplace transform* (5.1.1) *is uniquely determined for any function $f(x)$ admitting of such a transform. Conversely, two functions $f_1(x)$ and $f_2(x)$ whose Laplace transforms coincide, coincide themselves for all $x > 0$, except for, maybe, a set of zero measure; $f_1(x) = f_2(x)$ for all x where both of these functions are continuous.*

THEOREM 5.1.2 (inversion theorem). *Let $\tilde{f}(\lambda)$ be given by* (5.1.1) *for $\Re\lambda > \sigma_a$, and*

$$I(x) = \frac{1}{2\pi i}\lim_{R\to\infty}\int_{\sigma-iR}^{\sigma+iR} e^{\lambda x}\tilde{f}(\lambda)\,d\lambda, \qquad \sigma > \sigma_a. \tag{5.1.2}$$

Then in any open interval where $f(x)$ is bounded and possesses a finite number of maximum, minimum, and discontinuity points

$$I(x) = \begin{cases} (1/2)[f(x-0) + f(x+0)] & x > 0,\\ (1/2)f(0+0) & x = 0,\\ 0 & x < 0.\end{cases}$$

In particular, for any $x > 0$ at which $f(x)$ is continuous

$$I(x) = f(x).$$

The integration path in (5.1.2) lies to the right of all singularities of $\tilde{f}(\lambda)$. If the improper integral in (5.1.2) exists, then, provided that $f(x)$ is continuous,

$$f(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda x}\tilde{f}(\lambda)\,d\lambda. \tag{5.1.3}$$

The correspondence between the basic operations over functions and their transforms following from (5.1.1)–(5.1.3) is given in Table 5.1.

Table 5.1. Correspondence of operations over $f(x)$ and its Laplace transform $\tilde{f}(\lambda)$

	$f(x)$	$\tilde{f}(\lambda)$
1	$f(x-a), \quad f(x) = 0$ for $x \le 0$	$\exp(-a\lambda)\tilde{f}(\lambda)$
2	$f(bx), \quad b > 0$	$\tilde{f}(\lambda/b)/b$
3	$f'(x)$	$\lambda\tilde{f}(\lambda) - f(0+0)$
4	$\int_0^x f(x')dx'$	$\tilde{f}(\lambda)/\lambda$
5	$f_1 * f_2 \equiv \int_0^x f_1(x')f_2(x-x')dx'$	$\tilde{f}_1(\lambda)\tilde{f}_2(\lambda)$
6	$-xf(x)$	$\tilde{f}'(\lambda)$
7	$x^{-1}f(x)$	$\int_\lambda^\infty \tilde{f}(\lambda')d\lambda'$
8	$e^{ax}f(x)$	$\tilde{f}(\lambda - a)$

Calculation of the Laplace transform $\tilde{f}(\lambda)$ of a given $f(x)$ is merely calculation of the definite (improper) integral of $\exp\{-\lambda x\}f(x)$ along the semi-axis $(0, \infty)$. The reconstruction of the original from a given transform, which relates to integration in the complex plane, can appear to be a more complicated process. Before we dwell upon this problem, we note the existence of another type of Laplace transforms, namely two-sided ones.

DEFINITION OF TWO-SIDED LAPLACE TRANSFORM. Let a function $f(x)$ be defined on the whole axis, and let the integral

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda x} f(x)\, dx$$

converge along at least one line $\Re\lambda = c$. Then the function $\hat{f}(\lambda)$ is called the two-sided Laplace transform of the function $f(x)$.

While the one-sided transforms share many common properties with the power series, the two-sided transforms resemble the Laurent series. Its convergence domain is the strip $a < \Re z < b$, which is allowed to degenerate into a line as $a = b$.

It is worthwhile to notice that the Fourier transform $\varphi(k)$ of a function $f(x)$ is related to its two-sided Laplace transform $\hat{f}(\lambda)$ by

$$\varphi(k) = \hat{f}(-ik);$$

so, all properties of the Fourier transforms can be derived from the known properties of the Laplace transforms, and vice versa. The inversion formula for the two-sided Laplace transforms is of the same form as for the one-sided transform, because the latter can be considered as the two-sided transform of a function which is equal to zero at negative x.

5.2. Inversion of the Laplace transformation

As concerns analytic methods to inverse the Laplace transformation, the most popular of them are the residue method and the saddle point method. The former is based on representation of improper integral (5.1.3) in the form of the limit of a sequence of integrals I_n along closed contours C_n:

$$I_n = \frac{1}{2\pi i} \oint_{C_n} h(z)\, dz.$$

Here, the following assertion plays an important part.

LEMMA 5.2.1 (Jordan). *Let Γ_R be the arc $\{z\colon |z| = R,\ |\arg z - \varphi_0| < \pi/(2\nu)\}$, and let $h(z)$ satisfy, along this arc, the inequality*

$$|h\left(Re^{i\varphi}\right)| \le \varepsilon(R) \exp\left\{-R^\nu \cos[\nu(\varphi - \varphi_0)]\right\}.$$

If $\varepsilon(R)R^{1-\nu} \to 0$ as $R \to \infty$, then

$$\lim_{R\to\infty} \int_{\Gamma_R} |h(z)||dz| = 0.$$

This lemma allows us to go from the integral along the line to the integral along a closed contour; to evaluate the latter, we use the following theorem.

THEOREM 5.2.1 (theorem on residues). *Let a univalent function $h(z)$ be analytic in some domain D, except for isolated singularities, and let a closed contour C belong, together with its interior, to the domain D, enclose a finite number $z_1, z_2, \dots, z_n$ of singularities, and pass through none of them. Then*

$$\frac{1}{2\pi i} \oint_C h(z)\, dz = \sum_{k=1}^{n} \operatorname{Res} h(z_k),$$

where $\operatorname{Res} h(z_k)$ are the residues of the function $h(z)$ at the points z_k:

$$\operatorname{Res} h(z_k) = \frac{1}{(m-1)!} \lim_{z\to z_k} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_k)^m h(z)\right],$$

$z_k \ne \infty$ are poles of order m. In particular, if z_k is a simple pole and $h(z) = p(z)/q(z)$, $p(z)$ and $q(z)$ are analytic at z_k, and $p(z_k) \ne 0$, then $q'(z_k) \ne 0$ and

$$\operatorname{Res} h(z_k) = p(z_k)/q'(z_k).$$

Combining these assertions, we obtain the following theorem.

THEOREM 5.2.2 (expansion theorem). *Any meromorphic tame function $\tilde{f}(\lambda)$ in some domain $\Re\lambda > c_0$ satisfying the hypotheses of Jordan's lemma such that for any $c > c_0$ the integral $\int_{c-i\infty}^{c+i\infty} \tilde{f}(\lambda)\,d\lambda$ converges absolutely, can be considered as the transform of the function*

$$f(x) = \sum_k \operatorname{Res}\left[\tilde{f}(\lambda)e^{\lambda x}\right]_{\lambda=\lambda_k},$$

where the sum of residues is over all singularity points λ_k of the function $\tilde{f}(\lambda)$ arranged so that their absolute values do not decrease.

In contrast to the residue method, the saddle point method provides us with an asymptotic approximation of the function under consideration. Let the integral (5.1.3) be represented as

$$I(t) = \frac{1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty} e^{t\varphi(\lambda)}H(\lambda)\,d\lambda, \tag{5.2.1}$$

where t is a real-valued positive parameter, the functions $H(\lambda)$ and

$$\varphi(\lambda) \equiv u(\lambda) + iv(\lambda)$$

take real values along the real axis, and the function $u(\lambda)$ attains its minimum at the real axis. At the minimum point $\lambda = \lambda^0$, the derivatives of the function $u(\lambda) \equiv u(x + iy)$ with respect to x and y become zero:

$$(\partial u/\partial x)_{\lambda^0} = (\partial u/\partial y)_{\lambda^0} = 0. \tag{5.2.2}$$

Since the functions $u(\lambda)$ and $v(\lambda)$ are harmonic,

$$\partial^2 u/\partial x^2 = -\partial^2 u/\partial y^2, \qquad \partial^2 v/\partial x^2 = -\partial^2 v/\partial y^2, \tag{5.2.3}$$

while passing through the point λ^0 in a direction parallel to the imaginary axis, the function $u(\lambda) = u(\lambda^0 + iy)$ attains its maximum. This point is referred to as the saddle point. By the Cauchy–Riemann conditions

$$\partial v/\partial x = -\partial u/\partial y, \qquad \partial v/\partial y = \partial u/\partial x,$$

that is, the derivative of $v(\lambda)$ at the saddle point is equal to zero as well. Since $v(\lambda)$ itself is zero along the real axis,

$$\partial^2 v/\partial x^2 = -\partial^2 v/\partial y^2 = 0,$$

i.e., in a neighborhood of λ^0 the function $v(\lambda)$ is almost zero along a line parallel to the imaginary axis. We shift the integration contour in (5.2.1) so that it passes through the saddle point λ^0 but remains parallel to the imaginary axis. Then in a neighborhood of this point the function

$$\exp\{t\varphi(\lambda)\} = \exp\{tu(\lambda^0 + iy)\}$$

for large t quickly increases as y tends to zero (from the side of negative y), peaks sharply at $y = 0$ (i.e., $\lambda = \lambda^0$) and then quickly decreases. The greater is t, the narrower and higher is the peak of $\exp\{t\varphi(\lambda)\}$, yielding more grounds to ignore the variation of the continuous function $H(\lambda)$ under the integral sign, i.e., to factor out its value at the point λ^0 from the integral:

$$I(t) \sim \frac{H(\lambda^0)}{2\pi i} \int_{\lambda^0 - i\infty}^{\lambda^0 + i\infty} \exp\{tu(\lambda)\}\, d\lambda, \qquad t \to \infty.$$

Expanding $u(\lambda_0 + iy)$ into series in y at the saddle point

$$u(\lambda^0 + iy) = u(\lambda^0) + y[\partial u(\lambda^0, y)/\partial y]_{y=0} + (y^2/2)[\partial^2 u(\lambda^0, y)/\partial y^2]_{y=0} + \ldots$$

and taking (5.2.2) and (5.2.3), into account, we obtain

$$u(\lambda^0, y) = \varphi(\lambda^0) - |\varphi''(\lambda^0)| y^2/2 + \ldots,$$

where

$$|\varphi''(\lambda^0)| = \left[\partial^2 u(x, 0)/\partial x^2\right]_{\lambda^0}.$$

Since $d\lambda = i\, dy$, we arrive at the expression

$$I(t) \sim \frac{H(\lambda^0)}{2\pi} e^{t\varphi(\lambda^0)} \int_{-\infty}^{\infty} e^{-|\varphi''(\lambda^0)| y^2 t/2}\, dy.$$

Setting

$$|\varphi''(\lambda^0)| y^2 t/2 = z^2,$$

the integral is reduced to the Poisson integral

$$\int_{-\infty}^{\infty} e^{-z^2}\, dz = \sqrt{\pi},$$

and we obtain the final result

$$I(t) \sim \frac{1}{\sqrt{2\pi|\varphi''(\lambda^0)| t}} H(\lambda^0) e^{t\varphi(\lambda^0)}, \qquad t \to \infty, \tag{5.2.4}$$

where the saddle point λ^0 is determined by the saddle point condition

$$\varphi'(\lambda^0) = 0. \tag{5.2.5}$$

5.3. Tauberian theorems

Obviously, if we take as $f(x)$ a distribution density $p(x)$, then the integral

$$\tilde{p}(\lambda) = \int_0^\infty e^{-\lambda x} p(x)\, dx$$

converges in the half-plane $\Re\lambda \geq 0$.

Based on the definition of the mathematical expectation, this formula can be rewritten as

$$\tilde{p}(\lambda) = \mathsf{E} e^{-\lambda X},$$

where X is a random variable with distribution density $p(x)$. We take the last relation as the definition of the Laplace–Stieltjes transform of the distribution $F(x)$, in other words,

$$\tilde{p}(\lambda) = \int_0^\infty e^{-\lambda x}\, dF(x). \tag{5.3.1}$$

For our purposes, it is convenient to extend definition (5.3.1), to the whole class of measures, i.e., to measures which do not have to be probability distributions (as was done by Feller):

$$\tilde{m}(\lambda) = \int_0^\infty e^{-\lambda x}\, dM(x), \tag{5.3.2}$$

where $M(x)$ is an analogue of distribution function, i.e., a measure concentrated in the interval $[0, x]$. The difference between (5.3.2) and (5.3.1) consists in that $M(x)$ does not need to tend to one as $x \to \infty$ and therefore, $\tilde{m}(\lambda)$ does not necessarily tend to one as $\lambda \to 0$. The Tauberian theorems just establish a connection between the asymptotic behavior of the measure $M(x)$ or its density $m(x) = M'(x)$ as $x \to \infty$ and the behavior of its Laplace transform $\tilde{m}(\lambda)$ as $\lambda \to 0$.

We consider a simple example. Let

$$M(x) = Ax^\mu, \quad \mu \geq 0, \quad x > 0; \tag{5.3.3}$$

then

$$m(x) = \mu A x^{\mu-1},$$

and

$$\tilde{m}(\lambda) = \mu A \int_0^\infty e^{-\lambda x} x^{\mu-1}\, dx = A\Gamma(\mu+1)\lambda^{-\mu}.$$

The Tauberian theorems deal with arbitrary functions $M(x)$ which satisfy (5.3.3) asymptotically, as $x \to \infty$. Moreover, it is possible to weaken the requirements imposed on the asymptotic form and rewrite it as

$$M(x) \sim L(x)x^\mu, \qquad x \to \infty,$$

where $L(x)$ is a slowly varying function as $x \to \infty$, i.e., a function that satisfies the condition

$$L(tx)/L(t) \to 1, \qquad t \to \infty, \tag{5.3.4}$$

for any fixed x (for example, $\ln x$ or any its power satisfy this condition).

THEOREM 5.3.1. *If $L(x)$ slowly varies at infinity and $0 \le \mu < \infty$, then the relations*

$$M(x) \sim L(x)x^{\mu}, \qquad x \to \infty, \tag{5.3.5}$$

and

$$\tilde{m}(\lambda) \sim L(1/\lambda)\Gamma(\mu + 1)\lambda^{-\mu}, \qquad \lambda \to 0 \tag{5.3.6}$$

are equivalent.

THEOREM 5.3.2. *Tauberian Theorem* 5.3.1 *remains valid if we reverse the roles of zero and infinity, that is, if we let $x \to 0$ and $\lambda \to \infty$.*

THEOREM 5.3.3. *Let $0 < \mu < \infty$. If $M(x)$ possesses a monotone, beginning with some place, derivative $m(x)$, then*

$$m(x) \sim \mu L(x)x^{\mu-1}, \qquad x \to \infty, \tag{5.3.7}$$

if and only if

$$\tilde{m}(\lambda) \sim L(1/\lambda)\Gamma(\mu + 1)\lambda^{-\mu}, \qquad \lambda \to 0. \tag{5.3.8}$$

The product of a slowly varying function $L(x)$ and a power function x^{μ} is referred to as regularly varying with parameter μ. Thus, the long tails of stable distributions $G(x; \alpha, \beta)$ as $x \to -\infty$ and $1 - G(x; \alpha, \beta)$ as $x \to \infty$ appear to be regularly varying functions at infinity with parameter coinciding with the characteristic parameter of the stable law. In view of the limiting part of stable distributions, the following assertion becomes obvious.

PROPOSITION 5.3.1. *Let $F(x)$ be a distribution function such that $\bar{F}(x) \equiv 1 - F(x)$ regularly varies at infinity:*

$$\bar{F}(x) \sim L(x)x^{-\mu}, \qquad x \to \infty,$$

and let $F_n(x)$ be the multi-fold convolution of this function ($F_1(x) = F(x)$). Then

$$\bar{F}_n(x) \sim nL(x)x^{-\mu}, \qquad x \to \infty.$$

As concerns one-sided stable laws, $\mu = \alpha \in [0, 1)$, $L(x) = \text{const}$, and

$$F_n(x) = F(x/b_n),$$

which immediately yields

$$nx^{-\alpha} = (x/b_n)^{-\alpha}$$

or

$$b_n = n^{1/\alpha}.$$

In conclusion, we give a theorem containing the conditions which guarantee that a monotone function is regularly varying.

THEOREM 5.3.4. *A function $U(x)$ which is monotone in $(0, \infty)$ regularly varies at infinity if and only if*

$$\lim_{t \to \infty} U(tx)/U(t) = \psi(x) \tag{5.3.9}$$

on an everywhere dense set of x, where the limit ψ is finite and positive in some interval.

The proof of this theorem can be found in (Feller, 1966, §8, Chapter VIII).

5.4. One-sided stable distributions

As we have seen, Tauberian theorems lead us to one-sided stable laws in a quite natural way.

Let $F(x)$ be a distribution function, and let $\tilde{p}(\lambda)$ be the Laplace transform (5.3.1) of the corresponding density. Then, integrating by parts, we easily see that

$$\int_0^\infty e^{-\lambda x}[1 - F(x)]\,dx = [1 - \tilde{p}(\lambda)]/\lambda.$$

Since the function $1 - F(x)$ is monotone, the relations

$$1 - F(x) \sim L(x)x^{-\alpha}, \qquad x \to \infty, \tag{5.4.1}$$

and

$$1 - \tilde{p}(\lambda) \sim L(1/\lambda)\Gamma(1 - \alpha)\lambda^{\alpha}, \qquad \lambda \to 0, \tag{5.4.2}$$

are equivalent ($\alpha < 1$).

This version of Tauberian theorems was used by Feller to prove the two following theorems concerning one-sided stable distributions.

THEOREM 5.4.1. *For any $\alpha \in (0,1)$ the function*

$$\tilde{q}(\lambda;\alpha) = e^{-b\lambda^\alpha}, \qquad b > 0, \tag{5.4.3}$$

appears to be the Laplace transform of the one-sided stable distribution $G(x;\alpha)$ possessing the properties

$$1 - G(x;\alpha) \sim cx^{-\alpha}, \quad x \to \infty, \quad c = b/\Gamma(1-\alpha), \tag{5.4.4}$$

and

$$q(x;\alpha) = G'(x;\alpha) \sim \frac{1}{\sqrt{2\pi(1-\alpha)\alpha}} b^{1/(2-2\alpha)}(x/\alpha)^{(\alpha-2)/(2-2\alpha)}$$
$$\times \exp\left\{-(1-\alpha)b^{1/(1-\alpha)}(x/\alpha)^{-\alpha/(1-\alpha)}\right\}, \qquad x \to 0. \tag{5.4.5}$$

The proof of the theorem is based on the correspondence 5 (see Table 5.1) and the stability property (2.2.15)–(2.2.16):

$$[\tilde{q}(\lambda;\alpha)]^n = \tilde{q}(\lambda b_n;\alpha).$$

Function (5.4.3) satisfies this condition with

$$b_n = n^{1/\alpha}.$$

Relation (5.4.4) is a particular case of (5.4.1) because

$$1 - \tilde{q}(\lambda;\alpha) = 1 - e^{-b\lambda^\alpha} \sim b\lambda^\alpha, \qquad \lambda \to 0.$$

Relation (5.4.5) can be obtained with the use of the saddle point method by inverting the Laplace transformation

$$q(x;\alpha) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{-b\lambda^\alpha+\lambda x} d\lambda.$$

Setting

$$\lambda = zx^{-1-\gamma}, \qquad \gamma > 0,$$

we obtain

$$q(x;\alpha) = \frac{1}{2\pi i} x^{-1-\gamma}\int_{\sigma'-i\infty}^{\sigma'+i\infty} \exp\left\{-bx^{-(1+\gamma)\alpha}z^\alpha + zx^{-\gamma}\right\} dz.$$

Setting

$$\gamma = \alpha/(1-\alpha)$$

so that

$$(1+\gamma)\alpha = \gamma,$$

we obtain

$$q(x;\alpha) = x^{-1/(1-\alpha)} I(x^{-\alpha/(1-\alpha)}),$$

where

$$I(t) = \frac{1}{2\pi i}\int_{\sigma'-i\infty}^{\sigma'+i\infty} \exp\left\{t\varphi(z)\right\}\,dz,$$

$$\varphi(z) = z - bz^{\alpha}, \qquad t = x^{-\alpha/(1-\alpha)} \to \infty$$

if $x \to 0$. Now, the saddle point is

$$z^0 = (\alpha b)^{1/(1-\alpha)}$$

and, after obvious transformations, we arrive at (5.4.5). Comparing it with (4.7.13), one can see that the Laplace transform $\tilde{q}(\lambda;\alpha)$ with $b = 1$ corresponds to form B:

$$\int_0^\infty e^{-\lambda x} q^B(x;\alpha,1)\,dx = e^{-\lambda^\alpha}. \tag{5.4.6}$$

In the case $\alpha = 1/2$ the inverse transformation is expressed in terms of elementary function and leads us to a known distribution

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} &\exp\left\{\lambda x - \lambda^{1/2}\right\} d\lambda \\
&= \frac{1}{\pi i x}\int_{\sigma'-i\infty}^{\sigma'+i\infty} \exp\left\{s^2 - 2(x^{-1/2}/2)s\right\} s\,ds \\
&= \frac{1}{\pi i x}\exp\left\{-\frac{1}{4x}\right\}\int_{\sigma''-i\infty}^{\sigma''+i\infty} \exp\left\{(s - x^{-1/2}/2)^2\right\} s\,ds \\
&= \frac{1}{2\sqrt{\pi}} x^{-3/2} \exp\left\{-1/(4x)\right\}.
\end{aligned}$$

As concerns the behavior of (5.4.3) as $\alpha \to 1$, in the limit we have the degenerate distribution concentrated at $x = 1$.

The second theorem concerns the summation of independent random variables.

THEOREM 5.4.2. *Let $F(x)$ be a distribution function on $(0,\infty)$ and let its multifold convolution converge,*

$$F_n(b_n x) \to G(x), \qquad n \to \infty, \tag{5.4.7}$$

where the limiting distribution G is not degenerate at any point. Then

(a) *there exist a function $L(x)$ slowly varying at infinity and a constant $\alpha \in (0,1)$ such that*

$$1 - F(x) \sim L(x)x^{-\alpha}, \qquad x \to \infty; \tag{5.4.8}$$

(b) *vice versa, if* $F(x)$ *is of form* (5.4.8)*, then there exist* b_n *such that*

$$nb_n^{-\alpha}\Gamma(1-\alpha)L(b_n) \to 1, \qquad n \to \infty. \tag{5.4.9}$$

In this case, (5.4.7) *holds with* $G(x) = G(x;\alpha,1)$.

PROOF. Let $\tilde{p}(\lambda)$ and $\tilde{q}(\lambda)$ be the Laplace transforms of the distributions $F(x)$ and $G(x)$ respectively. Then (5.4.7) yields

$$-n\ln\tilde{p}(\lambda/b_n) \to -\ln\tilde{q}(\lambda) \tag{5.4.10}$$

It is clear that the function $\ln\tilde{p}(\lambda)$ satisfies condition (5.3.9), and therefore, it regularly varies at zero:

$$-\ln\tilde{p}(\lambda) \sim \lambda^\alpha L(1/\lambda), \qquad \lambda \to 0, \tag{5.4.11}$$

where $L(x)$ slowly varies at infinity and $\alpha \geq 0$. Then from (5.4.10) it follows that

$$-\ln\tilde{q}(\lambda) = c\lambda^\alpha, \qquad 0 < \alpha < 1,$$

while (5.4.11) yields

$$1-\tilde{p}(\lambda) \sim \lambda^\alpha L(1/\lambda), \quad \lambda \to 0. \tag{5.4.12}$$

and, using (5.4.1)–(5.4.2), we obtain (5.4.8). Finally, if (5.4.8) holds, then there exist b_n satisfying (5.4.9). Then from (5.4.12) it follows that

$$1-\tilde{p}(\lambda/b_n) \sim \lambda^\alpha b_n^{-\alpha}L(b_n/\lambda) \sim c\lambda^\alpha/n.$$

It follows herefrom that the left-hand side of (5.4.10) tends to λ^α, which completes the proof.

Now, let us consider the problem of the contribution of the maximum term

$$M_n = \max\{X_1,\ldots,X_n\}$$

to the sum

$$S_n = \sum_{i=1}^n X_i$$

in the case where the normalized sum

$$Z_n = \frac{S_n}{b_n}$$

converges to a one-sided stable distribution with characteristic $\alpha < 1$. We follow Feller's method (Feller, 1966).

Assuming that the terms X_i belong to the normal domain of attraction of the stable law, i.e., satisfy (5.4.8) with $L(x)$ = const = A, we represent the Laplace transform of the ratio S_n/M_n as

$$\begin{aligned}\tilde{p}_n(\lambda) &= \mathsf{E}e^{-\lambda S_n/M_n} \\ &= \int_0^\infty dx_1 \ldots \int_0^\infty dx_n \exp\left\{-\lambda \sum_{i=1}^n x_i/\max\{x_i\}\right\} p(x_1)\ldots p(x_n) \\ &= \sum_{j=1}^n \int_0^\infty dx_j \underbrace{\int dx_1 \ldots \int dx_i \ldots \int dx_n}_{x_i<x_j} \\ &\quad \times \exp\left\{-\lambda\left[1 + \sum_{i\neq j} x_i/x_j\right]\right\} p(x_1)\ldots p(x_n) \\ &= ne^{-\lambda}\int_0^\infty dx\, p(x)\left[\int_0^x \exp\{-\lambda y/x\} p(y)\, dy\right]^{n-1}. \end{aligned} \tag{5.4.13}$$

Changing the variable $t = y/x$ in the inner integral and introducing the variable $s = x/b_n$ in the outer one, we bring (5.4.13) into the form

$$f_n(\lambda) = ne^{-\lambda}\left\{b_n \int_0^\infty ds\, p(b_n s)[b_n s \int_0^1 e^{-\lambda t} p(b_n st)\, dt]^{n-1}\right\}. \tag{5.4.14}$$

It is convenient to represent the expression in the square brackets as

$$b_n s \int_0^1 e^{-\lambda t} p(b_n st)\, dt = 1 - [1 - F(b_n s)] - b_n s \int_0^1 [1 - e^{-\lambda t}] p(b_n st)\, dt, \tag{5.4.15}$$

where

$$F(b_n s) = b_n s \int_0^1 p(b_n st)\, dt.$$

Using formulae (5.4.8) and (5.4.9), we obtain

$$b_n s \int_0^1 e^{-\lambda t} p(b_n st)\, dt \sim 1 - (b_n s)^\alpha A\psi(\lambda), \qquad n \to \infty, \tag{5.4.16}$$

where

$$\psi(\lambda) \equiv 1 + \alpha \int_0^1 (1 - e^{-\lambda t}) t^{-\alpha-1}\, dt.$$

Substituting the obtained expressions into (5.4.14), we arrive at

$$\begin{aligned} f_n(\lambda) &\sim n^{1+1/\alpha} e^{-\lambda} \int_0^\infty ds\, p(n^{1/\alpha}s)\left[1 - \frac{s^{-\alpha}A}{n}\psi(\lambda)\right]^{n-1} \\ &\sim n^{1+1/\alpha} e^{-\lambda} \int_0^\infty ds\, p(n^{1/\alpha}s) \exp\left\{-s^{-\alpha}A\psi(\lambda)\right\}, \quad n \to \infty. \end{aligned} \tag{5.4.17}$$

As we have seen while deriving (5.4.16),

$$p\left(n^{1/\alpha}s\right) \sim \alpha A \left(n^{1/\alpha}s\right)^{-\alpha-1},$$

so we find the limiting expression for (5.4.17):

$$\lim_{n\to\infty} f_n(\lambda) = e^{-\lambda}\alpha A \int_0^\infty ds\, s^{-\alpha-1}\exp\left\{-s^{-\alpha}A\psi(\lambda)\right\}.$$

Setting

$$\sigma = s^{-\alpha}A\psi(\lambda),$$

we arrive at

$$f(\lambda) \equiv \lim_{n\to\infty} f_n(\lambda) = \frac{e^{-\lambda}}{\psi(\lambda)} = \frac{e^{-\lambda}}{1+\alpha\int_0^1(1-e^{-\lambda t})t^{-\alpha-1}dt}, \qquad (5.4.18)$$

which was first obtained (in terms of characteristic functions) in (Darling, 1952).

Formula (5.4.17) allows us to find the mathematical expectation of the ratio under consideration:

$$\mathsf{E}S_n/M_n = -f'(0) = \frac{1}{1-\alpha}.$$

Thus, a more rigorous analysis confirms the qualitative reasoning given in Section 2.5: if $\alpha < 1$, then the sum S_n is comparable to its maximum term. As is seen from the last formula, this property is violated as $\alpha \to 1$. The square of relative fluctuations is also expressed in a simple form:

$$\frac{\mathrm{Var}(S_n/M_n)}{[\mathsf{E}(S_n/M_n)]^2} = \frac{f''(0)}{[f'(0)]^2} = \frac{\alpha}{2-\alpha}.$$

As α decreases, so does the fluctuation, because the influence of the maximum term on the sum increases. As α approaches 1, the fluctuations are growing, coming close to 1.

5.5. Laplace transformation of two-sided distributions

We turn to the consideration of the one-sided Laplace transforms of two-sided stable distributions

$$\tilde{q}(\lambda;\alpha,\beta) = \int_0^\infty e^{-\lambda x}q(x;\alpha,\beta)\,dx. \qquad (5.5.1)$$

By virtue of property (c) (Section 4.9), in order to give a complete description of the density $q(x;\alpha,\beta)$ on the whole axis x, it suffices to evaluate (5.5.1) for all $\beta \in [-1,1]$.

For the sake of convenience, we slightly alter the expression of a stable density.

Let $\alpha < 1$ and $x > 0$. By virtue of (4.4.1),

$$\begin{aligned} q(x;\alpha,\beta) &= \pi^{-1}\Re\int_0^\infty e^{izx}g^+(z;\alpha,-\beta)\,dz \\ &= \pi^{-1}\Re\int_0^\infty \exp\left\{izx - z^\alpha \exp(i\beta\alpha\pi/2)\right\}\,dz \\ &= -\pi^{-1}\Im\int_0^\infty \exp\left\{-xu - u^\alpha \exp[i(1+\beta)\alpha\pi/2]\right\}\,du \\ &= \pi^{-1}\Im\int_0^\infty \exp\left\{-xu - u^\alpha \exp(-i\rho\pi)\right\}\,du, \end{aligned}$$

where $\rho = (1+\beta)\alpha/2$. By rotating the integration contour through $\rho\pi/(2\alpha)$, we obtain

$$q(x;\alpha,\beta) = \pi^{-1}\Im\int_0^\infty \exp\left\{-xue^{i\rho\pi/(2\alpha)} - u^\alpha e^{-i\rho\pi/2} + i\rho\pi/(2\alpha)\right\}\,du. \tag{5.5.2}$$

We substitute (5.5.2) into the right-hand side of (5.5.1) and change the integration order (this operation is valid because the double integral converges absolutely), thus obtaining

$$\begin{aligned} &\tilde{q}(\lambda;\alpha,\beta) \\ &= \pi^{-1}\Im\int_0^\infty du\exp\left\{-u^\alpha e^{-i\rho\pi/2} + i\rho\pi/(2\alpha)\right\}\int_0^\infty \exp\left\{-x[\lambda + ue^{i\rho\pi/(2\alpha)}]\right\}\,dx \\ &= \pi^{-1}\Im\int_0^\infty \exp\left\{-(ue^{-i\rho\pi/(2\alpha)})^\alpha + i\rho\pi/(2\alpha)\right\}\left[\lambda + ue^{i\rho\pi/(2\alpha)}\right]^{-1}\,du. \end{aligned}$$

Rotating the integration contour through $-\rho\pi/(2\alpha)$ and introducing a new variable, we obtain

$$\tilde{q}(\lambda;\alpha,\beta) = \pi^{-1}\int_0^\infty \exp\{-u^\alpha\}\frac{e^{i\rho\pi/\alpha} + u}{|\lambda + ue^{i\rho\pi/\alpha}|^2}\,du.$$

After one more passage to a new variable (from λu to u) we arrive at

$$\tilde{q}(\lambda;\alpha,\beta) = \pi^{-1}\int_0^\infty \exp\{-(\lambda u)^\alpha\}\frac{\sin(\rho\pi/\alpha)}{u^2 + 2u\cos(\rho\pi/\alpha) + 1}\,du, \tag{5.5.3}$$

which, as was established in (Zolotarev, 1986), remains true for any $\alpha \neq 1$ and any admissible β.

Similar reasoning proves that for $\alpha = 1$ and any admissible β

$$\tilde{q}(\lambda;1,\beta) = \pi^{-1}\int_0^\infty \left[\lambda\cos(\beta u\ln u) - u\sin(\beta u\ln u)\right]\frac{\exp\{-u\pi/2\}}{\sqrt{\lambda^2 + u^2}}\,du, \tag{5.5.4}$$

whereas for $\beta > 0$

$$\tilde{q}(\lambda; 1, \beta) = \pi^{-1} \int_0^\infty \exp\{-\beta u \ln u\} \frac{\sin(u\rho\pi)}{\lambda + u} du. \quad (5.5.5)$$

To elucidate the behavior of transform (5.5.3) for $\alpha < 1$ and $\beta \to 1$, that is, for $\rho \to \alpha$, we consider the function

$$\psi_\rho(u) = \frac{\alpha \sin(\rho\pi/\alpha)}{\rho\pi[u^2 + 2u\cos(\rho\pi/\alpha) + 1]}.$$

Since

$$\int_0^\infty \frac{du}{u^2 + 2u\cos(\rho\pi/\alpha) + 1} = \frac{\rho\pi}{\alpha \sin(\rho\pi/\alpha)},$$

we obtain

$$\int_0^\infty \psi_\rho(u) du = 1.$$

At the same time, for $u \neq 1$

$$\psi_\rho(u) \to 0, \qquad \rho \to \alpha.$$

Thus, $\lim \psi_\rho(u)$ as $\rho \to \alpha$ can be considered as the Dirac δ-function $\delta(u - 1)$, and formula (5.5.3) takes the form

$$\tilde{q}(\lambda; \alpha, 1) = e^{-\lambda^\alpha},$$

which coincides with that of the preceding section (5.4.6).

As concerns the two-sided Laplace transforms, since for $\alpha < 2$ and $\beta \neq \pm 1$ both tails of a stable law behave as a power function, $\hat{f}(\lambda)$ exists for $\lambda = -ik$, $k \in \mathbb{R}$ only, i.e., in the only case where it coincides with the characteristic function. In the case where $\beta = 1$ and $\Re\lambda \geq 0$, though,

$$\ln \hat{f}(\lambda; \alpha, 1) = \begin{cases} \lambda^\alpha, & \alpha > 1, \\ \lambda \ln \lambda, & \alpha = 1, \\ -\lambda^\alpha, & \alpha < 1. \end{cases}$$

5.6. The Mellin transformation

DEFINITION OF THE MELLIN TRANSFORM. The Mellin transform of a function $g(x)$ defined on the positive semi-axis and satisfying the condition

$$\int_0^\infty |g(x)| x^\sigma dx < \infty$$

is the function

$$\bar{g}(s) = \int_0^\infty g(x) x^s dx, \qquad \Re s = \sigma.$$

If we denote by $\hat{f}(\lambda)$ the two-sided Laplace transform of

$$f(x) = g(e^x),$$

then we obtain the relation

$$\hat{f}(-s-1) = \bar{g}(s).$$

This relation allows us to derive all properties of the Mellin transforms from the corresponding properties of the Laplace transforms; in particular, the inversion formula is of the form

$$g(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \bar{g}(s)x^{-s-1}\tilde{\mu}ds, \qquad x > 0,$$

whereas the convolution formulae are

$$\overline{\int_0^\infty f(x')g(x/x')\,dx'/x'} = \bar{f}(s)\bar{g}(s),$$

$$\overline{\int_0^\infty f(x')g(xx')\,dx'} = \bar{f}(1-s)\bar{g}(s).$$

Let us dwell on the Mellin transforms of strictly stable densities:

$$\bar{q}(s;\alpha,\delta) = \int_0^\infty x^s q^C(x;\alpha,\delta)\,dx. \tag{5.6.1}$$

Let $\alpha \neq 1$. We substitute (5.5.2) coinciding with the density $q^C(x;\alpha,\delta)$ for $\rho = (\delta+\alpha)/2$ $(-1 < s < 0)$ into (5.6.1), and obtain

$$\bar{q}(s;\alpha,\delta) = \pi^{-1}\Im\int_0^\infty dx\,x^s\int_0^\infty\left\{-xue^{i\rho\pi/(2\alpha)} - u^\alpha e^{-i\rho\pi/2} + i\rho\pi/(2\alpha)\right\}du.$$

Changing the integration order, we obtain

$$\begin{aligned}
\bar{q}(s;\alpha,\delta) &= \pi^{-1}\Im\int_0^\infty du\exp\left\{-u^\alpha e^{-i\rho\pi/2} + i\rho\pi/(2\alpha)\right\}\\
&\quad\times\int_0^\infty x^s\exp\left\{-xue^{i\rho\pi/(2\alpha)}\right\}dx\\
&= \pi^{-1}\Gamma(1+s)\Im\int_0^\infty\left[ue^{i\rho\pi/(2\alpha)}\right]^{-s}\exp\left\{-u^\alpha e^{-i\rho\pi/2}\right\}\frac{du}{u}.
\end{aligned}$$

Rotating the integration contour through $-\rho\pi/(2\alpha)$, after an appropriate change of variable we arrive at the following expression for the Mellin transform of the density $q^C(x;\alpha,\delta)$:

$$\begin{aligned}
\bar{q}(s;\alpha,\delta) &= \pi^{-1}\Gamma(1+s)\sin(\rho s\pi/\alpha)\int_0^\infty u^{-s-1}\exp\{-u^\alpha\}\,du\\
&= (\alpha\pi)^{-1}\Gamma(1+s)\Gamma(-s/\alpha)\sin(\rho s\pi/\alpha)\\
&= \frac{\sin(\rho s\pi/\alpha)}{\sin(s\pi)}\frac{\Gamma(1-s/\alpha)}{\Gamma(1-s)}.
\end{aligned} \tag{5.6.2}$$

Either sides of the last equality are analytic in the strip $-1 < \Re s < \alpha$; therefore, the equality validated for $-1 < s < 0$ remains true for all values of s belonging to that strip.

We have just considered the case $\alpha \neq 1$, but the formula

$$\bar{q}(s;\alpha,\delta) = \frac{\sin(\rho s\pi/\alpha)\Gamma(1-s/\alpha)}{\sin(s\pi)\Gamma(1-s)}, \quad \rho = (\delta+\alpha)/2, \quad -1 < \Re s < \alpha, \quad (5.6.3)$$

remains good for $\alpha = 1$ as well due to the continuity of function (5.6.1) at the point $\alpha = 1$.

Transform (5.6.3) can also be represented as

$$\bar{q}(s;\alpha,\delta) = \frac{\Gamma(s)\Gamma(1-s/\alpha)}{\Gamma(\rho s/\alpha)\Gamma(1-\rho s/\alpha)}. \quad (5.6.4)$$

5.7. The characteristic transformation

The characteristic transformation will be the last of the transformations we consider. It is more convenient to formulate the action of the transformation in terms of random variables.

DEFINITION. Let

$$w_0(k)_X = \mathsf{E}|X|^{ik} \quad (5.7.1)$$

and

$$w_1(k)_X = \mathsf{E}|X|^{ik}\operatorname{sign} X \quad (5.7.2)$$

with $k \in \mathbb{R}$. Then the 2×2 diagonal matrix

$$W_X(k) = \begin{pmatrix} w_0(k)_X & 0 \\ 0 & w_1(k)_X \end{pmatrix} \quad (5.7.3)$$

is referred to as the characteristic transform of the density $p_X(x)$.

Transform (5.7.3) was first introduced in (Zolotarev, 1962b) and later found an application in multiplicative problems of number theory. The characteristic transforms of random variables play the same role in the scheme of multiplication of random variables as the characteristic functions do in the scheme of summation. It is not hard to reveal this similarity in the light of the following properties of characteristic transforms.

(1) The characteristic transform exists for any random variable X.

This immediately follows from the definition (5.7.1)–(5.7.2) of the functions $w_j(k)_X$.

(2) The distribution F_X is uniquely determined by the characteristic transform W_X.

Indeed, let

$$c^+ = \mathsf{P}\{X > 0\}, \qquad c^- = \mathsf{P}\{X < 0\},$$
$$c^+ f^+(k) = \mathsf{E}|X|^{ik} e(X), \qquad c^- f^-(k) = \mathsf{E}|X|^{ik} e(-X),$$

where f^+ and f^- are some characteristic functions that uniquely relate to the parts of the distribution $F_X(x)$ on the semi-axis $x > 0$ and $x < 0$, respectively, while the corresponding coefficients c^+ and c^- are non-zero, because $e(x)$ is a step function. We obtain

$$c^+ + c^- = 1 - \mathsf{P}\{X = 0\}, \qquad w_j(k)_X = c^+ f^+(k) + (-1)^j c^- f^-(k), \quad j = 0, 1,$$
$$c^+ f^+(k) = [w_0(k)_X + w_1(k)_X]/2, \quad c^- f^-(k) = [w_0(k)_X - w_1(k)_X)]/2.$$

Therefore, the distribution F_X can indeed be reconstructed if we are given the functions w_0 and w_1.

(3) If U and V are independent random variables, then for their product $X = UV$

$$W_X(k) = W_U(k) W_V(k), \qquad k \in \mathbb{R}. \tag{5.7.4}$$

Indeed,

$$w_j(k)_X = \mathsf{E}|UV|^{ik} (\operatorname{sign} UV)^j = w_j(k)_U w_j(k)_V, \qquad j = 0, 1,$$

which implies (5.7.4).

We assume that $q = q^C(x; \alpha, \delta)$ and s is a complex number in the strip $-1 < \Re < \alpha$. Let

$$W(s; \alpha, \delta) = \begin{pmatrix} w_0(s; \alpha, \delta) & 0 \\ 0 & w_1(s; \alpha, \delta) \end{pmatrix}.$$

This function exists for any s lying in the strip and for any admissible values of the parameters; it closely relates to the Mellin transforms. Indeed,

$$w_0(s; \alpha, \delta) = \bar{q}(s; \alpha, \delta) + \bar{q}(s; \alpha, -\delta),$$
$$w_1(s; \alpha, \delta) = \bar{q}(s; \alpha, \delta) - \bar{q}(s; \alpha, -\delta).$$

Therefore, the assertions concerning the transforms $\bar{q}$ remain true also for the elements w_j of the transform W. Formula (5.6.3) allows us to obtain easily an explicit expression for the functions $w_j(s;\alpha,\delta)$. For $j = 0$,

$$\begin{aligned} w_0(s;\alpha,\delta) &= \bar q(s;\alpha,\delta) + \bar q(s;\alpha,-\delta) \\ &= \frac{\sin[(1+\delta/\alpha)s\pi/2]}{\sin(\pi s)}\frac{\Gamma(1-s/\alpha)}{\Gamma(1-s)} \\ &\quad + \frac{\sin[(1-\delta/\alpha)s\pi/2]}{\sin(\pi s)}\frac{\Gamma(1-s/\alpha)}{\Gamma(1-s)} \\ &= \frac{\cos[\delta s\pi/(2\alpha)]}{\cos(s\pi/2)}\frac{\Gamma(1-s/\alpha)}{\Gamma(1-s)}. \end{aligned} \tag{5.7.5}$$

This is nothing but the absolute moment of the random variable $Y_C(\alpha,\delta)$ of order s:

$$\mathsf{E}|Y_C(\alpha,\delta)|^s \equiv \int_{-\infty}^{\infty} |x|^s q^C(x;\alpha,\delta)\,dx = \frac{\cos[\delta s\pi/(2\alpha)]}{\cos(s\pi/2)}\frac{\Gamma(1-s/\alpha)}{\Gamma(1-s)}. \tag{5.7.6}$$

In the extreme cases $\alpha < 1$, $|\delta| = \alpha$, the moments take the more simple form

$$\mathsf{E}|Y_C(\alpha,\pm\alpha)|^s = \frac{\Gamma(1-s/\alpha)}{\Gamma(1-s)}, \tag{5.7.7}$$

which can be continued to the whole negative semi-axis ($-\infty < s < \alpha$).

In a perfectly similar way, we obtain

$$\begin{aligned} w_1(s;\alpha,\delta) &= \bar q(s;\alpha,\delta) - \bar q(s;\alpha,-\delta) \\ &= \frac{\sin[\delta s\pi/(2\alpha)]}{\sin(s\pi/2)}\frac{\Gamma(1-s/\alpha)}{\Gamma(1-s)}. \end{aligned} \tag{5.7.8}$$

5.8. The logarithmic moments

Although the random variables $|Y(\alpha,\delta)| \equiv |Y_C(\alpha,\delta)|$ have finite moments only of order less than α, their logarithms have moments of all orders, and this sometimes (for example, in certain statistical problems) turns out to be a very valuable property. It happens that moments of logarithmic type have a relatively simple form of expression. Below, we dwell upon the computation of

$$y_{jn}(\alpha,\delta) = \mathsf{E}(\ln|Y(\alpha,\delta)|)^n(\operatorname{sign} Y(\alpha,\delta))^j, \quad j = 0,1, \quad n = 1,2,\dots$$

It turns out that the logarithmic moments y_{jn} are polynomials of degree n in $1/\alpha$ and polynomials of degree at most $n+1$ in δ. Explicit expressions for them can be written with the use of the Bell polynomials $C_n(u_1,\dots,u_n)$ which were introduced in Section 4.7.

We introduce $Q_1, Q_2, \dots$ by setting $Q_1 = (1/\alpha - 1)\mathsf{C}$, where C is Euler's constant, and

$$Q_j = A_j\pi^j|B_j|/j + (1/\alpha^j - 1)\Gamma(j)\zeta(j) \qquad \text{for integer } j \ge 2,$$

where B_j are the Bernoulli numbers, $\zeta(j)$ is the value of the Riemann ζ-function at j, and A_j are defined for the logarithmic moments as follows:

$$A_j = (2^j - 1)[1 - (\delta/\alpha)^j] \quad \text{for the moments } y_{0n},$$
$$A_j = 1 - (\delta/\alpha)^j \quad \text{for the moments } y_{1n}.$$

THEOREM 5.8.1. *For any admissible values of the parameters* (α, δ) *the following equalities are valid* ($n = 1, 2, \ldots$):

$$y_{0n} = C_n(Q_1, Q_2, \ldots, Q_n), \tag{5.8.1}$$
$$y_{1n} = C_n(Q_1, Q_2, \ldots, Q_n)\delta/\alpha.$$

PROOF. Since

$$w_j(s; \alpha, \delta) = \mathsf{E}|Y(\alpha, \delta)|^s (\operatorname{sign} Y(\alpha, \delta))^j,$$

we conclude that

$$y_{jn} = (d/ds)^n w_j(s; \alpha, \delta)|_{s=0}$$

are the coefficients in the series expansion of the functions w_j in powers of s. However, it is more convenient to expand these functions not directly but after expanding their logarithms in series. We observe that in the case $\delta = 0$ we obtain $w_1(s; \alpha, \delta) = 0$, so the computation of y_{1n} should be carried out under the additional condition that $\delta \neq 0$. We obtain

$$\ln w_0(s; \alpha, \delta) = \ln \cos[s\delta\pi/(2\alpha)] - \ln \cos(s\pi/2) + \ln \Gamma(1 - s/\alpha) - \ln \Gamma(1 - s).$$

The function $\ln \cos x$ can be expanded in a power series just like $\ln \Gamma(1-x)$ (see (Gradshtein & Ryzhik, 1963, 1.518, 8.342)):

$$\ln \cos x = -\sum_{k=0}^{\infty} \frac{2^k(2^k - 1)|B_k|}{k\Gamma(k+1)} x^k,$$
$$\ln \Gamma(1 - x) = Cx + \sum_{k=2}^{\infty} \frac{\zeta(k)}{k!} x^k.$$

With the use of these formulae, we obtain

$$\ln w_0(s; \alpha, \delta) = \sum_{j=1}^{\infty} Q_j \frac{s^j}{j!},$$

where Q_j are the same as above. If we now use Bruno's formula, we find that

$$w_0(s; \alpha, \delta) = \exp\left(\sum_{j=1}^{\infty} Q_j \frac{s^j}{j!}\right) = 1 + \sum_{n=1}^{\infty} C_n(Q_1, Q_2, \ldots, Q_n) \frac{s^n}{n!},$$

which implies (5.8.1).

Considering $w_1(s;\alpha,\delta)$ with $\delta \neq 0$, it is convenient to rewrite it as

$$w_1(s;\alpha,\delta) = \delta \frac{\sin[s\delta\pi/(2\alpha)]}{\delta \sin(s\pi/2)} \frac{\Gamma(1-s/\alpha)}{\Gamma(1-s)} = \delta h(s)$$

and expand the function $h(s)$ in a series in powers of s. Of course, all series expansion coefficients for w_1 are proportional to δ. The expansion of $h(s)$ is carried out just like that of w_0, with, however, the difference that the formula (Gradshtein & Ryzhik, 1963, 1.518)

$$\ln\left(\frac{x}{\sin x}\right) = \sum_{k=2}^{\infty} \frac{2^k |B_k|}{k\Gamma(k+1)} x^k$$

is used here.

Since Q_j, $j \geq 2$, are polynomials of degree j in $1/\alpha$ and δ, by turning back to the explicit expression for the Bell polynomials it is not hard to see that $C_n(Q_1, \dots, Q_n)$ possesses the same property. In the case of the logarithmic moment y_{1n} the maximal power of δ is equal to $n+1$, because $y_{1n} = C_n\delta/\alpha$.

In conclusion, we give expressions for the logarithmic moments when $n = 1$ and $n = 2$, assuming that $C_1(Q_1) = Q_1$ and $C_2(Q_1, Q_2) = Q_1^2 + Q_2$:

$$y_{01} \equiv \mathsf{E} \ln |Y(\alpha,\delta)| = (1/\alpha - 1)C, \tag{5.8.2}$$

$$y_{11} \equiv \mathsf{E}[\ln |Y(\alpha,\delta)| \operatorname{sign} Y(\alpha,\delta)] = (1/\alpha - 1)2C\delta/\alpha, \tag{5.8.3}$$

$$y_{02} \equiv \mathsf{E} \ln^2 |Y(\alpha,\delta)| = (1/\alpha - 1)^2 C^2 + (2/\alpha^2 + 1)\pi^2/12 - [\pi\delta/(2\alpha)]^2, \tag{5.8.4}$$

$$y_{12} \equiv \mathsf{E}[\ln^2 |Y(\alpha,\delta)| \operatorname{sign} Y(\alpha,\delta)] = [(1/\alpha - 1)^2 C^2 + (2/\alpha^2 - 1)\pi^2/12 + (\pi\delta/(2\alpha))^2/3]\delta/\alpha. \tag{5.8.5}$$

5.9. Multiplication and division theorems

Using the transformations considered above, we can supplement our collection of the relations between random variables given in Section 3.7.

To avoid awkward formulae, it makes sense to agree on some more notation. If X is a random variable, then for any complex number s we assume that

$$X^s = |X|^s \operatorname{sign} X \tag{5.9.1}$$

with $0^s = 0$ for any s. If X is not negative with probability one, then this generalized understanding of a power obviously coincides with the traditional one. Powers of random variables in the sense of (5.9.1), obviously, possess the basic properties of ordinary powers:

$$(X^s)^r = X^{sr}, \qquad X_1^s X_2^s = (X_1X_2)^s,$$

but there is also an essential difference, because

$$X^0 = \operatorname{sign} X, \qquad (-X)^s = -X^s$$

for any s.

Here it is appropriate to point out a peculiarity of equalities between random variables in the sense of $\stackrel{d}{=}$. If $X'+X \stackrel{d}{=} X''+X$ or $X'X \stackrel{d}{=} X''X$, then, generally speaking, the relation $X' \stackrel{d}{=} X''$ is not necessarily true. However, if the characteristic function $f_X(t)$ is non-zero for almost all t or, what is the same, $w_k(t)_X$ are non-zero almost everywhere, then this equality becomes valid. Of course, the reverse is valid without any restrictions imposed on the distribution of X.

The use of relations between random variables in place of writing out relations between the corresponding distributions makes even more sense in the present chapter than in Section 3.7. The point is that we have to deal with products $X = X_1X_2$ of independent random variables, and the distribution function F of the product X is expressed in terms of the distribution functions F_1 and F_2 of the factors in a form being considerably more cumbersome than, say, the convolution of F_1 and F_2. Even in the simplest case, where F_1 and F_2 are continuous,

$$F(x) = [1 - F_2(0)] \int_0^\infty F_1(x/y)\, dF_2(y) + F_2(0) \int_{-\infty}^0 [1 - F_1(x/y)]\, dF_2(y).$$

We use below form C with parameter $\theta = \delta/\alpha$ instead of δ, and assume that the random variables written separately on one side of an equality are independent.

THEOREM 5.9.1. *For any admissible* (α, θ) *and* (α, θ'),

$$\frac{Y(\alpha, \theta)}{Y(\alpha, \theta')} \stackrel{d}{=} \frac{Y(\alpha, \theta')}{Y(\alpha, \theta)}. \tag{5.9.2}$$

PROOF. All such relations can be proved by a universal method. The characteristic transforms of the distributions of the left-hand and right-hand sides of the equalities are computed and compared. Since the characteristic transforms, like the characteristic functions, uniquely determine the distributions related to them, coincidence of the characteristic transforms means coincidence of the corresponding distributions.

In this case, the characteristic transform of the left-hand side is

$$w_j(s; \alpha, \theta) w_j(-s; \alpha, \theta') = \frac{\cos[(j - s\theta)\pi/2]\Gamma(1 - s/\alpha)}{\cos[(j - s)\pi/2]\Gamma(1 - s)} \times \frac{\cos[(j + s\theta')\pi/2]\Gamma(1 + s/\alpha)}{\cos[(j + s)\pi/2]\Gamma(1 + s)}, \qquad j = 0, 1.$$

Since

$$\frac{\cos[(j-s\theta)\pi/2]}{\cos[(j-s)\pi/2]} = \frac{\cos[(j+s\theta)\pi/2]}{\cos[(j+s)\pi/2]},$$

for $j = 0, 1$ we obtain

$$w_j(s;\alpha,\theta)w_j(-s;\alpha,\theta') = w_j(-s;\alpha,\theta)w_j(s;\alpha,\theta'), \qquad j = 0, 1,$$

i.e., the distributions of the left- and right-hand sides of (5.9.2) coincide.

THEOREM 5.9.2. *Let (α, θ) and (α', θ') be pairs of admissible values of parameters, and let ν be a number such that*

$$\alpha'(1 + |\theta|)/2 \le \nu \le [\alpha(1 + |\theta'|)/2]^{-1}. \tag{5.9.3}$$

Then

$$Y(\alpha,\theta)Y^{\nu}(\alpha',\theta') \stackrel{d}{=} Y(\alpha'/\nu,\theta)Y^{\nu}(\alpha\nu,\theta'). \tag{5.9.4}$$

PROOF. Let us compute the characteristic transform of the left-hand side of (5.9.4). Using (5.7.4), (5.7.5), and (5.7.8), we obtain

$$w_j(s;\alpha,\theta)w_j(s\nu;\alpha',\theta') = \frac{\cos[(j-s\theta)(\pi/2)]}{\cos[(j-s)(\pi/2)]}\frac{\Gamma(1-s/\alpha)}{\Gamma(1-s)} \times \frac{\cos[(j-s\nu\theta')(\pi/2)]}{\cos[(j-s\nu)(\pi/2)]}\frac{\Gamma(1-s/\alpha')}{\Gamma(1-s\nu)}.$$

Similarly, for the right-hand side

$$w_j\left(s;\alpha'/\nu,\theta\right)w_j(s\nu;\alpha\nu,\theta') = \frac{\cos[(j-s\theta)(\pi/2)]}{\cos[(j-s)(\pi/2)]}\frac{\Gamma(1-s\nu/\alpha')}{\Gamma(1-s)} \times \frac{\cos[(j-s\nu\theta')(\pi/2)]}{\cos[(j-s\nu)(\pi/2)]}\frac{\Gamma(1-s/\alpha)}{\Gamma(1-s\nu)}.$$

It is obvious that the expressions coincide. It remains to verify that the parameters in the right-hand side of (5.9.4) lie in the domain of admissible values. We know that $|\theta| \le \min(1, 2\nu/\alpha - 1)$, and therefore, the conditions

$$|\theta| \le \min\left\{1, 2\nu/\alpha' - 1\right\}, \qquad |\theta'| \le \min\left\{1, 2/(\alpha\nu) - 1\right\}$$

imposed on the right-hand side hold if and only if

$$|\theta| \le 2\nu/\alpha' - 1, \qquad |\theta'| \le 2/(\alpha\nu) - 1.$$

These inequalities are equivalent to (5.9.3), though.

THEOREM 5.9.3. *Let* (α, θ) *and* (α', θ') *be pairs of admissible values of the parameters, and let* μ *be any real number such that*

$$|\theta|/\min\{1, 2/\alpha' - 1\} \le |\mu| \le \min\{1, 2/\alpha - 1\}/|\theta'|. \tag{5.9.5}$$

Then

$$\begin{aligned} Y(\alpha, \theta)Y^{\mu}(\alpha', \theta') &\stackrel{d}{=} Y(\alpha, \theta'\mu)Y^{\mu}(\alpha', \theta/\mu) \\ &\stackrel{d}{=} Y(\alpha, -\theta'\mu)Y^{\mu}(\alpha', -\theta/\mu). \end{aligned} \tag{5.9.6}$$

The proof of the first part of (5.9.6) is similar to that of (5.9.4). The second part follows from (3.7.3)

$$-Y(\alpha, \theta) \stackrel{d}{=} Y(\alpha, -\theta).$$

From (5.9.6) and (5.9.4), we derive the following assertion.

COROLLARY 5.9.1. *For any pairs* (α, θ) *and* (α', θ') *of admissible values of the parameters and any number* $\mu > 0$ *such that*

$$\begin{aligned} |\theta|/\mu &\le \min\{1, 2/(\alpha\mu) - 1\}, \\ |\theta'|/\mu &\le \min\{1, 2\mu/\alpha' - 1\}, \end{aligned} \tag{5.9.7}$$

the relation

$$\begin{aligned} Y(\alpha, \theta)Y^{\mu}(\alpha', \theta') &\stackrel{d}{=} Y(\alpha'/\mu, \theta'\mu)Y^{\mu}(\alpha\mu, \theta/\mu) \\ &\stackrel{d}{=} Y(\alpha'/\mu, -\theta'\mu)Y^{\mu}(\alpha\mu, -\theta/\mu) \end{aligned} \tag{5.9.8}$$

is valid.

THEOREM 5.9.4. *Let* (α, θ) *be a pair of admissible values of the parameters, and let* $0 < \alpha' \le 1$. *Then*

$$Y(\alpha, \theta)Y^{1/\alpha}(\alpha', 1) \stackrel{d}{=} Y(\alpha\alpha', \theta). \tag{5.9.9}$$

COROLLARY 5.9.2. *If* $\alpha \le 1$ *and* $\alpha' \le 1$, *then*

$$Y(\alpha, 1)Y^{1/\alpha}(\alpha', 1) \stackrel{d}{=} Y(\alpha\alpha', 1). \tag{5.9.10}$$

COROLLARY 5.9.3. *In* (5.9.2), *set* $\alpha = 1/2$, $\theta = 1$ *and* $\alpha' = 1/2^k$. *Then*

$$Y(1/2, 1)Y^{2}(1/2^k, 1) \stackrel{d}{=} Y(1/2^{k+1}, 1). \tag{5.9.11}$$

By (2.3.17), which yields

$$Y(1/2,1) \stackrel{d}{=} Y^{-2}(2,0) \stackrel{d}{=} (2N^2)^{-1},$$

where N is a random variable with standard normal distribution (2.2.5), this relation provides us with

$$Y(1/2^n,1) \stackrel{d}{=} 2^{1-2^n} N^{-2} N^{-2^2} \dots N^{-2^n}.$$

If we set $\alpha = 1$, $\theta = 0$, and $\alpha' = 1/2^n$ in (5.9.9), then the relation

$$NN^{-1} Y(1/2^n,1) \stackrel{d}{=} Y(1/2^n,0)$$

follows from (2.3.19), and yields the relation (Brown & Tukey, 1946)

$$Y(1/2^n,0) \stackrel{d}{=} 2^{1-2^n} NN^{-1} N^{-4} \dots N^{-2^n}.$$

PROOF. We begin with (5.9.9). The components of the characteristic transform of its left-hand side are of the form

$$w_j(s;\alpha,\theta) w_j(s/\alpha;\alpha',1) = \frac{\cos[(j-s\theta)\pi/2]\Gamma(1-s/(\alpha\alpha'))}{\cos[(j-s)\pi/2]\Gamma(1-s)} = w_j(s;\alpha\alpha',\theta).$$

Moreover, since $\alpha' \le 1$, we obtain

$$|\theta| \le \min\{1, 2\alpha-1\} \le \min\{1, 2/(\alpha\alpha')-1\},$$

i.e., the pair $(\alpha\alpha', \theta)$ belongs to the domain of admissible values.

The proof of the following theorem is carried out in the same way.

THEOREM 5.9.5. *For any pairs (α,θ) and $(1,\theta')$ of admissible values,*

$$Y(\alpha,\theta) Y^{\theta}(1,\theta') \stackrel{d}{=} Y(\alpha,\theta\theta'). \tag{5.9.12}$$

COROLLARY 5.9.4. *For any pair (α,θ) of admissible values,*

$$Y(\alpha,\theta) \stackrel{d}{=} Y(\alpha,\theta_\alpha) Y^{\theta_\alpha}(1,\theta/\theta_\alpha),$$

where

$$\theta_\alpha = \min\{1, 2/\alpha - 1\}.$$

In particular, if $\alpha \le 1$, then

$$Y(\alpha,\theta) \stackrel{d}{=} Y(\alpha,1) Y(1,\theta),$$

i.e., the random variable $Y(\alpha,\theta)$ can be factorized in two independent stable components $Y(\alpha,1)$ and $Y(1,\theta)$.

In the remaining part of this section, we deal with the extreme strictly stable distribution with parameters (α, θ_α). As a matter of fact, the main role among them is played by the distributions with $\alpha < 1$, because $\theta = 1$ corresponds to the distribution degenerate at the point $x = 1$, while the distributions with $\alpha > 1$ (more precisely, the part of them concentrated on the semi-axis $x > 0$) can, by the duality property, be expressed in terms of the distributions of the first group. The random variables $Y(\alpha, 1)$, $0 < \alpha < 1$, whose distributions form this group (and only they) are positive with probability one. The abbreviated notation $Y(\alpha)$ will be used below for the variables $Y(\alpha, 1)$.

THEOREM 5.9.6. *Assume that* $\omega_1, \dots, \omega_n$ *are chosen so that* $0 < \omega_j < 1$. *Then for any* $n \geq 2$

$$Y^{\alpha_n}(\alpha_n) \stackrel{d}{=} Y^{\omega_1}(\omega_1) Y^{\omega_1 \omega_2}(\omega_2) \dots Y^{\omega_1 \cdots \omega_2}(\omega_n), \tag{5.9.13}$$

where $\alpha_n = \omega_1 \omega_2 \dots \omega_n$.

We prove (5.9.13) by induction. For $n = 2$, (5.9.13) coincides with (5.9.10). If it is true for a set of $n - 1$ variables, then, by virtue of (5.9.10),

$$Y(\alpha_n) \stackrel{d}{=} Y(\alpha_{n-1}\omega_n) \stackrel{d}{=} Y(\alpha_{n-1}) Y^{1/\alpha_{n-1}}(\omega_n).$$

If we now replace $Y(\alpha_{n-1})$ by its expression given by (5.9.13), we arrive at (5.9.13) with n random variables.

Despite the fact that (5.9.13) is a simple corollary to (5.9.10), it serves as the source of a great body of interesting interconnections between the distributions under consideration.

We will frequently come up against infinite products of positive random variables; by convention, these products converge if the series of logarithms of their factors converge with probability one.

The condition that the product $\mathsf{E}X_1^s \mathsf{E}X_2^s \dots$ converges for some real $s \neq 0$ is a quite convenient necessary and sufficient condition for the convergence of an infinite product $X_1 X_2 \dots$ of independent and positive (with probability one) random variables. It is not difficult to see if one reduces the question of convergence of a product of random variables to the question of convergence of a sum of random variables.

Let E be a random variable with exponential distribution

$$\mathsf{P}\{E > x\} = e^{-x}, \qquad x \geq 0.$$

THEOREM 5.9.7. *Assume that* $\omega_1, \omega_2, \dots$ *is a numerical sequence,* $0 < \omega_j \leq 1$, *and* $\alpha = \omega_1 \omega_2 \dots$ *is their product. If* $\alpha > 0$, *then*

$$Y(\alpha) \stackrel{d}{=} Y(\omega_1) Y^{1/\omega_1}(\omega_2) Y^{1/(\omega_1 \omega_2)}(\omega_3) \dots, \tag{5.9.14}$$

$$Y^{\alpha}(\alpha) \stackrel{d}{=} Y^{\omega_1}(\omega_1) Y^{\omega_1 \omega_2}(\omega_2) \dots. \tag{5.9.15}$$

If $\alpha = 0$, *then*

$$Y^{\omega_1}(\omega_1)Y^{\omega_1\omega_2}(\omega_2)\ldots \stackrel{d}{=} 1/E;$$

in particular, for any $0 < \omega < 1$

$$Y^{\omega}(\omega)Y^{\omega^2}(\omega)Y^{\omega^3}(\omega)\ldots \stackrel{d}{=} 1/E.$$

The proof of (5.9.14) and (5.9.15) is reduced to a passage to the limit as $n \to \infty$ in (5.9.13), with the use of the Mellin transforms of both sides of the equality and the criterion given above for convergence of infinite products of independent random variables.

The following relation stands rather along. Denote by Γ_ν, $\nu > 0$, a random Γ-distributed variable with parameter ν, i.e., whose density is of the form

$$p_\nu(x) = \frac{1}{\Gamma(\nu)}x^{\nu-1}e^{-x}, \qquad x \ge 0.$$

THEOREM 5.9.8. *For any integer* $n \ge 2$

$$1/Y(1/n) \stackrel{d}{=} \Gamma_{1/n}\Gamma_{2/n}\ldots\Gamma_{(n-1)/n}n^n. \tag{5.9.16}$$

COROLLARY 5.9.5. *In view of relation* (5.9.9) *with* $\alpha = 1$ *and* $\alpha' = 1/n$, *from* (5.9.16) *it follows that a stable random variable with parameters* $\alpha = 1/n$ *and* $|\theta| \le 1$ *can be represented as*

$$Y^{-1}(1/n,\theta) \stackrel{d}{=} Y^{-1}(1,\theta)\Gamma_{1/n}\Gamma_{2/n}\ldots\Gamma_{(n-1)/n}n^n \stackrel{d}{=} Y(1,\theta)\Gamma_{1/n}\Gamma_{2/n}\ldots\Gamma_{(n-1)/n}n^n.$$

PROOF. The Mellin transform of the distribution of $1/Y(1/n)$ is, in view of (5.6.4), of the form

$$\mathsf{E}Y^{-s}(1/n) = \frac{\Gamma(1+ns)}{\Gamma(1+s)}.$$

This fraction can be expanded in a product of ratios of Γ-functions by using the so-called multiplication theorem for the functions $\Gamma(x)$, $x > 0$:

$$\Gamma(x)\Gamma(x+1/n)\ldots\Gamma(x+(n-1)/n) = (2\pi)^{(n-1)/2}n^{(1-2nx)/2}\Gamma(nx).$$

We thus obtain

$$\frac{\Gamma(1+ns)}{\Gamma(1+s)} = n^{ns}\frac{\Gamma(s+1/n)}{\Gamma(1/n)}\frac{\Gamma(s+2/n)}{\Gamma(2/n)}\ldots\frac{\Gamma(s+(n-1)/n)}{\Gamma((n-1)/n)}. \tag{5.9.17}$$

Since the ratio $\Gamma(s+\nu)/\Gamma(\nu)$ is the Mellin transform of Γ_ν, the composition of the Mellin transforms in the left- and right-hand sides of (5.9.17) yields (5.9.16).

So, we have just completed the introduction to the 'multiplicative' properties of strictly stable laws. Here something should be said about where these numerous and diverse interrelations between distributions (which, for conciseness, we have clothed as relations between random variables) might turn out to be useful.

First of all, such relations can be a convenient tool in computing distributions of various kinds of functions of independent random variables $Y(\alpha, \theta)$. If we supplement the relations of a multiplicative nature in this chapter by the additive relations of Section 3.7, we obtain a kind of distinctive 'algebra' in the set of independent random variables of a special form. In particular, this helps us to get expressions of a new form for stable densities as integrals of positive functions. For example, the representations of the one-sided densities

$$q(x; 1/4, 1) = \frac{1}{2\pi} x^{-4/3} \int_0^\infty \exp\left\{-\frac{1}{4} x^{-1/3}(y^4 + y^{-2})\right\} dy,$$

$$q(x; 1/3, 1) = \frac{1}{2\pi} x^{-3/2} \int_0^\infty \exp\left\{-\frac{1}{3\sqrt{3x}}(y^3 + y^{-3})\right\} dy$$

can be obtained in such a way (Zolotarev, 1986).

The second domain of application of the 'multiplicative' properties of stable laws is the construction of various kinds of useful statistics with explicitly computable distributions.

Finally, some of the relations turn out to be useful in stochastic modeling problems to generate sequences of random variables with stable distributions.

6

Special functions and equations

6.1. Integrodifferential equations

As we might expect, there exists a large body of various integral and integrodifferential equations that link stable distributions. The investigation of these equations allows us to obtain a useful information concerning the properties of stable laws.

In Section 2.5, we gave a canonical form of expression for the characteristic function $g(k; \alpha, \beta)$ (valid both for forms A and B); let us transform it as follows:

$$g(k) \equiv g(k; \alpha, \beta) = \exp\left\{ \int_{x \neq 0} (e^{ikx} - 1 - ik \sin x)\, dH(x) \right\}$$
$$= \exp\left\{ - \int_{x \neq 0} (e^{ikx} - 1 - ik \sin x) R(x) \frac{dx}{x} \right\}, \tag{6.1.1}$$

where $R(x) = -xH'(x)$, $x \neq 0$, is a function that is non-decreasing on the semiaxis $x < 0$ and $x > 0$. It has the simplest form if the parameterization is taken in form A:

$$R(x) = \begin{cases} -\pi^{-1}\Gamma(1+\alpha)\sin(\alpha\pi/2)(1+\beta)x^{-\alpha}, & x > 0, \\ \pi^{-1}\Gamma(1+\alpha)\sin(\alpha\pi/2)(1-\beta)|x|^{-\alpha}, & x < 0. \end{cases} \tag{6.1.2}$$

The function $R(x)$ corresponding to form B is obtained by replacing the parameter $\beta = \beta^A$ in (6.1.2) by its expression in terms of α and β^B. In what follows, let

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t} dt$$

be the integral sine function.

THEOREM 6.1.1. *For either of the two forms of parameterization A or B the* *distribution function $G(x;\alpha,\beta)$ and the density $q(x;\alpha,\beta)$ satisfy the equations*

$$xG'(x;\alpha,\beta) = -\int_{y\neq 0} [G'(x;\alpha,\beta)\,\mathrm{Si}(y) - G(x-y;\alpha,\beta) + G(x;\alpha,\beta)]\,dR(y), \tag{6.1.3}$$

$$xq(x;\alpha,\beta) = \int_{y\neq 0} [q(x;\alpha,\beta)\frac{\sin y}{y} - q(x-y;\alpha,\beta)]R(y)\,dy. \tag{6.1.4}$$

PROOF. We transform (6.1.1), integrating the integral in the exponential by parts and keeping in mind that

$$\left|\int_0^u (e^{ikv} - 1 - ik\sin v)\frac{dv}{v}\right| = O(u^2) \qquad u \to 0,$$

$$\left|\int_{1\le|v|\le u} (e^{ikv} - 1 - ik\sin v)\frac{dv}{v}\right| = O(\ln u) \qquad u \to \infty.$$

As a result, we obtain

$$g(k) = \exp\left\{\int_{y\neq 0}\left[\int_0^{ky}(e^{iv}-1)\frac{dv}{v} - ik\,\mathrm{Si}(y)\right]dR(y)\right\}. \tag{6.1.5}$$

Differentiating both sides of (6.1.5) with respect to k, $k \neq 0$, we obtain

$$g'(k) = \frac{1}{k}g(k)\int (e^{iky} - 1 - ik\,\mathrm{Si}(y))\,dR(y).$$

Hence, after the change of variables $k = s/x$, $x \neq 0$, we obtain

$$\frac{d}{dx}g\left(\frac{s}{x}\right) = -\frac{1}{x}g\left(\frac{s}{x}\right)\int\left(e^{isy/x} - 1 - \frac{is}{x}\,\mathrm{Si}(y)\right)dR(y). \tag{6.1.6}$$

We transform the inversion formula

$$q(x;\alpha,\beta) = \frac{1}{2\pi}\int e^{-ikx}g\,(k)\,dk, \qquad x \neq 0,$$

by carrying out the change of variable $kx = s$:

$$xq(x;\alpha,\beta) = \frac{\operatorname{sign} x}{2\pi}\int e^{-is}g\,(s/x)\,ds.$$

Differentiating both sides of this equality with respect to x and replacing $(d/dx)g(s/x)$ by its expression (6.1.6), we obtain

$$(xq(x;\alpha,\beta))' = -\frac{\operatorname{sign} x}{2\pi}\int e^{-is}g\,(s/x)\,\frac{ds}{x}\int\left(e^{isy/x} - 1 - \frac{is}{x}\,\mathrm{Si}(y)\right)dR(y).$$

A change in the order of integration and the inverse change of variable $s = kx$ leads to the equality

$$(xq(x;\alpha,\beta))' = -\frac{1}{2\pi}\int dR(y)\int (e^{iky} - 1 - ik\,\mathrm{Si}(y))e^{-ikx}g(k)\,dk.$$

The inner integral is a linear combination of integrals, each of which is simply the inversion formula for the density or its derivative, i.e.,

$$(xq(x;\alpha,\beta))' = -\int [q(x-y;\alpha,\beta) - q(x;\alpha,\beta) + q'(x;\alpha,\beta)\,\mathrm{Si}(y)]\,dR(y). \tag{6.1.7}$$

Equality (6.1.3) is obtained by integration of both sides of the equality with respect to x from $-\infty$ to x and making use of the fact that $xg(x;\alpha,\beta) \to 0$ as $x \to -\infty$. Integration by parts in (6.1.3) leads us to (6.1.4).

REMARK 6.1.1. Although the theorem is connected with stable distributions normalized by the conditions $\gamma = 0$ and $\lambda = 1$, the assertion of the theorem extends easily to the general case. Indeed, since

$$G(x;\alpha,\beta,\gamma,\lambda) = G((x-l)\lambda^{-1/\alpha};\alpha,\beta,0,1)$$

by (3.7.2), where the quantity l is related to β, γ, and λ by simple formulae, we obtain equations for $G(x;\alpha,\beta,\gamma,\lambda)$ and $q(x;\alpha,\beta,\gamma,\lambda)$ from (6.1.3) and (6.1.4) by replacing x by $(x-l)\lambda^{-1/\alpha}$.

REMARK 6.1.2. Note that in the course of the proof we obtain an integrodifferential equation (6.1.7) for the densities of stable laws.

Integral and integrodifferential equations for stable distributions can differ (sometimes very strongly) in form, though they are all equivalent in the final analysis. The integrodifferential equation (for densities with $\alpha \neq 1$) obtained in (Medgyessy, 1958) is apparently the closest to equation (6.1.7), which was taken from (Kanter, 1975) along with (6.1.3) and (6.1.4).

6.2. The Laplace equation

Now we consider the function

$$S_\alpha(u,v) = \begin{cases} xq(x;\alpha,\beta) & \text{if } \alpha \neq 1,\ x > 0, \\ \beta q(x;\alpha,\beta) & \text{if } \alpha = 1,\ \beta > 0, \end{cases}$$

where $q(x;\alpha,\beta)$ is given in form B and the variables u and v are related to x and β as follows:

$$u = \begin{cases} -\ln x & \text{if } \alpha \neq 1, \\ x/\beta - \ln\beta & \text{if } \alpha = 1, \end{cases}$$

$$v = \begin{cases} \beta\Phi(\alpha)/\alpha & \text{if } \alpha \neq 1, \\ \pi/(2\beta) & \text{if } \alpha = 1. \end{cases}$$

Let $z = u + iv$ and define the function $\Psi_\alpha(z)$ as follows: if $0 < \alpha < 1$, then

$$\Psi_\alpha(z) = \frac{i}{\pi}\int_0^\infty \exp\left\{-k - k^\alpha \exp(-i\alpha\pi/2 + \alpha z)\right\} dk;$$

if $1 < \alpha \le 2$, then

$$\Psi_\alpha(z) = \frac{1}{\pi}\int_0^\infty \exp\left\{ike^{-z} - z - k^\alpha\right\} dk;$$

and if $\alpha = 1$, then

$$\Psi_1(z) = \frac{i}{\pi}\int_0^\infty \exp\left\{-(\pi/2 + z)k - k\ln k\right\} dk.$$

Recalling representation (4.4.1) of the density $q(x;\alpha,\beta)$, we see that

$$S_\alpha(u,v) = \Re\Psi_\alpha(z).$$

It is clear that $\Psi_\alpha(x)$ is an entire analytic function for all α.

We thus arrive at the following assertion.

THEOREM 6.2.1. *For each admissible value of α the function $S_\alpha(u,v)$ is a solution of the first boundary value problem of the Laplace equation $\Delta S_\alpha = 0$ in the strip*

$$-\infty < u < \infty, \qquad -\pi\delta_\alpha/(2\alpha) \le v \le \pi\delta_\alpha/(2\alpha)$$

for the case $\alpha \neq 1$, and in the half-plane $-\infty < u < \infty$, $v \ge \pi/2$ for the case $\alpha = 1$. The boundary conditions are

$$S_\alpha(\pm\infty, v) = 0,$$
$$S_\alpha(v, \pm\pi\delta_\alpha/(2\alpha)) = e^{-u}q\left(e^{-u};\alpha,\pm\operatorname{sign}(1-\alpha)\right)$$

for $\alpha \neq 1$, and

$$S_1(\pm\infty, v) = S_1(u,\infty) = 0, \qquad S_1(u,\pi/2) = q(u;1,1)$$

for $\alpha = 1$.

REMARK 6.2.1. Here the domain of variation of the variables u and v arises, of course, due to constraints imposed on the parameter β. But if $S_\alpha(u,v)$ is defined directly as the real part of $\Psi_\alpha(z)$, then it is harmonic at each point of the (u,v)-plane.

6.3. Fractional integrodifferential equations

A number of equations for densities $q(x;\alpha,\delta)$ of strictly stable distributions can be written with the use of the fractional integral operator $\mathbb{I}^r$:

$$\mathbb{I}^r h(x) \equiv e^{-i\pi r} I^r_- h(x) = \frac{e^{-i\pi r}}{\Gamma(r)} \int_x^\infty (t-x)^{r-1} h(t)\,dt.$$

In the functions $\psi(x) = xq(x;\alpha,\delta)$, $x > 0$, considered below, the densities $q(x;\alpha,\delta)$ are parameterized in form C (but we omit the index C).

We mention several special properties of the operators $\mathbb{I}^r$ that are needed in what follows.

LEMMA 6.3.1. *Let z be a complex number. If $r \geq 0$ and $\Re z > 0$, or if $r < 0$ and $\Re z \geq 0$, then*

$$\mathbb{I}^r \exp(-zx) = \exp(-i\pi r - zx) z^{-r}. \tag{6.3.1}$$

In particular, for $r > 0$ and $z = -ik$, $k > 0$,

$$\mathbb{I}^{-r} \exp(ikx) = \exp(i\pi r/2 + ikx) k^r. \tag{6.3.2}$$

Let s be a real number, and n, the smallest integer such that $n+s > 0$. Then for any $r < s$

$$\mathbb{I}^r x^{-s} = \exp(-i\pi r) \frac{s(s+1)\dots(s+n-1)\Gamma(s-r)}{\Gamma(s+n)} x^{r-s}. \tag{6.3.3}$$

In particular, if $s > 0$, then for any $r < s$

$$\mathbb{I}^r x^{-s} = \exp(i\pi r) \frac{\Gamma(s-r)}{\Gamma(s)} x^{r-s}. \tag{6.3.4}$$

PROOF. The case $r = 0$ clearly does not need to be analyzed. If $r > 0$ and $\Re z > 0$, then

$$\begin{aligned}\mathbb{I}^r \exp(-zx) &= \frac{\exp(-i\pi r)}{\Gamma(r)} \int_x^\infty e^{-zt}(t-x)^{r-1} dt \\ &= \frac{\exp(-i\pi r)}{\Gamma(r)} e^{-zx} \int_0^\infty e^{-zt} t^{r-1} dt,\end{aligned}$$

which, as is not difficult to see, implies (6.3.1).

If $-1 < r < 0$ and $\Re z \geq 0$, integration by parts gives us

$$\begin{aligned}\mathbb{I}^r \exp(-zx) &= \frac{\exp(-i\pi r)}{\Gamma(r)} \int_x^\infty (e^{-zt} - e^{-zx})(t-x)^{r-1} dt \\ &= \frac{\exp(-i\pi r)}{\Gamma(1+r)} z \int_x^\infty e^{-zt}(t-x)^r dt.\end{aligned}$$

The integral is then transformed as in the preceding case.

If r is a negative integer, then $\mathbb{I}^r$ is the operator of r-fold differentiation, and verification of (6.3.1) presents no difficulties.

But if r is negative and not an integer, then, choosing a positive integer n such that $-1 < r + n < 0$, we can factor the operator, $\mathbb{I}^r = \mathbb{I}^{r+n}\mathbb{I}^{-n}$, and obtain

$$\begin{aligned}\mathbb{I}^r \exp(-zx) &= \mathbb{I}^{r+n}\frac{d^n}{dx^n}\exp(-zx)\\ &= \exp(i\pi n)z^n\mathbb{I}^{r+n}\exp(-zx),\end{aligned}$$

i.e., we arrive at the case analyzed above.

The verification of (6.3.3) is similar.

THEOREM 6.3.1. *Let $x > 0$, and let $\alpha \neq 1$ and δ be some pair of admissible parameter values. Then for any $r > -1/\alpha$ the function $\psi(x) = xq(x;\alpha,\delta)$ is the real part of a function $\chi(\xi)$, $\xi = x^{-\alpha}$, satisfying the equation*

$$x\mathbb{I}^{-\alpha r}(x^{-1}\chi(x^{-\alpha})) = \exp\{-i\pi r + i(\alpha-\delta)r\pi/2\}\,\xi^r\mathbb{I}^{-r}\chi(\xi). \tag{6.3.5}$$

PROOF. Consider the function

$$\chi(\xi) = \frac{1}{\pi}\int_0^\infty \exp\{it - \xi t^\alpha \exp(i\delta\pi/2)\}\,dt. \tag{6.3.6}$$

A simple transformation of the integral in (4.1.1) shows that $\psi(x) = \Re\chi(\xi)$. Let us demonstrate that $\chi(\xi)$ satisfies (6.3.5). Using (6.3.1), we obtain

$$\begin{aligned}\mathbb{I}^{-r}\chi(\xi) &= \frac{1}{\pi}\int_0^\infty \exp(ik)\mathbb{I}^{-r}\exp\{-\xi k^\alpha \exp(i\delta\pi/2)\}\,dk\\ &= \exp(ir\pi + ir\delta\pi/2)\frac{1}{\pi}\int_0^\infty k^{\alpha r}\exp\{ik - \xi k^\alpha\exp(i\delta\pi/2)\}\,dk.\end{aligned} \tag{6.3.7}$$

Further, since

$$\mu(x) \equiv x^{-1}\chi(x^{-\alpha}) = \frac{1}{\pi}\int_0^\infty \exp\{ikx - k^\alpha\exp(i\delta\pi/2)\}\,dk, \tag{6.3.8}$$

from (6.3.2) it follows that

$$\begin{aligned}x\mathbb{I}^{-\alpha r}\mu(x) &= \frac{x}{\pi}\int_0^\infty \exp\{-k^\alpha\exp(i\delta\pi/2)\}\,\mathbb{I}^{-\alpha r}\exp(ikx)\,dk\\ &= \exp(i\alpha r\pi/2)\frac{1}{\pi}\int_0^\infty k^{\alpha r}\exp\{ikx - k^\alpha\exp(i\delta\pi/2)\}\,d(kx)\\ &= \xi^r\exp(i\alpha r\pi/2)\frac{1}{\pi}\int_0^\infty k^{\alpha r}\exp\{ik - \xi k^\alpha\exp(i\delta\pi/2)\}\,dk.\end{aligned} \tag{6.3.9}$$

Comparison of (6.3.7) and (6.3.9) validates (6.3.5). The condition $\alpha r > 1$ ensures the existence of the integrals in these equalities.

A slight change in the reasoning allows us to derive another equation of type (6.3.5).

Let us carry out integration by parts in expression (6.3.6) for the function $\chi(\xi)$:

$$\chi(\xi) \;=\; i/\pi + (\alpha\xi/\pi)\exp\{i(\delta-1)\pi/2\}\int_0^\infty k^{\alpha-1}\exp\{ik-\xi k^\alpha\exp(i\delta\pi/2)\}\,dk. \tag{6.3.10}$$

Let $\tau(\xi)=\xi^{-1}\chi(\xi)$. The functions τ, χ, and ψ are related as follows:

$$\tau(\xi)=x^\alpha\chi(x^{-\alpha})=x^{1+\alpha}\mu(x),$$
$$\mu(x)=\xi^{1/\alpha}\chi(\xi)=\xi^{1+/\alpha}\tau(\xi), \tag{6.3.11}$$

whereas the differential operators with respect to the variables x and ξ are related by the equalities

$$\frac{d}{d\xi}=-\frac{1}{\alpha}x^{1+\alpha}\frac{d}{dx},\qquad \frac{d}{dx}=-\alpha\xi^{1+1/\alpha}\frac{d}{d\xi}. \tag{6.3.12}$$

THEOREM 6.3.2. *For any $r>-1$ and any pair of admissible values of the parameters $\alpha\neq1$ and δ the function $x^{1+\alpha}q(x;\alpha,\delta)$ is the real part of a function $\tau(\xi)$ satisfying the equation*

$$\xi^{r+1}\mathbb{I}^{-r}\tau(\xi)+(i/\pi)\Gamma(1+r)\exp\{i\pi(1+r)\}$$
$$=\alpha\exp\{i\pi r-i(r+1)(\alpha-\delta)\pi/2\}\,\mathbb{I}^{1-\alpha(r+1)}(x^{-(1+\alpha)}\tau(x^{-\alpha})). \tag{6.3.13}$$

PROOF. By virtue of (6.3.10),

$$\tau(\xi)-(i\pi\xi)^{-1}=\frac{\alpha}{i\pi}\exp(i\delta\pi/2)\int_0^\infty k^{\alpha-1}\exp\{ik-\xi k^\alpha\exp(i\delta\pi/2)\}\,dk.$$

Herefrom, making use of properties (6.3.1) and (6.3.4) of the operators $\mathbb{I}^r$, we obtain

$$\begin{aligned}\mathbb{I}^{-r}(\tau(\xi)-(i\pi\xi)^{-1})&=\mathbb{I}^{-r}\tau(\xi)+\frac{1}{i\pi}\exp(i\pi r)\Gamma(1+r)\xi^{-(1+r)}\\&=\frac{\alpha}{i\pi}\exp\{i\pi r+i(r+1)\delta\pi/2\}\\&\quad\times\int_0^\infty k^{\alpha(r+1)-1}\exp\{ik-\xi k^\alpha\exp(i\delta\pi/2)\}\,dk.\end{aligned} \tag{6.3.14}$$

On the other hand, by virtue of (6.3.2) and (6.3.8),

$$\begin{aligned}&\mathbb{I}^{1-\alpha(r+1)}\mu(x)=\mathbb{I}^{1-\alpha(r+1)}(x^{-(1+\alpha)}\tau(x^{-\alpha}))\\&=\exp\{i[\alpha(r+1)-1]\pi/2\}\,\frac{1}{\pi}\int_0^\infty k^{\alpha(r+1)-1}\exp\{ikx-k^\alpha\exp(i\delta\pi/2)\}\,dk\\&=\xi^{r+1}\exp\{i[\alpha(r+1)-1]\pi/2\}\,\frac{1}{\pi}\int_0^\infty k^{\alpha(r+1)-1}\exp(ik-\xi k^\alpha\exp(i\delta\pi/2))dk.\end{aligned} \tag{6.3.15}$$

Comparison of (6.3.14) and (6.3.15) yields (6.3.13).

COROLLARY 6.3.1. *Let* $r = 1/\alpha - 1$. *In this case,* (6.3.13) *takes the following more simple form:*

$$\mathbb{I}^{1-1/\alpha}\tau(\xi) + \frac{i}{\pi}\Gamma(1/\alpha)\exp(i\pi/\alpha)\xi^{-1/\alpha} = \alpha\exp\left\{i[(2+\delta)\pi - 3\alpha]\pi/(2\alpha)\right\}\xi\tau(\xi). \tag{6.3.16}$$

We present another corollary as an independent theorem, because it is connected with the case where integrodifferential equation (6.3.13) becomes an ordinary differential equation.

THEOREM 6.3.3. *If* $\alpha = m/n$ *is a rational number differing from 1, then for any pair* α, δ *of admissible parameter values the density* $q(x;\alpha,\delta)$ *is the real part of a function* $\mu(x)$ *satisfying the equation*

$$n^n\left(x^{1+m/n}d/dx\right)^{n-1}(x^{1+m/n}\mu(x))$$
$$= m^n\exp\left\{-im(1-\delta/\alpha)\pi/2\right\}x^m(d/dx)^{m-1}\mu(x) + (i/\pi)\Gamma(1+n)m^{n-1}x^m. \tag{6.3.17}$$

The proof of (6.3.17) reduces to a transformation of (6.3.13) in the case where $\alpha = m/n$ and $r = n-1$. The passage from the variable ξ to x is by means of (6.3.11) and (6.3.12). With their help, equation (6.3.17) can be given another form if we write it for the function $\tau(\xi)$ and pass from the variable x to ξ:

$$n^m\xi^n\left(d/d\xi\right)^{n-1}\tau(\xi) + (i/\pi)(-1)^n\Gamma(n)n^m$$
$$= (-1)^{m+n}m^m\exp\left\{-im(1-\delta/\alpha)\pi/2\right\}\left(\xi^{1+n/m}d/dx\right)^{m-1}(\xi^{1+n/m}\tau(\xi)). \tag{6.3.18}$$

It is not hard to see that (6.3.17) looks simpler than (6.3.18) for $n < m$, and more complicated for $n > m$. This is very clear, for example, in the case $m = 1$, where (6.3.18) is transformed into the equation

$$n\left(\frac{d}{d\xi}\right)^{n-1}\tau(\xi) + (-1)^n\exp\left\{-i(1-\delta/\alpha)\pi/2\right\}\xi\tau(\xi)$$
$$+ i(-1)^m\frac{\Gamma(m+1)}{\pi}\xi^{-m} = 0. \tag{6.3.19}$$

6.4. Splitting of the differential equations

The above equations, beginning with (6.3.5), share a common feature. They are connected not with the density $q(x;\alpha,\delta)$ itself but with a certain complex-valued function whose real part is expressed in terms of the density. This means that each of these equations is, generally speaking, a system of two equations for two functions, of which only one is of interest to us. The linear nature of both equations allows us to write an equation for each of these functions, but only at the cost of complicating the equation. Thus, the order of equations (6.3.17) and (6.3.18) is equal to $\max(m-1, n-1)$, while the order of the equation for the density $q(x;\alpha,\delta) = \Re\mu(x)$ in the general case is $2\max(m-1, n-1)$. Sometimes, however, the complication does not occur, because the coefficients of the operators turn out to be real.

We consider this phenomenon in more detail for the example of equation (6.3.13) and its particular cases (6.3.17) and (6.3.18). The number

$$\exp\{i\pi r - i(r+1)(\alpha-\delta)\pi/2\}$$

is the factor in (6.3.13) which we shall be concerned with. It is real if and only if the number $2r-(r+1)(\alpha-\delta)$ is even, i.e., is equal to $2k$ for some integer k:

$$2r - \alpha(r+1)(1-2\eta/\pi) = 2k.$$

We hence obtain

$$\delta = \alpha - 2(k-r)/(r+1). \tag{6.4.1}$$

Determination of the numbers δ satisfying (6.4.1) means requiring that δ belongs to the domain of admissible values, i.e., $|\delta| \le \delta_\alpha$. It is not hard to see that this is equivalent to the conditions

$$\begin{aligned} r \le k \le r + \alpha(r+1) &\quad \text{if } 0 < \alpha < 1, \\ (\alpha-1)(r+1) + r \le k \le 2r+1 &\quad \text{if } 1 < \alpha \le 2. \end{aligned}$$

For equations (6.3.17) and (6.3.18) with $\alpha = m/n \ne 1$ and $r = n-1$ representation (6.4.1) is equivalent (since r is an integer) to

$$\delta = \delta^{(k)} = \alpha(1-2k/m) \tag{6.4.2}$$

with the following constraints imposed on the integer k:

$$\begin{aligned} 0 \le k \le m &\quad \text{if } m < n, \\ m-n \le k \le n &\quad \text{if } m > n. \end{aligned}$$

For each pair $\alpha = m/n$, $\delta^{(k)}$ satisfying (6.4.2), equations (6.3.17)–(6.3.18) break up into pairs of mutually unconnected equations for the real and imaginary parts of the corresponding function. Furthermore, the most interesting equation for the real part is always homogeneous.

It is not hard to see that for each $\alpha = m/n \ne 1$ there are at least two values δ satisfying (6.4.2): $\delta = \delta_\alpha$ and $\delta = -\delta_\alpha$. The set of $\alpha = m/n \ne 1$ for which there are no cases of splitting of the differential equations other than cases of extremal laws is made up of numbers of the form $1/(p+1)$ and $(2p+1)/(p+1)$, $p = 1, 2, \dots$

If the order of the equations is at most two, then the form of the equations themselves allows us to hope that they can be solved, at least with the help of some special functions.

6.5. Some special cases

1. If $\alpha = 1/2$ and δ is any admissible value, then from (6.3.19) we obtain

$$\tau'(\xi) - \tfrac{1}{2}\exp(i\zeta)\xi\tau(\xi) + i\xi^{-2}/\pi = 0, \qquad \zeta = (1 + 8/\alpha)\pi/2.$$

This equation can be solved without difficulty as we recall that $\tau(\infty) = 0$:

$$\tau(\xi) = \frac{i}{\pi}\exp\left(\frac{\xi^2}{4}e^{i\zeta}\right)\int_\xi^\infty \exp\left(-\frac{t^2}{4}e^{i\zeta}\right)\frac{dt}{t^2}.$$

Consequently,

$$\begin{aligned} x^{3/2}q(x; 1/2, \delta) &= \Re\tau(\xi) \\ &= -\frac{1}{\pi}\Im\left(z\exp(z^2/4)\int_z^\infty \exp(-t^2/4)\frac{dt}{t^2}\right), \end{aligned} \tag{6.5.1}$$

where $z = x^{-1/2}\exp(i\zeta/2)$.

In particular, if $\delta = 0$ (i.e., $\zeta = \pi/2$, which corresponds to a symmetric distribution), then after appropriate transformations (6.5.1) takes the form

$$\begin{aligned} q(x; 1/2, 0) = \frac{x^{-3/2}}{2\sqrt{2\pi}}\Bigg\{&\cos\left[\frac{1}{4x}\left(\frac{1}{2} - C\left(\sqrt{2/(\pi x)}\right)\right)\right] \\ &+ \sin\left[\frac{1}{4x}\left(\frac{1}{2} - S\left(\sqrt{2/(\pi x)}\right)\right)\right]\Bigg\}, \end{aligned} \tag{6.5.2}$$

where

$$C(z) = \int_0^z \cos(\pi t^2/2)dt, \qquad S(z) = \int_0^z \sin(\pi t^2/2)dt$$

are special functions called Fresnel integrals.

2. If $\alpha = 1/3$ and $\delta = \alpha$ (a case where equation (6.3.19) splits), then for $y(\xi) = \Re\tau(\xi)$ we obtain the equation

$$y''(\xi) - \tfrac{1}{3}\xi y(\xi) = 0.$$

The equation is solved (to within a constant factor) by the so-called Airy integral, which can be expressed, in turn, in terms of the Macdonald function of order 1/3:

$$y(\xi) = 3c\int_0^\infty \cos(t^3 + t\xi)dt = c\sqrt{\xi}K_{1/3}\left(2\xi^{3/2}/\sqrt{27}\right).$$

To establish the value c we are aided by expansion (4.2.4) of the function $x^{4/3}q(x; 1/3, 1)$ in powers of $\xi = x^{-1/3}$. On the one hand (see (Gradshtein & Ryzhik, 1963, 8.4.32(5))),

$$c\sqrt{\xi}K_{1/3}\left(2\xi^{3/2}/\sqrt{27}\right) \to c\sqrt{3}\Gamma(1/3)/2$$

as $\xi \to 0$, and on the other hand,

$$y(\xi) = x^{4/3}q\,(x; 1/3, 1/3) \to (3\pi)^{-1}\sqrt{3}\Gamma(1/3)/2$$

as $\xi \to 0$; therefore, $c = (3\pi)^{-1}$. Consequently,

$$q\,(x; 1/3, 1/3) = \frac{x^{3/2}}{3\pi}K_{1/3}\left(\frac{2}{\sqrt{27}}x^{-1/2}\right). \tag{6.5.3}$$

There are four more cases where the density $g(x; \alpha, \delta)$ can be expressed in terms of special functions.

3. If $\alpha = \delta = 2/3$, from (6.3.18) it follows that that $y(\xi) = \Re\tau(\xi)$ satisfies the equation

$$y''(\xi) + \tfrac{4}{9}\xi^2 y''(\xi) + \tfrac{10}{9}\xi y(\xi) = 0.$$

4. If $\alpha = 2/3$ and $\delta = 0$, then for $y(\xi) = \Re\tau(\xi)$

$$y''(\xi) - \tfrac{4}{9}\xi^2 y'(\xi) - \tfrac{10}{9}\xi y(\xi) = 0.$$

5. If $\alpha = 3/2$ and $\delta = \delta_{3/2} = 1/2$, then, by virtue of (6.3.19), the function $y(x) = \Re\mu(x)$ is a solution of the equation

$$y''(x) + \tfrac{4}{9}x^2 y'(x) + \tfrac{10}{9}xy(x) = 0.$$

6. If $\alpha = 3/2$ and $\delta = -1/2$, then for $y(x) = \Re\mu(x)$

$$y''(x) - \tfrac{4}{9}x^2 y'(x) - \tfrac{10}{9}xy(x) = 0.$$

Recall that in the last two cases $\delta = 1/2$ corresponds to $\beta = -1$ in forms A and B, while $\delta = -1/2$ corresponds to $\beta = 1$.

6.6. The Whittaker functions

All the equations considered above are connected with a single type of special functions, the so-called Whittaker functions

$$W_{\lambda,\mu}(z) = \frac{z^{\lambda}e^{-z/2}}{\Gamma(\mu-\lambda+1/2)}\int_0^{\infty} e^{-t}t^{\mu-\lambda-1/2}(1+t/z)^{\mu+\lambda-1/2}dt,$$
$$\Re(\mu-\lambda) > -1/2, \qquad |\arg z| < \pi.$$

We restrict our consideration to the particular case $\alpha = 2/3$, $\delta = 0$. The other cases are treated similarly.

It is well known (see, for example, (Kamke, 1959, 2.273(5))) that in the chosen case a function of the form

$$y(\xi) = C\xi^{-1}\exp\left(\tfrac{2}{27}\xi^3\right)W_{-1/2,1/6}\left(\tfrac{4}{27}\xi^3\right)$$

is a solution of the equation. As in the analysis of the preceding case, the value of the constant C can be determined by comparing the asymptotic behavior of $y(\xi)$ as $\xi = x^{-2/3} \to \infty$, obtained in two different ways. On the one hand (see (Gradshtein & Ryzhik, 1963, 9.227))

$$y(\xi) \sim C\frac{\sqrt{27}}{2}\xi^{-(1+3/2)} = C\frac{\sqrt{27}}{2}x^{1+2/3}.$$

On the other hand, on the basis of (4.9.1)

$$y(\xi) = x^{1+2/3}q\,(x;2/3,0) \sim x^{1+2/3}q\,(0;2/3,0) = \frac{3}{4\sqrt{\pi}}x^{1+2/3}.$$

This implies that $C = 1/(2\sqrt{3\pi})$, i.e., for any $x > 0$

$$\begin{aligned} q\,(x;2/3,0) &= q\,(-x;2/3,0)\\ &= \frac{x^1}{2\sqrt{3\pi}}\exp\left(\tfrac{2}{27}x^{-2}\right)W_{-1/2,1/6}\left(\tfrac{4}{27}x^{-2}\right). \end{aligned} \tag{6.6.1}$$

In the remaining cases we obtain in the same way the following expressions for the densities on the semi-axis $x > 0$:

$$q\,(x;2/3,2/3) = \frac{x^{-1}}{\sqrt{3\pi}}\exp\left(-\tfrac{2}{27}x^{-2}\right)W_{1/2,1/6}\left(\tfrac{4}{27}x^{-2}\right), \tag{6.6.2}$$

$$q\,(x;3/2,1/2) = \frac{x^{-1}}{\sqrt{3\pi}}\exp\left(-\tfrac{2}{27}x^{3}\right)W_{1/2,1/6}\left(\tfrac{4}{27}x^{3}\right), \tag{6.6.3}$$

$$q\,(x;3/2,-1/2) = \frac{x^{-1}}{2\sqrt{3\pi}}\exp\left(\tfrac{2}{27}x^{3}\right)W_{-1/2,1/6}\left(\tfrac{4}{27}x^{3}\right). \tag{6.6.4}$$

According to the duality law (4.6.2), the densities under consideration on the positive semi-axis must be related by the equalities

$$xq\,(x;3/2,1/2) = x^{-3/2} q\left(x^{-3/2}, 2/3, 2/3\right),$$
$$xq\,(x;3/2,-1/2) = x^{-3/2} q\left(x^{-3/2}, 2/3, 0\right),$$

and this can indeed be observed in the above formulas (therefore, in particular, we can obtain (6.6.4) as a corollary to (6.6.1), and (6.6.2), to (6.6.3)).

The integral representation

$$W_{-1/2,1/6}(z) = \frac{z^{-1/2}}{\Gamma(7/6)} e^{-z/2} \int_0^\infty e^{-t} t^{1/6} \left(1 + \frac{t}{z}\right)^{-5/6} dt$$

of the function $W_{-1/2,1/6}(z)$ in the z-plane with a cut along the negative semi-axis shows that this is a multi-valued function with third-order algebraic branch points $z = 0$ and $z = \infty$. Consequently, the function $W_{-1/2,1/6}(z^3)$ is analytic on the whole complex plane, and we can speak of its values $W_{-1/2,1/6}(z^3)$ on the negative semi-axis, although the formula given is unsuitable for expressing them. This consideration, together with the consequence

$$q\,(x;3/2,1/2) = q\,(-x,3/2,-1/2)$$

of (3.7.3), leads to new expressions for the densities $q\,(x;3/2,1/2)$ and $q\,(x;2/3,2/3)$:

$$q\,(x;3/2,1/2) = -\frac{x^{-1}}{2\sqrt{3\pi}} \exp\left(\tfrac{2}{27}x^3\right) W_{-1/2,1/6}\left(-\tfrac{4}{27}x^3\right),$$
$$q\,(x;2/3,2/3) = \frac{x^{-1}}{2\sqrt{3\pi}} \exp\left(\tfrac{2}{27}x^{-2}\right) W_{-1/2,1/6}\left(-\tfrac{4}{27}x^{-2}\right).$$

A comparison of these expressions with (6.6.3) and (6.6.2) shows that in the complex plane

$$W_{-1/2,1/6}(-z^3) = -2W_{-1/2,1/6}(z^3).$$

6.7. Generalized incomplete hypergeometrical function

In (Hoffmann-Jørgensen, 1993), a generalized incomplete hypergeometrical function ${}_pG_q(\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q;\gamma,z)$ is used, where p, q are non-negative integers, α_i, β_j are positive numbers, and $0 < \gamma \le 1$; the function is defined as the incomplete hypergeometrical series

$${}_pG_q = \sum_{n=0}^{\infty} \left(\prod_{i=1}^{p} \frac{\Gamma(\alpha_i + \gamma n)}{\Gamma(\alpha_i)} \prod_{j=1}^{q} \frac{\Gamma(\beta_j)}{\Gamma(\beta_j + \gamma n)} \right) \frac{z^n}{n!}. \qquad (6.7.1)$$

In the case $0 < \gamma < 1$, this power series converges in all the complex plane, in the case $\gamma = 1$ it converges in the disk $|z| < 1$.

The theorem given in (Hoffmann-Jørgensen, 1993) gives the representation of the densities of strongly stable laws in terms of the function ${}_1G_0$. Actually, it is equivalent to the following.

If $1 < \alpha \le 2$, $x \ne 0$, then

$$xq(x; \alpha, \delta) = -\frac{1}{\pi}\Im {}_1G_0(1; 1/\alpha, -xe^{i\pi\rho/\alpha}).$$

If $0 < \alpha < 1$, $x > 0$, then

$$xq(x; \alpha, \rho) = -\frac{1}{\pi}\Im {}_1G_0(1; \alpha, -x^{-\alpha}e^{i\pi\rho}).$$

Following (Zolotarev, 1994), we make some remarks concerning the functions ${}_pG_q$ introduced in (Hoffmann-Jørgensen, 1993).

The functions ${}_pG_q$ have some pre-history. In (Wright, 1935), the asymptotic behavior of a more general series than (6.7.1) was investigated. Namely (see (Bateman & Erdelyi, 1953, §4.1))

$${}_pW_q = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + \mu_1 n)\dots\Gamma(\alpha_p + \mu_p n)}{\Gamma(\beta_1 + \nu_1 n)\dots\Gamma(\beta_q + \nu_q n)} \frac{z^n}{n!}, \tag{6.7.2}$$

where p, q are non-negative integers, the product over an empty set is assumed equal to 1, $\alpha_i, \beta_j \in \mathbb{R}$, and μ_i, ν_j are positive numbers satisfying the condition

$$1 + \sum_{j=1}^{q} \nu_j - \sum_{i=1}^{p} \mu_i > 0. \tag{6.7.3}$$

Later series (6.7.2) were not considered by anyone, and little is known about their properties. Apparently, this circumstance explains why they did not receive any name and were not included in the class of acknowledged special functions.

The next remark concerns the description of the domain of convergence of series (6.7.1) in the case $0 < \gamma \le 1$. Elementary calculations of the convergence radius R of this series shows that for any $\gamma > 0$

$$R = \begin{cases} \infty, & 1 + \gamma(q - p) > 0, \\ -(p - q)^{-1} & 1 + \gamma(q - p) = 0, \end{cases}$$

and $R = 0$ otherwise (the first case quite conforms to (6.7.3)). Therefore, the equality $R = \infty$ is not ensured by the condition $0 < \gamma < 1$ just like the equality $R = 1$ is not ensured by $\gamma = 1$.

6.8. The Meijer and Fox functions

In the problems related to higher transcendental functions, a distinguished position belongs to the so-called Meijer's G-functions. They appeared and became an object of intensive investigations in the second half of the thirties.

DEFINITION OF A G-FUNCTION. Let p, q, and $m \le q$, $n \le p$, $p + q < 2(m + n)$, be some non-negative numbers, and let $a_1, \dots, a_p$, $b_1, \dots, b_q$ be systems of complex numbers satisfying the condition

$$\max_i \Re a_i - \min_j \Re b_j < 1.$$

The Meijer function $G^{mn}_{pq}\left(z \,\middle|\, {}^{a_1,\dots,a_p}_{b_1,\dots,b_q}\right)$ of order (m, n, p, q) is uniquely defined in the sector $\{z = re^{i\varphi} : |\varphi| < \pi[(m + n) - (p + q)/2]\}$ by the equality

$$\int_0^\infty x^{s-1} G^{mn}_{pq}\left(z \,\middle|\, \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) dx = \frac{\prod_{j=1}^{m} \Gamma(b_j + s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j - s)} \frac{\prod_{i=1}^{n} \Gamma(1 - a_i - s)}{\prod_{i=n+1}^{p} \Gamma(a_i + s)}, \tag{6.8.1}$$

$$-\min_j \Re b_j < \Re s < 1 - \max_i \Re a_i$$

(the product over an empty set of indices is assumed equal to 1).

The comparison of equalities (5.6.4) and (6.8.1) suggests an idea of relating the densities of strongly stable laws with Meijer's functions in some cases.

Consider this possibility in more detail. Assume that the parameters $\alpha = P/Q \ne 1$, $(P, Q) = 1$, and $\rho = \alpha U/V > 0$, $(U, V) = 1$, are rational numbers[1]. Let M be the least positive integer for which $M = PM_1$ and $M = VM_2$, where M_1, M_2 are positive integers (least common multiple). Put $N = QM_1$, $L = UM_2$. Obviously, $\alpha = M/N$, $\rho = L/M$. The following assertion holds see ((Zolotarev, 1994)).

THEOREM 6.8.1. *Let $\alpha = P/Q \ne 1$ and $\rho = \alpha U/V > 0$ be rational numbers. Then*

$$xq(x; \alpha, \rho) = xq(x; M/N, \alpha L/M) = AG^{mn}_{pq}\left(x^M B^{-1} \,\middle|\, \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) \tag{6.8.2}$$

[1] $(P, Q) = 1$ means that P and Q are relatively prime, i.e., they possess no common divisor different from 1.

for any $x > 0$*, where* $m = M - 1$, $n = N - 1$, $p = N + L - 2$, $q = M + L - 2$*;*

$$(a_1, \dots, a_n) = \left(\frac{1}{N}, \dots, \frac{N-1}{N}\right), \qquad (a_{n+1}, \dots, a_p) = \left(\frac{1}{L}, \dots, \frac{L-1}{L}\right);$$
$$(b_1, \dots, b_m) = \left(\frac{1}{M}, \dots, \frac{M-1}{M}\right), \qquad (b_{m+1}, \dots, b_q) = \left(\frac{1}{L}, \dots, \frac{L-1}{L}\right);$$
$$A = N\sqrt{\alpha}(2\pi)^{N[\rho-(1+\alpha)/2]}, \qquad B = \frac{M^M}{N^N}. \tag{6.8.3}$$

PROOF. Using (5.6.4) and the well-known Gauss–Legendre multiplication formula for Γ-functions, we obtain

$$\begin{aligned} M\int_0^\infty x^{Ms} q(x;\alpha,\rho)\,dx &= \frac{MN}{L}\frac{\Gamma(Ms)\Gamma(-Ns)}{\Gamma(-Ls)\Gamma(Ls)} \\ &= MAB^s \frac{\Gamma(1/M+s)\dots\Gamma((M-1)/V+s)\Gamma(1/N-s)\dots\Gamma((N-1)/N-s)}{\Gamma(1/L-s)\dots\Gamma((L-1)/L-s)\Gamma(1/L+s)\dots\Gamma((L-1)/L+s)} \\ &= M\int_0^\infty x^{Ms}\left[AG_{pq}^{mn}\left(x^M B^{-1}\middle|\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right)\frac{dx}{x}\right] \end{aligned}$$

Comparing the left- and right-hand sides of the last equality, we arrive at representation (6.8.2).

Using the properties of Meijer's function, we easily see that particular cases of formula (6.8.2) give us the known analytical expressions of stable densities. In the normal case $\alpha = 2$, $\rho = 1$, we obtain $M = 2$, $N = 1$, $L = 1$, $B = 4$, $A = \pi^{-1/2}$, etc.; thus

$$xq(x;2,1) = \pi^{-1/2} G_{01}^{10}\left(x^2/4\middle|\begin{matrix}-\\ 1/2\end{matrix}\right). \tag{6.8.4}$$

According to formula (1.7.2) of (Mathai & Saxena, 1978),

$$G_{01}^{10}\left(z\middle|\begin{matrix}-\\ b\end{matrix}\right) = z^b e^{-z}.$$

Applying this formula to (6.8.4), we obtain

$$q(x;2,1) = \frac{1}{\sqrt{4\pi}} e^{-x^2/4}.$$

Similar reasoning, with the use of the formulae

$$G_{11}^{11}\left(z^\alpha\middle|\begin{matrix}\beta/\alpha\\ \beta/\alpha\end{matrix}\right) = \frac{z^\beta}{1+z^\alpha},$$
$$G_{10}^{01}\left(z\middle|\begin{matrix}a\\ -\end{matrix}\right) = G_{01}^{10}\left(1/z\middle|\begin{matrix}-\\ 1-a\end{matrix}\right),$$

yields the Cauchy and Lévy densities.

Representation (6.8.2) will obviously add little to those few particular cases where the densities q happen to be represented by Fresnel integrals, Bessel functions , and Whittaker functions. However this representation together with numerous connections in the class of G-functions may lead to new interesting formulae for the densities of stable laws, such as the expression in terms of G-functions of various integral transforms of the densities $q(x;\alpha,\rho)$ with rational values of parameters, for example $\int_0^1 x^u(1-x)^{v-1}q(xy;\alpha,\rho)\,dx$, $\int_1^\infty x^{-u}(x-1)^{v-1}q(xy;\alpha,\rho)\,dx$, and $\int_0^\infty k(x)q(xy;\alpha,\rho)\,dx$ where $k(x)$ are various kernel functions (see (Erdélyi *et al.*, 1954)): $k(x) = x^{u-1}(x+\lambda)^{-v}$, $x^{-u}\exp(-vx)$, $x^{1/2}\exp(-vx^{1/2})$, $x > 0$, and do on, as well as the integral of the form $\int_0^\infty q(x;\alpha_1,\rho_1)(q(xy;\alpha_2,\rho_2)\,dx$.

The above-mentioned integral transforms with Euler kernels together with (6.8.2) make it possible to obtain the expressions of distribution functions of strongly stable laws $G(x;\alpha,\rho)$ with rational values of parameters in terms of G-functions. Namely, if $\alpha = P/Q \neq 1$, $\rho = \alpha U/V > 0$, then for $x > 0$

$$\begin{aligned} G(x;\alpha,\rho) &= 1-\rho+\int_0^x q(u;\alpha,\rho)\,du \\ &= 1-\rho+\frac{A}{M}G^{m,n+1}_{p+1,q+1}\left(x^M B^{-1}\left|\begin{matrix}1,a_1,\dots,a_p\\ b_1,\dots,b_q,0\end{matrix}\right.\right), \end{aligned}$$

where the parameters a_i, b_j, m, n, p, q, and others are the same as in (6.8.2), and for $x < 0$

$$\begin{aligned} G(x;\alpha,\rho) &= \int_{-\infty}^x q(u;\alpha,\rho)\,du = \int_{|x|}^\infty q(u;\alpha,\alpha-\rho)\,du \\ &= \frac{A'}{M}G^{m+1,n}_{p'+1,q'+1}\left(|x|^M B^{-1}\left|\begin{matrix}a'_1,\dots,a'_{p'},1\\ 0,b'_1,\dots,b'_{q'}\end{matrix}\right.\right), \end{aligned}$$

where $a'_i = a_i$, $1 \le i \le n = N-1$, $b'_j = b_j$, $1 \le j \le m = M-1$,

$$a'_i = \frac{i-n}{M-L}, \qquad n+1 \le i \le p' = N+M-L-2,$$
$$b'_j = \frac{j-m}{M-L}, \qquad m+1 \le j \le q' = 2M-L-2,$$

A' is derived from A in (6.8.3) by replacing $\rho = \alpha L/M$ with $\rho' = \alpha L'/M$, $L' = M-L$.

The above results concern only rational values of the characteristic α. In actual computations, this constraint is not heavy, because in numerical calculations we always deal with rational numbers. Nevertheless, it is possible to eliminate this constraint by passing to the Fox's H-functions (Schneider, 1986).

The Fox function, or H-function, also called the generalized G-function or generalized Mellin–Barnes function, represents a rich class of functions which contains functions such as Meijer's G-function, hypergeometric functions, Wright's hypergeometric series, Bessel functions, Mittag–Leffler functions, etc., as special cases.

As Meijer's G-functions, the Fox's H-functions can be determined by the Mellin transformation

$$\int_0^\infty x^{s-1} H_{pq}^{mn}\left(ax \left| \begin{matrix} (a_1,\alpha_1),\dots,(a_p,\alpha_p) \\ (b_1,\beta_1),\dots,(b_q,\beta_q) \end{matrix} \right. \right) dx = \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)} \frac{\prod_{i=1}^{n}\Gamma(1-a_i-\alpha_i s)}{\prod_{i=n+1}^{p}\Gamma(a_i+\alpha_i s)}, \quad (6.8.5)$$

where m, n, p, q, a_i, b_j are defined above, and $\alpha_1,\dots,\alpha_p,\beta_1,\dots,\beta_q$ are positive numbers satisfying the condition

$$\alpha_j(b_h+\nu) \neq \beta_h(a_j-\lambda-1)$$

for $\nu,\lambda = 0,1,2,\dots, h=1,\dots,m, j=1,\dots,n$.

The theorem on residues enables one to express the Fox function as the infinite series

$$H_{pq}^{mn}(z) = \sum_{j=1}^{m}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}C_{jk}z^{s_{jk}}/\beta_j, \quad (6.8.6)$$

where

$$\begin{aligned} s_{jk} &= \frac{b_j+k}{\beta_j}, \\ C_{jk} &= \frac{A_j(s_{jk})B(s_{jk})}{C(s_{jk})D(s_{jk})}, \\ A_j(s) &= \prod_{\substack{l=1\\ l\neq j}}\Gamma(b_l-\beta_l s), \\ B(s) &= \prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s), \\ C(s) &= \prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s), \\ D(s) &= \prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s). \end{aligned}$$

Formula (6.8.6) can be used for calculation of particular values of Fox functions and to study their asymptotic behavior as $z \to \infty$.

6.9. Stable densities as a class of special functions

Many facts indicate that, by virtue of their richness of analytic properties, the functions $q(x; \alpha, \beta)$ merit being distinguished as an independent class and accorded 'civil rights' in the theory of special functions. Some of these facts are presented below.

The functions $q(x; \alpha, 1)$ (form B) with $0 < \alpha < 1$ turn out to be useful in the theory of Laplace transforms and the operational calculus connected with it. Let

$$V(\lambda) = \int_0^\infty \exp(-\lambda x) v(x)\, dx, \qquad \Re\lambda \geq 0,$$

denote the Laplace transform of a function $v(x)$ (we write $V(\lambda) \rightleftharpoons v(x)$). The following formulae are well known (for example, see (Erdélyi *et al.*, 1954)). If $V(\lambda) \rightleftharpoons v(x)$, then, for instance,

$$V(\sqrt{\lambda}) \rightleftharpoons \frac{1}{2\sqrt{\pi}} x^{-3/2} \int_0^\infty \exp\left(-\frac{u^2}{4x}\right) v(u)\, du; \tag{6.9.1}$$

for any $c > 0$

$$V(c\lambda + \sqrt{\lambda}) \rightleftharpoons \frac{1}{2\sqrt{\pi}} \int_0^{x/c} u(x - cu)^{-3/2} \exp\left(-\frac{u^2}{4(x - cu)}\right) v(u)\, du. \tag{6.9.2}$$

The functions $q(x; \alpha, 1)$ with $0 < \alpha < 1$ provide an opportunity for generalizing these relations.

THEOREM 6.9.1. *If $V(\lambda) \rightleftharpoons v(x)$, then the following relations hold true for any $0 < \alpha < 1$ and any $c > 0$:*

$$V(\lambda^\alpha) \rightleftharpoons \int_0^\infty u^{-1/\alpha} q(xu^{-1/\alpha}; \alpha, 1) v(u)\, du, \tag{6.9.3}$$

$$V(c\lambda + \lambda^\alpha) \rightleftharpoons \int_0^{x/c} u^{-1/\alpha} q(u^{-1/\alpha}(x - cu); \alpha, 1) v(u)\, du. \tag{6.9.4}$$

In particular, for $\alpha = 1/3$

$$V(\lambda^{1/3}) \rightleftharpoons \frac{x^{-3/2}}{3\pi} \int_0^\infty u^{3/2} K_{1/3}\left(\frac{2}{\sqrt{27}} u^{3/2} x^{-1/2}\right) v(u)\, du, \tag{6.9.5}$$

$$V(c\lambda + \lambda^{1/3}) \rightleftharpoons \frac{1}{3\pi} \int_0^x \left(\frac{u}{x - cu}\right)^{3/2} K_{1/3}\left(\frac{2}{\sqrt{27}} \left(\frac{u^3}{x - cu}\right)^{1/2}\right) v(u)\, du, \tag{6.9.6}$$

where $K_{1/3}$ is the Macdonald function of order $1/3$.

PROOF. By virtue of (3.7.2) and (5.4.6), for any $x > 0$

$$\exp(-x\lambda^{\alpha}) = \int_0^{\infty} e^{-\lambda u} u^{-1/\alpha} q(x^{-1/\alpha}u; \alpha, 1)\, du. \tag{6.9.7}$$

Using this equality and the integral expression for $V(\lambda^{\alpha})$, we obtain (6.9.3) after changing the order of integration.

Further, by the same equality (6.9.7),

$$\begin{aligned} V(c\lambda + \lambda^{\alpha}) &= \int_0^{\infty} \exp(-c\lambda y - \lambda^{\alpha}) v(y)\, dy \\ &= \int_0^{\infty}\int_0^{\infty} \exp(-\lambda(cy + u)) y^{-1/\alpha} q(y^{-1/\alpha}u; \alpha, 1)\, dy\, du. \end{aligned}$$

The change of variable $u = x - cy$ yields an integral of the form

$$\int_0^{\infty} \exp(-\lambda x) U(x)\, dx,$$

where the function $U(x)$ coincides with the right-hand side of (6.9.4).

Relations (6.9.6) and (6.9.7) are derived from (6.9.3) and (6.9.4) after replacing $q(u; 1/3, 1)$ by its expression in (6.5.3).

We saw above that the functions $q(x; \alpha, \delta)$ are solutions of various types of integral and integrodifferential equations, and even of special types of ordinary differential equations in the case of rational $\alpha \neq 1$. Though the function $q(x; \alpha, \delta)$ represents only one of the solutions, a detailed analysis of its analytic extensions may possibly reveal (as can be seen by the example of an analysis of the Bessel and Whittaker equations) other linearly independent solutions.

Above all else, solutions of the equations with the densities

$$q(x; m/n, 1) \equiv q^B(x; m/n, 1), \qquad m < n,$$

should be considered.

A good illustration is the case $m = 1$ and $n = p + 1 \geq 2$ connected with the equation (see (6.3.19))

$$y^{(p)}(\xi) = (-1)^p \xi y(\xi)/(p + 1), \qquad \xi > 0,$$

whose solution is the function

$$y(\xi) = \xi^{-(p+2)} q(\xi^{-(p+1)}; 1/(p + 1), 1).$$

Of interest is the connection between the densities of extremal stable laws and the Mittag–Leffler function

$$E_{\sigma}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n\sigma + 1)}, \qquad \sigma > 0.$$

THEOREM 6.9.2. *For any* $0 < \alpha < 1$ *and any complex-valued* λ

$$\alpha E_\alpha(-\lambda) = \int_0^\infty \exp(-\lambda x) x^{-1-1/\alpha} q(x^{-1/\alpha}; \alpha, 1)\, dx. \tag{6.9.8}$$

If $1/2 \le \alpha < 1$, *then*

$$\alpha E_\alpha(-\lambda) = \int_0^\infty \exp(-\lambda x) q(x; 1/\alpha, -1)\, dx. \tag{6.9.9}$$

PROOF. Let $\lambda > 0$. It is known (Bieberbach, 1931) that in this case the function $E_\alpha(-\lambda)$ has the representation

$$E_\alpha(-\lambda) = \frac{1}{2\pi i \alpha} \int_L \exp\left(z^{1/\alpha}\right) \frac{dz}{z+\lambda}, \tag{6.9.10}$$

where the integration contour L consists of the following three parts ($z = x+iy$): the line L_1 given by $y = -(\tan\varphi)x$, where x runs from $x = \infty$ to $x = h$, with $h > 0$ and $\alpha\pi/2 < \varphi < \alpha\pi$; the circular arc L_2 given by $|z| = h/\cos\varphi$, $-\varphi \le \arg z \le \varphi$; and the reflection L_3 of L_1 with respect to the x-axis. We replace $(z+\lambda)^{-1}$ in (6.9.10) by the equivalent integral

$$\int_0^\infty \exp\{-(z+\lambda)u\}\, du.$$

The double integral thus obtained converges absolutely; hence we can change the order of integration. Then

$$E_\alpha(-\lambda) = \int_0^\infty \exp(-\lambda u) f_\alpha(u)\, du,$$

where the function $f_\alpha(u)$ can be transformed by integration by parts as follows:

$$\begin{aligned} f_\alpha(u) &= \frac{1}{2\pi i \alpha} \int_L \exp\left(z^{1/\alpha} - zu\right) dz \\ &= \frac{1}{2\pi i \alpha u} \int_L \exp\left(z^{1/\alpha} - zu\right) dz^{1/\alpha}. \end{aligned}$$

We make a change of variable, setting $z = \zeta^\alpha / u$:

$$f_\alpha(u) = \frac{1}{\alpha} u^{-1-1/\alpha} \left(\frac{1}{2\pi i} \int_{L^*} \exp\left(-\zeta^\alpha + \zeta u^{-1/\alpha}\right) d\zeta \right).$$

The contour L^* is the image of L under the change of variable. The integral in parentheses is the inverse Laplace transform of the function $\exp(-\lambda^\alpha)$ ($0 < \alpha < 1$) and, consequently, represents the function $q(u^{-1/\alpha}; \alpha, 1)$ according to (5.4.6).

Equality (6.9.9) is obtained from (6.9.8) by using the duality law for densities (4.6.2) and the fact that the pair $\alpha < 1$, $\delta = \alpha$ in form C corresponds to the pair $\alpha' = 1/\alpha$, $\beta = -1$ in form B.

The left-hand sides of (6.9.8) and (6.9.10) can be extended analytically from the half-line $\lambda > 0$ to the whole complex plane, as is clear from the definition of the function $E_\alpha(x)$. On the other hand, $x^{-1-1/\alpha}q(x^{-1/\alpha};\alpha,1)$ decreases with increasing x more rapidly than any function $\exp(-cx)$, $c > 0$. Therefore, the right-hand side can also be extended to the whole complex plane, i.e., equalities (6.9.8) and (6.9.9) are valid for any complex s.

Another interesting connection between the functions $E_\sigma(x)$ and $q(x;\alpha,1)$ reveals itself when the following equality (obtained in (Humbert, 1953) for any $\sigma > 0$) is generalized:

$$E_{\sigma/2}(x) = \frac{1}{\sqrt{\pi}}\int_0^\infty E_\sigma(xu^\sigma)\exp\left(-\frac{u^2}{4}\right)du. \tag{6.9.11}$$

THEOREM 6.9.3. *Suppose that* $0 < \alpha < 1$ *and* $\sigma > 0$. *Then for any complex s*

$$E_{\alpha\sigma}(\lambda) = \int_0^\infty E_\sigma(\lambda u^{-\alpha\sigma})q(u;\alpha,1)\,du. \tag{6.9.12}$$

The proof of the theorem can be found in (Zolotarev, 1986).

6.10. Transstable functions

The class of strictly stable laws is the starting object for the following generalizations. As we know, each law in this class is characterized in form C by three parameters (α,ρ,λ), where

$$\rho = (\delta + \alpha)/2.$$

The parameter λ is purely a scale parameter, because

$$q^C(x;\alpha,\rho,\lambda) = \lambda^{-1/\alpha}q^C(x\lambda^{-1/\alpha};\alpha,\rho,1) \equiv \lambda^{-1/\alpha}q^C(x\lambda^{-1/\alpha};\alpha,\rho).$$

According to (4.2.4) and (4.2.9), the densities $q^C(x;\alpha,\rho)$ can be represented by convergent power series. We use these representations here in a somewhat extended version including also the case $\alpha = 1$. Namely, if $0 < \alpha < 1$, $0 \le \rho \le \alpha$ and $x > 0$, or if $\alpha = 1$, $0 < \rho < \alpha$ and $x > 1$, then

$$q^C(x;\alpha,\rho) = \pi^{-1}\sum_{n=1}^\infty \frac{(-1)^{n-1}\Gamma(\alpha n+1)}{n!}\sin(n\rho\pi)x^{-\alpha n-1}. \tag{6.10.1}$$

If $1 < \alpha \le 2$, $\alpha - 1 \le \rho \le 1$ and $x \in \mathbb{R}$, or if $\alpha = 1$, $0 < \rho < \alpha$, and $|\alpha| < 1$, then

$$q^C(x;\alpha,\rho) = \pi^{-1}\sum_{n=1}^\infty(-1)^{n-1}\frac{\Gamma(n/\alpha+1)}{n!}\sin(n\rho\pi/\alpha)x^{n-1}. \tag{6.10.2}$$

It is not hard to see that series (6.10.1) and (6.10.2) remain convergent if, keeping the restrictions on the variation of x, we extend the domain of variation of (α, ρ) in the first case to the strip $0 < \alpha \le 1$, $\rho \in \mathbb{R}$, and in the second case to the half-plane $\alpha \ge 1$, $\rho \in \mathbb{R}$.

This enables us to define in the complex plane $\mathbb{Z}$ (possibly with cuts) the family of analytic functions

$$T = \{\sigma(z; \alpha, \rho) : \alpha > 0, \rho \in \mathbb{R}\},$$

by setting them equal to the corresponding functions $q(x; \alpha, \rho)$ on the parts of the real axis $x = \Re z$ where the latter were defined.

DEFINITION. Let T_0 be the set of analytic functions formed all the functions $\sigma(z; \alpha, \rho)$ in T with $\alpha \ge 1$ or with $0 < \alpha < 1$ and ρ an integer together with the principal branches of all the functions $\sigma(z; \alpha, \rho)$ in T with $0 < \alpha < 1$ and ρ not an integer. The functions in T_0 are called transstable functions.

Let us proceed to the properties of the functions.

(1) If $\alpha > 1$, then, for any ρ, $\sigma(z, \alpha, \rho)$ is an entire analytic function.

(2) If $\alpha = 1$ and ρ is not an integer, then $\sigma(z, 1, \rho)$ is a meromorphic analytic function with two simple conjugate poles which are solutions of the equation

$$z^2 + 2z\cos(\rho\pi) + 1 = 0.$$

The function $\sigma(z, 1, \rho)$ is equal to zero for any integer ρ.

(3) If $0 < \alpha < 1$, $x > 0$, and ρ is an integer, then

$$\sigma(x, \alpha, \rho) = 0.$$

(4) If $\alpha > 1$ and $x > 0$ or if $\alpha = 1$ and $0 < x < 1$, then

$$x^{-\alpha}\sigma(x^{-\alpha}, 1/\alpha, \alpha\rho) = x\sigma(x, \alpha, \rho).$$

(5) Each function $\sigma(z, \alpha, \rho)$ is periodic in the variable ρ with period $T_\alpha = 2\min\{1, \alpha\}$, i.e.,

$$\sigma(z, \alpha, \rho + T_\alpha) = \sigma(z, \alpha, \rho).$$

(6) For any transstable function

$$\sigma(z, \alpha, -\rho) = -\sigma(z, \alpha, \rho).$$

(7) If $\alpha \ge 1$ then for any complex number z

$$\sigma(-z, \alpha, \rho) = \sigma(z, \alpha, \alpha - \rho),$$
$$z\sigma(z, \alpha, \rho) = z^{-\alpha}\sigma(z^{-\alpha}, 1/\alpha, \alpha\rho).$$

(8) Suppose that $\alpha \geq 1$ and $|\rho| \leq \alpha$, i.e., we consider points (α, ρ) in a strip where the periodicity with respect to ρ does not yet manifest itself. The Fourier transform $\hat{\sigma}(k; \alpha, \rho)$ of the function $\sigma(x; \alpha, \rho)$ exists if and only if $|\rho \pm \alpha/2| \leq 1/2$, which corresponds to the domain bounded by the curves

$$\rho = (\alpha + 1)/2$$

and

$$\rho = (\alpha - 1)/2$$

and the domain symmetric to it with respect to the α-axis. In the first domain

$$\ln \hat{\sigma}(k; \alpha, \rho) = -|k|^{\alpha} \exp\left\{-i(\rho - \alpha/2)\pi \operatorname{sign} k\right\}.$$

In particular, for the symmetric function

$$\hat{\sigma}(k; \alpha, \alpha/2) = \exp\left\{-|k|^{\alpha}\right\}, \quad \alpha \geq 1.$$

(9) Suppose that $\alpha > 0$ and $|\rho| \leq T_{\alpha}/2$. Consider the Mellin transform of the function $x\sigma(x; \alpha, \rho)$:

$$R(s; \alpha, \rho) = \int_0^{\infty} x^s \sigma(x; \alpha, \rho)\, dx.$$

If $0 < \alpha \leq 1$, then $R(s; \alpha, \rho)$ exists for values in some neighborhood of the point $s = 0$ for any ρ. But if $\alpha > 1$ then $R(s; \alpha, \rho)$ exists for s in some neighborhood of zero if and only if

$$|\rho| \leq (\alpha + 1)/2.$$

If $R(s; \alpha, \rho)$ exists, then it exists for all $s \in (-1, \alpha)$ and is given by the function (5.6.3).

These and some other properties of transstable functions are listed and discussed in (Zolotarev, 1986).

6.11. Concluding remarks

We conclude this chapter with brief walk through some other generalizations of stable distribution.

Let us fix some subsequence $\{k(n)\}$ of natural numbers, and consider the asymptotic behavior of the sequence of distributions of the sums

$$S_n^* = B_n^{-1} \sum_{i=1}^{k(n)} X_i + C_n.$$

Khinchin demonstrated that any infinitely divisible distribution can play the role of the limit distribution in this scheme. Thus, in order to distinguish some non-trivial subclass of limit distributions, we need to impose some additional constraints on $\{k(n)\}$. Such a generalization was considered by Shimizu, Pillai, and Kruglov, who dwell on the case where

$$k(n) \to \infty, \quad k(n) \le k(n+1), \quad k(n+1)/k(n) \to r \in [1, \infty).$$

The limit distributions arisen are referred to as semi-stable. Kruglov established that a probability distribution is semi-stable as soon as it is either Gaussian or infinitely divisible with no Gaussian component with the spectral function $H(x)$ of the form

$$H(x) = \begin{cases} |x|^{-\alpha}\theta_1(\ln|x|), & x < 0, \\ -|x|^{-\alpha}\theta_2(\ln|x|), & x > 0, \end{cases}$$

where $\alpha \in (0, 2)$, $\theta_i(y)$ are periodic functions with one and the same period.

The so-called pseudo-stable distribution are close to semi-stable ones. A pseudo-stable distribution is either Gaussian or infinitely divisible with no Gaussian component whose Lévy spectral function is of the form

$$H(\pm x) = x^{-\alpha}\theta_\pm(\ln x), \qquad x > 0,$$

where $0 < \alpha < 2$, $\theta_\pm(y)$ are continuous almost periodical functions with finite Fourier spectrum. Moreover,

$$H(\pm x) = \mp \sum_{k=1}^{l} \theta_k(y)\varphi_k^\pm(x/y),$$

where $x > 0$, $y > 0$, the functions $\theta_k(y)$ and $\varphi_k^\pm(x)$ are non-negative, continuous, monotone decreasing, and allowing for the representation

$$\theta_k(x) = x^{-\alpha}\left[a_{k0} + \sum_{m=1}^{l_k}\left(a_{km}\cos(\omega_m \ln x) + b_{km}\sin(\omega_m \ln x)\right)\right],$$

$$\varphi_k^\pm(y) = y^{-\alpha}\left[c_{k0} + \sum_{m=1}^{l_k}\left(c_{km}^\pm\cos(\omega_m \ln x) + d_{km}^\pm\sin(\omega_m \ln x)\right)\right],$$

$y > 0$, $1 + 2l_k \le l$, $0 < \alpha < 2$, $\omega_m > 0$. Recently, a series of new results along these directions are obtained by Yu.S. Khokhlov (Khokhlov, 1996).

Schneider suggested a generalization of one-sided stable distributions by specifying their Laplace transforms

$$\phi_{m,\alpha}(\lambda) = \int_0^\infty dx e^{-\lambda x} p_{m,\alpha}(x)$$

as

$$\begin{aligned}\phi_{m,\alpha}(\lambda) &= AbH_{0m}^{m0}\left(\frac{\lambda}{b}\,\middle|\, \begin{matrix} - \\ ((k-1)/a, 1/a), k = 1, \ldots, m \end{matrix}\right)\\ &= Aab\sum_{k=1}^{m}\sum_{n=0}^{\infty} c_{k,n}\frac{(-1)^n}{n!}\left(\frac{\lambda}{b}\right)^{k-1+na}\end{aligned}$$

with

$$\begin{aligned}c_{k,n} &= \prod_{j=1, j\neq k}^{m}\Gamma((j-k)/a - n),\\ a &= m + \alpha - 1,\\ b &= \left[\frac{a^m}{\Gamma(1-\alpha)}\right]^{1/a}.\end{aligned}$$

In particular,

$$\phi_{1,\alpha}(\lambda) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{\lambda}{b}\right)^{na} = \exp\left\{-(\lambda/b)^{\alpha}\right\}$$

and we have the one-sided stable distribution.

7

Multivariate stable laws

7.1. Bivariate stable distributions

The concept of stable laws can be naturally extended to the case of multidimensional (and even infinite-dimensional) spaces. We start with the consideration of strictly stable bivariate and (in the following section) trivariate distributions as the most important for physical applications.

Let $\mathbf{X} = (X_1, X_2)$ be a two-dimensional random vector on a plane. In the polar coordinates, it is represented through the variables

$$R = \sqrt{X_1^2 + X_2^2}, \qquad \Phi = \arctan(X_2/X_1).$$

If we denote by $w(\varphi)$ the probability density of the angle Φ between the vector $\mathbf{X}$ and the x-axis

$$w(\varphi)\,d\varphi = \mathsf{P}\{\Phi \in d\varphi\},$$

then the bivariate probability density of $\mathbf{X}$ can be rewritten as

$$p_2(\mathbf{r}) = p_2(r,\varphi) = p(r \mid \varphi)w(\varphi), \qquad r = |\mathbf{r}|. \tag{7.1.1}$$

By normalization,

$$\int_0^{2\pi} d\varphi \int_0^{\infty} dr\, r p_2(r,\varphi) = \int_0^{2\pi} d\varphi\, w(\varphi) \int_0^{\infty} p(r \mid \varphi) r\, dr = 1,$$

where

$$\int_0^{2\pi} d\varphi w(\varphi) = 1, \tag{7.1.2}$$

$$\int_0^{\infty} p(r \mid \varphi) r\, dr = 1. \tag{7.1.3}$$

The factor $p(r \mid \varphi)r$ is the conditional density function for the absolute value R of the vector $\mathbf{X}$, whereas

$$\int_0^r p(r' \mid \varphi)r' dr' = F_R(r \mid \varphi) \tag{7.1.4}$$

is the corresponding distribution function. The probability for the absolute value R to exceed a given value r by a fixed $\Phi = \varphi$ is expressed through this function by the relation:

$$\mathsf{P}\{R > r \mid \varphi\} = 1 - F_R(r \mid \varphi). \tag{7.1.5}$$

As well as in the univariate case, we consider the power distribution of the absolute value

$$\mathsf{P}\{R > r \mid \varphi\} = \begin{cases} c(\varphi)r^{-\alpha}, & r > \varepsilon(\varphi), \\ 1, & r < \varepsilon(\varphi), \end{cases} \tag{7.1.6}$$

where the positive α does not depend on φ, and $\varepsilon(\varphi)$ is determined by the normalization

$$1 = \mathsf{P}\{R > \varepsilon(\varphi) \mid \varphi\} = c(\varphi)\varepsilon^{-\alpha},$$

which yields

$$\varepsilon(\varphi) = [c(\varphi)]^{1/\alpha}.$$

Differentiating both parts of equality (7.1.5) with account of (7.1.4) and (7.1.6), we obtain:

$$p(r \mid \varphi) = \begin{cases} \alpha c(\varphi)r^{-\alpha-2}, & r > [c(\varphi)]^{1/\alpha}, \\ 0, & r < [c(\varphi)]^{1/\alpha}. \end{cases} \tag{7.1.7}$$

The characteristic function of the bivariate distribution $p_2(\mathbf{r})$

$$f_2(\mathbf{k}) = \mathsf{E}e^{i\mathbf{kX}} = \int_{\mathbb{R}^2} e^{i\mathbf{kr}} p_2(\mathbf{r})\, d\mathbf{r}, \qquad \mathbf{k} \in \mathbb{R}^2,$$

can be rewritten (see (7.1.1)) as

$$f_2(\mathbf{k}) = \int_0^{2\pi} d\varphi\, w(\varphi) \int_0^\infty e^{i\mathbf{kr}} p(r \mid \varphi) r\, dr.$$

Denoting the polar coordinates of the vector $\mathbf{k}$ by k and ϑ, one can rewrite the last expression as

$$f_2(k, \theta) = \int_0^{2\pi} d\varphi\, w(\varphi) \int_0^\infty e^{ikr\cos(\vartheta-\varphi)} p(r \mid \varphi) r\, dr.$$

Recalling (7.1.7), we obtain

$$f_2(k,\theta) = \alpha \int_0^{2\pi} d\varphi W(\varphi) \int_{[c(\varphi)]^{1/\alpha}}^{\infty} e^{ikr\cos(\vartheta-\varphi)} r^{-\alpha-1} dr, \tag{7.1.8}$$

where

$$W(\varphi) \equiv w(\varphi)c(\varphi) \tag{7.1.9}$$

is a non-negative function integrable on $[0, 2\pi)$. With $k = 0$, (7.1.8) turns into 1, which is in accordance with the normalization of density (7.1.1) and is a quite natural property of characteristic functions.

We denote by ϕ_+ and ϕ_- the ranges of φ where $\cos(\vartheta - \varphi)$ are positive and negative respectively; then characteristic function (7.1.8) can be represented as

$$f_2(k,\theta) = \alpha \int_{\phi_+} d\varphi W(\varphi) \int_{[c(\varphi)]^{1/\alpha}}^{\infty} e^{ikr|\cos(\vartheta-\varphi)|} r^{-\alpha-1} dr$$
$$+ \alpha \int_{\phi_-} d\varphi W(\varphi) \int_{[c(\varphi)]^{1/\alpha}}^{\infty} e^{-ikr|\cos(\vartheta-\varphi)|} r^{-\alpha-1} dr.$$

Passing to the new integration variable $z = kr|\cos(\vartheta - \varphi)|$ and using (3.3.14)–(3.3.15), we obtain

$$f_2(k,\theta) = \alpha k^\alpha \int_0^{2\pi} d\varphi W(\varphi)|\cos(\vartheta-\varphi)|^\alpha \left\{ I_c^{(-\alpha)}(k[c(\varphi)]^{1/\alpha}|\cos(\vartheta-\varphi)|) \right.$$
$$\left. + iI_s^{(-\alpha)}(k[c(\varphi)]^{1/\alpha}|\cos(\vartheta-\varphi)|)\,\mathrm{sign}[\cos(\vartheta-\varphi)] \right\}. \tag{7.1.10}$$

Let us turn to the behavior of characteristic function (7.1.10) in the domain of small k. We begin with the case $\alpha < 1$. According to (3.3.19)–(3.3.20) with $k \to 0$ and $\alpha < 1$,

$$I_s^{(-\alpha)}(k[c(\varphi)]^{1/\alpha}|\cos(\vartheta-\varphi)|) \to -\Gamma(-\alpha)\sin(\alpha\pi/2),$$
$$I_c^{(-\alpha)}(k[c(\varphi)]^{1/\alpha}|\cos(\vartheta-\varphi)|) \sim \alpha^{-1}k^{-\alpha}[c(\varphi)]^{-1}|\cos(\vartheta-\varphi)|^{-\alpha}$$
$$- \alpha^{-1}\Gamma(1-\alpha)\cos(\alpha\pi/2).$$

Substituting these expressions into (7.1.10) and using relations (7.1.2) and (7.1.5), we obtain the following formula for characteristic functions in the domain of small k:

$$f_2(k,\vartheta) \sim 1 - k^\alpha\Gamma(1-\alpha)\cos(\alpha\pi/2)\int_0^{2\pi} d\varphi\, W(\varphi)|\cos(\vartheta-\varphi)|^\alpha$$
$$- ik^\alpha\alpha\Gamma(-\alpha)\sin(\alpha\pi/2)\int_0^{2\pi} d\varphi\, W(\varphi)|\cos(\vartheta-\varphi)|^\alpha\,\mathrm{sign}[\cos(\vartheta-\varphi)]. \tag{7.1.11}$$

We denote

$$\Gamma(1-\alpha)\cos(\alpha\pi/2)\int_0^{2\pi} d\varphi\, W(\varphi)|\cos(\vartheta-\varphi)|^\alpha = \lambda(\vartheta), \tag{7.1.12}$$

$$\frac{\int_0^{2\pi} d\varphi\, W(\varphi)|\cos(\vartheta-\varphi)|^\alpha \operatorname{sign}[\cos(\vartheta-\varphi)]}{\int_0^{2\pi} d\varphi\, W(\varphi)|\cos(\vartheta-\varphi)|^\alpha} = \beta(\vartheta); \tag{7.1.13}$$

then (7.1.11) takes the form

$$f_2(k,\theta) \sim 1 - \lambda(\vartheta)k^\alpha[1 - i\beta(\vartheta)\tan(\alpha\pi/2)]. \tag{7.1.14}$$

Since k, being the absolute value of the vector $\mathbf{k}$, cannot be negative, we can immediately proceed to the derivation of the limiting characteristic function for the normalized sum of random vectors $\mathbf{S}_n$:

$$\mathbf{Z}_n = \mathbf{S}_n/b_n, \qquad b_n = b_1 n^{1/\alpha}.$$

Repeating the reasoning cited at the end of Section 3.3, we obtain

$$g_2(\mathbf{k};\alpha,\beta(\cdot),\lambda(\cdot)) = \lim_{n\to\infty} f_{\mathbf{Z}_n}(\mathbf{k}) = \exp\left\{-\lambda(\vartheta)k^\alpha[1 - i\beta(\vartheta)\tan(\alpha\pi/2)]\right\}. \tag{7.1.15}$$

The case $\alpha > 1$ is treated in the same way as for univariate distributions (Section 3.4) and provides us with the same form (7.1.15) for characteristic functions of the centered vector sums

$$\mathbf{Z}_n = (\mathbf{S}_n - n\mathbf{a})/b_n, \quad \mathbf{a} = \mathsf{E}\mathbf{X}. \tag{7.1.16}$$

Unlike (3.4.24), expression (7.1.15) includes the dependencies $\lambda(\vartheta)$ and $\beta(\vartheta)$ and, consequently, the set of bivariate strictly stable laws is a non-parametric family.

For axially symmetric distributions, the functions

$$w(\varphi) = 1/(2\pi), \qquad c(\varphi) = c$$

do not depend on the angle φ, therefore

$$\lambda(\vartheta) \equiv \lambda = 2^{\alpha+1}B\left[(\alpha+1)/2, (\alpha+1)/2\right](c/4\pi)\Gamma(1-\alpha)\cos(\alpha\pi/2)$$

and

$$\beta(\vartheta) = 0.$$

In this case, the characteristic function also does not depend on the azimuth angle ϑ and takes the form

$$g_2(\mathbf{k};\alpha,0,\lambda) = e^{-\lambda k^\alpha}.$$

Setting $\lambda = 1$, we obtain the characteristic function in the reduced (or standard) form

$$g_2(\mathbf{k}; \alpha) = e^{-k^\alpha}, \qquad k \geq 0. \tag{7.1.17}$$

One can demonstrate that the results obtained here are in accordance with those for univariate distributions. For this purpose, the vector $\mathbf{X}$ should be chosen so that its projection X_2 is always equal to zero. In this case, φ can take only two values: $\varphi = 0$ (the vector $\mathbf{X}$ is directed along the x-axis) and $\varphi = \pi$ (the vector $\mathbf{X}$ has the opposite direction). The angular distribution $w(\varphi)$ is of the form

$$w(\varphi) = p_1\delta(\varphi) + p_2\delta(\varphi - \pi), \tag{7.1.18}$$

where

$$p_1, p_2 \geq 0, \qquad p_1 + p_2 = 1.$$

It is clear that the sum of any number of such vectors lies on the x-axis as well. The corresponding characteristic function

$$\mathsf{E}e^{i(k_1X_1+k_2X_2)} = \mathsf{E}e^{ik_1X_1}$$

depends on a single variable k_1,

$$k_1 = k\cos\varphi = \begin{cases} k, & \varphi = 0, \\ -k, & \varphi = \pi. \end{cases}$$

Thus, from (7.1.15) for $g_2(\mathbf{k}; \dots) = g_2(k, \theta; \dots)$ we obtain

$$\begin{aligned} g_2(k_1, 0; \alpha, \beta(0), \lambda(0)) &= \exp\left\{-\lambda(0)k_1^\alpha[1 - i\beta(0)\tan(\alpha\pi/2)]\right\}, && k_1 > 0, \\ g_2(k_1, \pi; \alpha, \beta(\pi), \lambda(\pi)) &= \exp\left\{-\lambda(\pi)(-k_1)^\alpha[1 - i\beta(\pi)\tan(\alpha\pi/2)]\right\}, && k_1 < 0. \end{aligned}$$

Substituting (7.1.18) into (7.1.12)–(7.1.13), and taking (7.1.9) into account, we obtain

$$\begin{aligned} \lambda(0) &= \lambda(\pi) = \Gamma(1-\alpha)\cos(\alpha\pi/2)[p_1c(0) + p_2c(\pi)], \\ \beta(0) &= [p_1c(0) - p_2c(\pi)]/[p_1c(0) + p_2c(\pi)], \\ \beta(\pi) &= -\beta(0). \end{aligned}$$

Writing λ and β for $\lambda(0)$ and $\beta(0)$, respectively, we combine the two expressions into one:

$$\begin{aligned} g(k_1; \alpha, \beta(0), \lambda(0)) &= \exp\left\{-\lambda|k_1|^\alpha[1 - i\beta\tan(\alpha\pi/2)\operatorname{sign} k_1]\right\} \\ &= g_1(k_1; \alpha, \beta, 0, \lambda), \end{aligned}$$

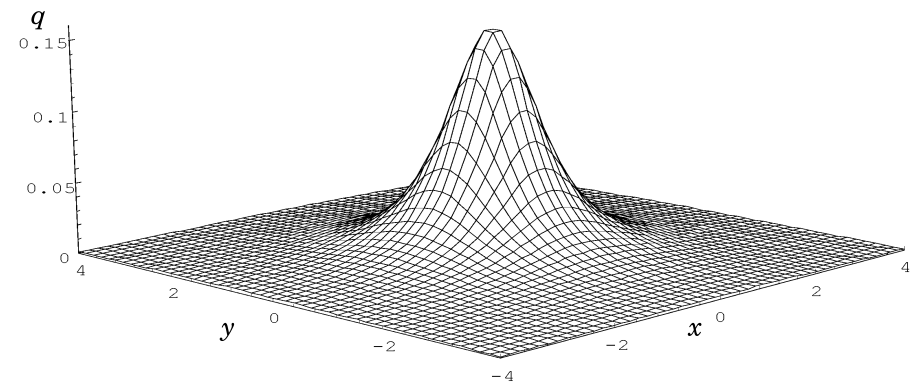


Figure 7.1. Bivariate Cauchy distribution density $q_2(\mathbf{r}; 1)$

which coincides with the univariate stable distribution with $\gamma = 0$.

The bivariate stable density is given by the corresponding inverse Fourier transformation:

$$q_2(\mathbf{r}; \alpha, \beta(\cdot), \lambda(\cdot)) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-i\mathbf{kr}} g_2(\mathbf{k}; \alpha, \beta(\cdot), \lambda(\cdot))\, d\mathbf{k}. \tag{7.1.19}$$

In particular, in the axially symmetric case ($\beta(\cdot) = 0$, $\lambda(\cdot) = 1$), (7.1.19) terns into

$$q_2(\mathbf{r}; \alpha) = \frac{1}{2\pi} \int_0^\infty e^{-k^\alpha} J_0(kr) k\, dk \tag{7.1.20}$$

(here and in what follows by $J_\varkappa$ we denote the Bessel function of order $\varkappa$). Setting here $\alpha = 1$ and $\alpha = 2$, we find the bivariate Cauchy and Gaussian distribution densities, respectively:

$$q_2(\mathbf{r}; 1) = \frac{1}{2\pi(1 + r^2)^{3/2}}, \tag{7.1.21}$$

$$q_2(\mathbf{r}; 2) = \frac{1}{4\pi} e^{-r^2/4}. \tag{7.1.22}$$

These functions are plotted in Figures 7.1, 7.2.

Expanding the functions under the integral (7.1.20) into the Taylor series,

$$e^{-k^\alpha} = \sum_{n=0}^\infty \frac{(-k^\alpha)^n}{n!},$$

$$J_0(kr) = \sum_{n=0}^\infty (-1)^n \frac{(kr)^{2n}}{2^{2n}(n!)^2},$$

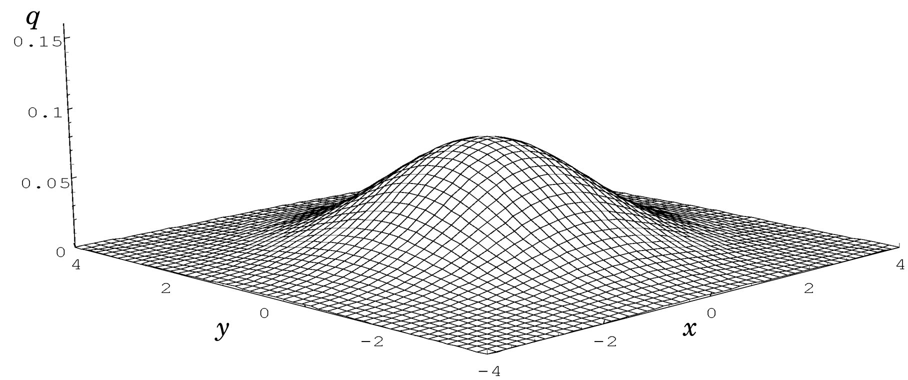


Figure 7.2. Bivariate Gauss distribution density $q_2(\mathbf{r}; 2)$

we obtain, respectively,

$$q_2(\mathbf{r};\alpha) = \frac{1}{\pi^2 r^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}\left[\Gamma(n\alpha/2+1)\right]^2 \sin(\alpha n\pi/2)(r/2)^{-n\alpha}, \tag{7.1.23}$$

$$q_2(\mathbf{r};\alpha) = \frac{1}{2\pi\alpha}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\Gamma((2n+2)/\alpha)(r/2)^{2n}. \tag{7.1.24}$$

The former of these series converges for $\alpha < 1$, and is the asymptotic series for $\alpha > 1$. The latter series, on the contrary, converges for $\alpha \geq 1$, and is asymptotic one for $\alpha < 1$.

Writing $\mathbf{Y}(\alpha) = \{Y_1(\alpha), Y_2(\alpha)\}$ for random vectors distributed by axially symmetric stable laws, we make some remarks.

The characteristic function of $\mathbf{Y}(\alpha)$, as a function of $k = |\mathbf{k}|$, is of the same form as the characteristic function of the univariate random variable $Y(\alpha, 0)$ with the same characteristic α and $\beta = 0$:

$$g_2(\mathbf{k};\alpha) = e^{-|\mathbf{k}|^\alpha}, \quad g_1(k;\alpha,0) = e^{-|k|^\alpha}. \tag{7.1.25}$$

In the case $\alpha = 2$ (and only in this case), the characteristic function is factorized into two components

$$g_2(\mathbf{k}_1;\alpha) = e^{-k^2} = e^{-k_1^2}e^{-k_2^2} = g_1(k_1;2,0)g_1(k_2;2,0), \tag{7.1.26}$$

each of which depends only on one of the components k_1, k_2.

As one can see from the expression of the characteristic function in terms of Cartesian coordinates

$$g_2(k_1,k_2;\alpha) = \iint_{\mathbb{R}^2} e^{ik_1x_1+ik_2x_2} q_2(x_1,x_2;\alpha)\,dx_1\,dx_2, \tag{7.1.27}$$

$g_2(k_1, 0; \alpha)$ and $g_2(0, k_2; \alpha)$ are the characteristic functions of the components $Y_1(\alpha)$ and $Y_2(\alpha)$ respectively. Thus, (7.1.26) says that the components $Y_1(2)$ and $Y_2(2)$ of the vector $\mathbf{Y}(2)$ are mutually independent. This is a well-known property of bivariate (in general—multivariate) symmetric Gaussian distributions. We want to stress here that this property takes place only for the Gaussian distribution. No other bivariate (in general—multivariate) stable law possess such a property.

In view of the abovesaid, the relations

$$g_2(k_1, 0; \alpha) = e^{-|k_1|^\alpha},$$
$$g_2(0, k_2; \alpha) = e^{-|k_2|^\alpha}$$

mean that each of the components $Y_1(\alpha)$ and $Y_2(\alpha)$ has univariate stable distribution with the same characteristic α and $\beta = 0$. This is true for all admissible α.

7.2. Trivariate stable distributions

To derive characteristic functions of trivariate stable laws, we follow the same way as before, We present a three-dimensional random vector $\mathbf{X}$ as

$$\mathbf{X} = R\mathbf{U},$$

where R is the absolute value of the vector $\mathbf{X}$, and $\mathbf{U}$ is the unit vector indicating the direction of $\mathbf{X}$. The trivariate distribution density can be written as

$$p_3(\mathbf{r}) = p(r \mid \mathbf{\Omega})w(\mathbf{\Omega}),$$

where $\mathbf{\Omega} = \mathbf{r}/r$, and

$$\int w(\mathbf{\Omega})d\mathbf{\Omega} = 1.$$

It follows from the normalization

$$\iint p_3(r, \mathbf{\Omega})r^2 \, dr \, d\mathbf{\Omega} = 1$$

that

$$\int_0^\infty p(r \mid \mathbf{\Omega})r^2 dr = 1.$$

The conditional density $p(r \mid \mathbf{\Omega})r^2$ of R corresponds to the distribution function

$$F_R(r \mid \mathbf{\Omega}) = \int_0^r p(r \mid \mathbf{\Omega})r^2 dr.$$

Taking this distribution in the form

$$1 - F_R(r \mid \mathbf{\Omega}) = \begin{cases} c(\mathbf{\Omega})r^{-\alpha}, & r > \varepsilon(\mathbf{\Omega}), \\ 1, & r < \varepsilon(\mathbf{\Omega}), \end{cases}$$

we obtain for the conditional density

$$p(r \mid \mathbf{\Omega}) = \begin{cases} \alpha c(\mathbf{\Omega})r^{-\alpha-3}, & r > [c(\mathbf{\Omega})]^{1/\alpha}, \\ 0, & r < [c(\mathbf{\Omega})]^{1/\alpha}. \end{cases} \tag{7.2.1}$$

The characteristic function of the three-dimensional vector $\mathbf{X}$ can be represented as

$$f_3(\mathbf{k}) = \int d\mathbf{\Omega} w(\mathbf{\Omega}) \int e^{i\mathbf{k\Omega}r} p(r \mid \mathbf{\Omega}) r^2 dr,$$

where $\mathbf{k}$ is a three-dimensional vector. Recalling (7.2.1), we obtain

$$f_3(\mathbf{k}) = \alpha \int d\mathbf{\Omega}\, W(\mathbf{\Omega}) \int_{[c(\mathbf{\Omega})]^{1/\alpha}} e^{i\mathbf{k\Omega}r} r^{-\alpha-1} dr, \tag{7.2.2}$$

where

$$W(\mathbf{\Omega}) = w(\mathbf{\Omega})c(\mathbf{\Omega}).$$

Formula (7.2.2) is similar to (7.1.8) with the only difference that we see here $\mathbf{\Omega}$ instead of φ, and the scalar product $\mathbf{k\Omega}$ occurs instead of $k\cos(\vartheta - \varphi)$. In this connection, there is no necessity to repeat all calculations of the preceding section. The result is rather obvious: instead of (7.1.15) we obtain

$$g_3(\mathbf{k}; \alpha, \beta(\cdot), \lambda(\cdot)) = \exp\left\{-\lambda(\mathbf{u})k^\alpha[1 - i\beta(\mathbf{u})\tan(\alpha\pi/2)]\right\}, \quad k = |\mathbf{k}|, \quad \mathbf{u} = \mathbf{k}/k. \tag{7.2.3}$$

Here

$$\lambda(\mathbf{u}) = \Gamma(1-\alpha)|\cos(\alpha\pi/2)| \int d\mathbf{\Omega}\, W(\mathbf{\Omega})|\mathbf{\Omega u}|^\alpha, \tag{7.2.4}$$

$$\beta(\mathbf{u}) = \frac{\int d\mathbf{\Omega}\, W(\mathbf{\Omega})|\mathbf{\Omega u}|^\alpha \operatorname{sign}(\mathbf{\Omega u})}{\int d\mathbf{\Omega}\, W(\mathbf{\Omega})|\mathbf{\Omega u}|^\alpha}. \tag{7.2.5}$$

Relation (7.2.3) holds true for all $\alpha \in (0, 2]$ except $\alpha = 1$: in the latter case, it is good only for spherically symmetric (isotropic) distributions when $\beta(\mathbf{u}) = 0$ (the trivariate Cauchy distribution).

Trivariate densities are obtained from characteristic functions by means of the trivariate inverse Fourier transformation

$$q_3(\mathbf{r}; \alpha, \beta(\cdot), \lambda(\cdot)) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{-i\mathbf{kr}} g_3(\mathbf{k}; \alpha, \beta(\cdot), \lambda(\cdot))\, d\mathbf{k}.$$

In the spherically symmetric case ($\beta(\cdot) = 0$, $\lambda(\cdot) = 1$) this expression takes the form

$$q_3(\mathbf{r};\alpha) = \frac{1}{2\pi^2 r}\int_0^\infty e^{-k^\alpha}\sin(kr)k\,dk. \tag{7.2.6}$$

For $\alpha = 1$ and $\alpha = 2$, from (7.2.6) we obtain the trivariate Cauchy and Gauss distributions, respectively:

$$q_3(\mathbf{r};1) = \frac{1}{\pi^2(1+r^2)^2}, \tag{7.2.7}$$

$$q_3(\mathbf{r};2) = \frac{1}{(4\pi)^{3/2}}e^{-r^2/4}. \tag{7.2.8}$$

The remarks concerning bivariate axially symmetric stable distributions in the preceding section can be reformulated for trivariate spherically symmetric stable distributions in an evident way. But in this case there exists a worthwhile fourth property. We rewrite (7.2.6) as

$$q_3(\mathbf{r};\alpha) = \rho_3(|\mathbf{r}|;\alpha),$$
$$\rho_3(r;\alpha) = \frac{1}{2\pi^2 r}\int_0^\infty e^{-k^\alpha}\sin(kr)k\,dk,$$

and compare it with the univariate analogue

$$\rho_1(r;\alpha) \equiv q(r;\alpha,0) = \pi^{-1}\int_0^\infty e^{-k^\alpha}\cos(kr)\,dk$$

which follows from (4.1.2). It is easy to see that these functions are related to each other by the equality

$$\rho_3(r;\alpha) = -\frac{1}{2\pi r}\frac{d\rho_1(r;\alpha)}{dr}. \tag{7.2.9}$$

This is not an inherent property of stable distribution. Relation (7.2.9) holds for any spherically symmetric distribution if ρ_1 is the distribution of one of the components of the random vector. The key point is that in the case under consideration ρ_1 turns out to be stable distribution as well.

As follows from (4.2.4), (4.2.9), (4.3.2), (4.3.3), and (4.4.14), we can present the following expressions for $\rho_1(r;\alpha) \equiv q(r;\alpha,0)$:

$$\rho_1(r;\alpha) = \frac{\alpha r^{1/(\alpha-1)}}{\pi|1-\alpha|}\int_0^{\pi/2}\exp\{-r^{\alpha/(\alpha-1)}U(\varphi;\alpha,0)\}U(\varphi;\alpha,0)\,d\varphi, \tag{7.2.10}$$

$$\rho_1(r;\alpha) = \frac{1}{\pi}\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n!}\Gamma(n/\alpha+1)\sin(n\pi/2)r^{n-1}, \tag{7.2.11}$$

$$\rho_1(r;\alpha) = \frac{1}{\pi}\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n!}\Gamma(n\alpha+1)\sin(n\alpha\pi/2)r^{-n\alpha-1}. \tag{7.2.12}$$

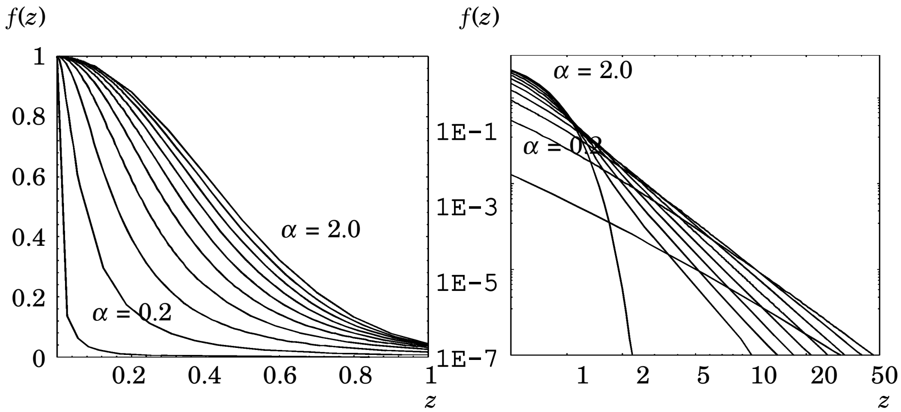


Figure 7.3. $f(z;\alpha) = a^{-3}\rho_3(z/a;\alpha)$, $a = [\rho_3(0;\alpha)]^{1/3}$, $\alpha = 0.2, 0.4, \dots, 1.8, 2.0$ for $z < 1$ and $z > 0.5$

Series (7.2.11) is convergent for $\alpha > 1$ and asymptotic ($x \to 0$) for $\alpha < 1$. It can be represented in a simpler form

$$\rho_1(r;\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\pi\alpha} \frac{\Gamma((2n+1)/\alpha)}{(2n)!} r^{2n}. \tag{7.2.13}$$

Series (7.2.12) is convergent for $\alpha < 1$ and asymptotic ($x \to \infty$) for $\alpha > 1$.

Substituting (7.2.10), (7.2.13), and (7.2.12) into (7.2.9), we obtain

$$\rho_3(r;\alpha) = \frac{\alpha r^{1/(\alpha-1)}}{2\pi^2 r^2 (\alpha-1)|1-\alpha|} \int_0^{\pi/2} \exp\left\{-r^{\alpha/(\alpha-1)} U(\varphi;\alpha,0)\right\} \times U(\varphi;\alpha,0)[\alpha r^{\alpha/(\alpha-1)} U(\varphi;\alpha,0) - 1]\, d\varphi, \tag{7.2.14}$$

$$\rho_3(r;\alpha) = \frac{1}{2\pi^2\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \Gamma\left(\frac{2n+3}{\alpha}\right) r^{2n}, \tag{7.2.15}$$

$$\rho_3(r;\alpha) = \frac{1}{2\pi^2 r} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \Gamma(n\alpha+2) \sin(n\alpha\pi/2) r^{-n\alpha-2} \tag{7.2.16}$$

respectively. The results obtained with the use of these expressions are shown in Figures 7.3.

7.3. Multivariate stable distributions

The concept of stable laws can be naturally extended to the case of spaces with arbitrary dimensions and even to infinitely-dimensional spaces. We consider a sequence of independent and identically distributed random variables $X_1, X_2, \dots$ taking values of the N-dimensional Euclidean space $\mathbb{R}^N$, and form the sequence of sums

$$Z_n = (X_1 + \dots + X_n - a_n)/b_n, \qquad n = 1, 2, \dots,$$

normalized by some sequences of positive numbers b_n and non-random elements a_n of $\mathbb{R}^N$. The set of all weak limits of the distributions of such sequences Z_n as $n \to \infty$ is called the family of stable distributions on $\mathbb{R}^N$, or the family of Lévy–Feldheim distributions. This is not the only way of generalizing the stable distributions. If the sums $X_1 + \dots + X_n$ are normalized by non-singular matrices σ_n but not by positive numbers b_n^{-1}, then the concept of stable laws becomes essentially more rich. Very little is known at present about the properties of multivariate stable laws (in particular, about their analytic properties). Neither the amount nor the diversity of the facts known here can compare in any way with what is known about the univariate distributions. Here we present and comment the canonical representation of the characteristic function $g_N(k)$, $k \in \mathbb{R}^N$, of finite-dimensional Lévy–Feldheim laws.

As was mentioned above, the distributions form a non-parametric set. The corresponding characteristic functions are of the form

$$g_N(k) = \exp\{i(k, a) - \psi_\alpha(k)\}, \qquad 0 < \alpha \leq 2, \tag{7.3.1}$$

where $a \in \mathbb{R}^N$ and the functions $\psi_\alpha(k)$, which are determined by the parameter α and by a certain finite measure $M(du)$ on the sphere $S = \{u : |u| = 1\}$, are as follows.

If $\alpha = 2$, then $\psi_\alpha(k) = (\sigma k, k)$, where σ is the so-called covariance matrix. If $0 < \alpha < 2$, then

$$\psi_\alpha(k) = \int_S |(k, u)|^\alpha \omega_\alpha(k, u) M(du), \tag{7.3.2}$$

where

$$\omega_\alpha(k, u) = \begin{cases} 1 - i\tan(\alpha\pi/2)\,\mathrm{sign}(k, u), & \alpha \neq 1, \\ 1 + i(2/\pi)\ln|(k, u)|\,\mathrm{sign}(k, u), & \alpha = 1. \end{cases}$$

Representation (7.3.1)–(7.3.2) is an analogue of form A of representation of characteristic functions of univariate stable laws (3.6.1).

This analogue is not the only one. If we use a spherical system of coordinates in $\mathbb{R}^N$ and write a vector k in the form $k = |k|u$, then it is not difficult to represent (7.3.1)–(7.3.2) as (cf. (7.2.3))

$$\ln g_N(k) = \begin{cases} \lambda[i|k|\gamma - |k|^\alpha(1 - i\beta\tan(\alpha\pi/2))], & \alpha \neq 1, \\ \lambda[i|k|\gamma - |k|(1 + i(2/\pi)\beta\ln|k|)], & \alpha = 1, \end{cases} \tag{7.3.3}$$

where $0 < \alpha \leq 2$ and β, γ, and λ are real-valued functions defined on the unit sphere S determined by the equalities

$$\lambda = \lambda(u) = \int_S |(u,u')|^\alpha M(du'), \qquad u \in S,$$

$$\lambda\beta = \lambda\beta(u) = \int_S |(u,u')|^\alpha \operatorname{sign}(u,u') M(du'),$$

$$\lambda\gamma = \lambda\gamma(u) = \begin{cases} (u,a), & \alpha \neq 1, \\ (u,a) - (2/\pi)\int_S (u,u') \ln|(u,u')| M(du'), & \alpha = 1. \end{cases}$$

We give some properties of the functions β, γ, and λ.

(1) They are continuous on S, and for a given α they uniquely determine the shift a and the measure $M(du)$ in the representation (7.3.1)–(7.3.2). In particular, for a given $\alpha \neq 1$ the functions β and λ uniquely determine the measure M.

(2) The domain of variation for the values of the function γ is the whole real axis.

(3) The following relations hold for any $u \in S$:

$$\beta(-u) = -\beta(u), \qquad \lambda(-u) = \lambda(u),$$
$$|\beta(u)| \leq 1, \qquad 0 \leq \lambda(u) \leq M_0,$$

where M_0 is the value of the complete measure $M(du)$ on S.

Here all inequalities are strict, unless $M(du)$ is concentrated entirely on some subspace of $\mathbb{R}^N$. This leads, in particular, to the conclusion that

$$\lambda_0 = \inf\{\lambda(u) \colon u \in S\} > 0, \qquad |g_N(k)| \leq \exp(-\lambda_0|k|^\alpha),$$

and hence the corresponding stable distribution has the density $q_N(x;\alpha,a,M)$ bounded by the quantity

$$\frac{\Gamma(1+N/\alpha)}{\Gamma(1+N/2)}(2\sqrt{\pi}\lambda_0^{1/\alpha})^{-N}.$$

Each of forms M and B also possesses two analogues obtained by transforming the corresponding analogues of form A. Namely, if $\alpha \neq 1$, then

$$\ln g_N(k) = i(k,a) - \int_S \{|(k,u)|^\alpha - i(k,u)\tan(\alpha\pi/2)(|(k,u)|^{\alpha-1} - 1)\} M(du),$$

$$\ln g_N(|k|u) = \lambda[i|k|\gamma - |k|^\alpha + i\beta|k|(|k|^{\alpha-1} - 1)\tan(\alpha\pi/2)]. \tag{7.3.4}$$

In the case where $\alpha = 1$, the functions $\ln g_N(k)$ and $\ln g_N(|k|u)$ are defined just as in (7.3.1)–(7.3.2). As a result, these representations turn out to be continuous functions of α in the whole domain $0 < \alpha \leq 2$ in which α varies.

The first analogue of form B is the representation (7.3.1)–(7.3.3) with $\omega_\alpha(k,u)$ replaced by the function

$$\tilde{\omega}_\alpha(k,u) = \begin{cases} \exp(-i\Phi(\alpha)\,\mathrm{sign}(k,u)), & \alpha \neq 1, \\ \pi/2 + i\ln|(k,u)|\,\mathrm{sign}(k,u), & \alpha = 1, \end{cases}$$

and $M(du)$ replaced by the measure

$$\tilde{M}(du) = \begin{cases} |\cos(\pi\alpha/2)|^{-1}M(du), & \alpha \neq 1, \\ (2/\pi)M(du), & \alpha = 1. \end{cases}$$

The second analogue of form B is obtained from (7.3.3) by the same transformations used in the univariate case to derive form B from A. This representation is

$$\ln g_N(|k|u) = \begin{cases} \lambda[i|k|\gamma - |k|^\alpha \exp(-i\Phi(\alpha)\beta)], & \alpha \neq 1, \\ \lambda[i|k|\gamma - |k|((\pi/2) + i\beta\ln|k|)], & \alpha = 1. \end{cases} \tag{7.3.5}$$

The elements α, β, γ, and λ in (7.3.5) determining the stable distributions are connected with the set of analogous determining elements in (7.3.3) by relations (3.6.4)–(3.6.6).

In those cases where the measure $\tilde{M}(du)$ happens to be concentrated on a half S^* of the sphere S, the first analogue of form B allows us to write the Laplace transform of the corresponding stable distribution $q(x,\alpha,a,\tilde{M})$ as a component-wise analytic continuation of g (the substitution $k = is$). Namely,

$$\int_{\mathbb{R}^N} \exp\{-(s,x)\}q(x;\alpha,a,\tilde{M})dx = \begin{cases} \exp\{-(s,a) - \varepsilon(\alpha)\int_{S^*}(s,u)^\alpha \tilde{M}(du)\}, & \alpha \neq 1, \\ \exp\{-(s,a) - \int_{S^*}(s,u)\ln(s,u)\tilde{M}(du)\}, & \alpha = 1, \end{cases}$$

where $\varepsilon(\alpha) = \mathrm{sign}(1-\alpha)$, and the vector s takes values in the half-space L^* containing S^*.

The representations given for the characteristic functions of stable distributions can serve as a starting point for analytic study of properties of univariate stable laws. Recently, forms (7.3.1)–(7.3.2) are most used in literature, apparently due to the manifestation of a peculiar 'inertia' created by the first investigations.

The analogues of forms B and M have not yet been used in general, although they unquestionably have some merits which may make them no less popular. For example, these representations make it possible to connect the N-dimensional density $q_N(x;\alpha,a,M)$ with the one-dimensional one $q_1(y;\alpha,\beta,\gamma,\lambda)$ in the case where N is odd: $N = 2m+1$. Namely, for all $x \in \mathbb{R}^N$

$$q_N(x;\alpha,a,M) = \frac{(-1)^m}{2(2\pi)^{2m}} \int_S q_1^{(2m)}((u,x),\alpha,\beta,\gamma,\lambda)\,du, \tag{7.3.6}$$

where β, γ, and λ are the functions in the representation of the characteristic function of the distribution $q_N(x; \alpha, a, M)$, and where the density $q_1(y; \alpha, \beta, \gamma, \lambda)$ corresponds to form A or B, depending on which of the representations, (7.3.3) or (7.3.5), is in use.

The functions β, γ, and λ (irrespective of the representation they are associated with) are, respectively, the generalized asymmetry, shift, and scale characteristics of the distribution, as in their interpretation in the univariate case. However, it should be kept in mind here that in carrying various concepts associated with univariate distributions over to the multivariate case we inevitably encounter a variety of possible generalizations. For instance, the analogue of the univariate symmetric laws, which have characteristic functions of the form

$$g_1(k) = \exp(-\lambda|k|^\alpha),$$

can be taken to be the spherically symmetric distributions on $\mathbb{R}^N$ having the same form of characteristic functions (with λ = const), but the analogues of the symmetric laws can also be taken (and this is more natural) to be the stable distributions with the functions $\beta(u) = \gamma(u) = 0$ for all $u \in S$, which is equivalent to the equality

$$\ln g_N(k) = -\lambda(u)|k|^\alpha = -\int_S |(k,u)|^\alpha M(du),$$

where $M(du)$ is a certain symmetric measure.

It is worth noting that the multivariate stable laws form a subset of multivariate infinitely divisible laws whose characteristic functions (in the Lévy–Khinchin representation)

$$f(k) = \exp\left\{ i(a,k) - (Bk,k) + \int_{\mathbb{R}^N} \left(e^{i(k,x)} - 1 - \frac{i(k,x)}{1+|x|^2} \right) \frac{1+|x|^2}{|x|^2} H(dx) \right\}$$

are generalizations of formula (3.5.14) to the N-dimensional case.

7.4. Spherically symmetric multivariate distributions

Below, we use the following notation. A function $f(x)$, $x \in \mathbb{R}^N$, is referred to as a radial function, if it depends on the distance $|x|$ only. A random vector $X \in \mathbb{R}^N$ and its probability density $p_N(x)$ are called spherically symmetric, if $p_N(x)$ is a radial function, that is,

$$p_N(x) = \rho_N(|x|), \tag{7.4.1}$$

where $\rho_N(r)$ is a function given in the semiaxis $r \geq 0$. The characteristic function $f_N(k)$ of a spherically symmetric vector $X \in \mathbb{R}^N$ is a radial function as well,

$$f_N(k) = \varphi_N(|k|). \tag{7.4.2}$$

It is clear that these functions satisfy the relations

$$\int_{\mathbb{R}^N} \rho_N(|x|)dx = \frac{2\pi^{N/2}}{\Gamma(N/2)} \int_0^\infty \rho_N(r)r^{N-1}dr = 1,$$
$$\varphi_N(0) = 1.$$

The Fourier transformation formulae

$$f_N(k) = \int_{\mathbb{R}^N} e^{i(k,x)} p_N(x)\,dx,$$
$$p_N(x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{-i(k,x)} f_N(k)\,dk$$

imply the following relations for the corresponding radial functions (Sneddon, 1951; Samko *et al.*, 1993):

$$\varphi_N(t) = (2\pi)^{N/2} t^{1-N/2} \int_0^\infty \rho_N(r) J_{N/2-1}(tr) r^{N/2} dr, \tag{7.4.3}$$
$$\rho_N(r) = (2\pi)^{-N/2} r^{1-N/2} \int_0^\infty \varphi_N(t) J_{N/2-1}(rt) t^{N/2} dt \tag{7.4.4}$$

(recall that $J_\varkappa$ stands for the Bessel function of order $\varkappa$).

Let us consider a sequence of distributions $p_1(x), p_2(x_1,x_2), p_3(x_1,x_2,x_3), \dots$ sharing a common characteristic radial function $\phi(t)$ and therefore being spherically symmetric.

THEOREM 7.4.1. *Let $X^N = \{X_1, \dots, X_N\}$ be a random N-dimensional vector with the spherically symmetric density*

$$p_N(x) = \rho_N(|x|), \qquad x \in \mathbb{R}^N.$$

Then its projection on the n-dimensional subspace $\{X_1, \dots, X_n\}$ has the density

$$p_n(x) = \rho_n(|x|), \qquad x \in \mathbb{R}^n,$$

possessing the same characteristic radial function $\varphi(t)$.

PROOF. It is quite obvious that

$$f_n(k_1, \dots, k_n) = f_N(k_1, \dots, k_n, 0, \dots, 0) = \varphi(\sqrt{k_1^2 + \dots + k_n^2}),$$

which completes the proof.

Now, let $\rho_1(r), \rho_2(r), \rho_3(r), \dots$ be some radial functions of the spherically symmetric densities and $\theta_i(s) = \rho_i(\sqrt{s})$, $i = 1, 2, 3, \dots$

THEOREM 7.4.2. *The following relations are true for any $N > 1$:*

$$\theta_N(s) = \frac{1}{\sqrt{\pi}} \left(D_-^{1/2} \theta_{N-1} \right)(s) = \pi^{(1-N)/2} \left(D_-^{(N-1)/2} \theta_1 \right)(s), \tag{7.4.5}$$

where D_-^ν is the fractional derivative. In particular, for any $N > 2$

$$\theta_N(s) = -\frac{1}{\pi} \theta'_{N-2}(s), \tag{7.4.6}$$

or

$$\rho_N(r) = -\frac{1}{2\pi r} \rho'_{N-2}(r). \tag{7.4.7}$$

PROOF. Since the characteristic function $f_N(k)$, $k \in R^N$ depends only on $|k|$, we set

$$f_N(k_1, 0, \dots, 0) = \varphi(t), \qquad t = |k_1| \geq 0;$$

then

$$\begin{aligned} \varphi(t) &= \int_{-\infty}^{\infty} dx_1 \, e^{itx_1} \int_{\mathbb{R}^{N-1}} \rho_N(|x|) dx_2 \dots dx_n \\ &= \int_{-\infty}^{\infty} dx_1 \, e^{itx_1} \int_{\mathbb{R}^{N-1}} \theta_N(x^2) dx_2 \dots dx_n. \end{aligned}$$

Passing to the polar coordinates in the inner integral, we obtain

$$\begin{aligned} \varphi(t) &= \int_{-\infty}^{\infty} dx_1 \, e^{itx_1} \Omega_{N-1} \int_0^{\infty} \theta_N(r^2 + x_1^2) r^{N-2} dr \\ &= \int_{-\infty}^{\infty} dx_1 \, e^{itx_1} \left(\Omega_{N-1}/2 \right) \int_{x_1^2}^{\infty} \theta_N(\sigma)(\sigma - x_1^2)^{(N-3)/2} d\sigma, \\ \Omega_N &= 2\pi^{N/2}/\Gamma(N/2), \end{aligned}$$

which yields

$$\begin{aligned} \theta_1(s) &= \frac{\pi^{(N-1)/2}}{\Gamma((N-1)/2)} \int_s^{\infty} \frac{\theta_N(\sigma) \, d\sigma}{(\sigma - s)^{1-(N-1)/2}} \\ &= \pi^{(N-1)/2} (I_-^{(N-1)/2} \theta_N)(s), \end{aligned}$$

and, after inversion, we arrive at (7.4.5).

For illustration, we apply (7.4.5) to the derivation of the multivariate Cauchy density from the univariate one

$$\rho_1(\sqrt{s}) = \theta_1(s) = \frac{1}{\pi(1+s)}.$$

By virtue of (7.4.5),

$$\theta_N(s) = \pi^{-(N+1)/2} \frac{(-1)^n}{\Gamma(n-(N-1)/2)} \frac{d^n I(s)}{ds^n},$$

where $n = [(N+1)/2]$ is the integer part of $(N+1)/2$, and

$$I(s) = \int_s^\infty \frac{d\sigma}{(1+\sigma)(\sigma-s)^{(N+1)/2-n}\mu}(s+1)^{-\mu} \int_0^\infty \frac{dx}{x^\mu(1+x)} = \frac{\pi}{(s+1)^\mu \sin(\pi\mu)}$$

with

$$\mu = (N+1)/2 - n < 1.$$

Since

$$\frac{d^n}{ds^n}(s+1)^{-\mu} = (-1)^n \mu(\mu+1)\dots(\mu+n-1)(s+1)^{-\mu-n},$$

we finally obtain

$$\theta_N(s) = \frac{\Gamma((N+1)/2)}{[\pi(1+s)]^{(N+1)/2}}$$

and, respectively,

$$q_N(x;1) = \frac{\Gamma((N+1)/2)}{[\pi(1+x^2)]^{(N+1)/2}} \tag{7.4.8}$$

for all N.

7.5. Spherically symmetric stable distributions

To find the characteristic functions of multivariate spherically symmetric stable laws, we make use of the N-dimensional analogue of the Zipf–Pareto symmetric distribution:

$$\mathsf{P}\{|X| > r\} = \begin{cases} Ar^{-\alpha}, & r > \varepsilon = A^{1/\alpha}, \\ 1, & r < \varepsilon. \end{cases}$$

The radial distribution density function is

$$\rho_N(r) = -\frac{1}{S_{N-1}} \frac{dP\{|x| > r\}}{dr} = \frac{\alpha A \Gamma(N/2)}{2\pi^{N/2}} r^{-\alpha-N}, \qquad r > \varepsilon,$$

while the radial characteristic function,

$$\varphi_N(t) = 2^{N/2-1}\alpha A\Gamma(N/2)t^\alpha \int_{\varepsilon t}^{\infty} \tau^{-\alpha-N/2} J_{N/2-1}(\tau)\,d\tau.$$

Integrating by parts, in view of the relation

$$\frac{d}{d\tau}[\tau^{-(N/2-1)}J_{N/2-1}(\tau)] = -\tau^{-N/2+1}J_{N/2}(\tau),$$

we obtain

$$\varphi_N(t) = 2^{N/2-1}A\Gamma(N/2)t^\alpha \left\{(r_0 t)^{-N/2-\alpha+1}J_{N/2-1}(r_0 t) - \int_{\varepsilon t}^{\infty} \tau^{-N/2-\alpha+1}J_{N/2}(\tau)\,d\tau\right\}.$$

As $t \to 0$,

$$\{\dots\} \sim (\varepsilon t)^{-N/2-\alpha+1}[(\varepsilon t/2)^{N/2-1}/\Gamma(N/2) - \dots] - \int_0^\infty \tau^{-N/2-\alpha+1}J_{N/2}(\tau)\,d\tau,$$

which results in

$$1 - \varphi_N(t) \sim \frac{A\Gamma(N/2)\Gamma(1-\alpha/2)}{\Gamma((N+\alpha)/2)}(t/2)^\alpha.$$

Now set

$$Z_n = (X_1 + \dots + X_n)/b_n, \tag{7.5.1}$$

and let $f_N(k)$ be the characteristic function of the normalized vector sum Z_n of independent summands X_i:

$$f_N^{(n)}(k) = \mathsf{E}\exp\left\{ik\sum_{j=1}^{n} X_j/b_n\right\} = \varphi_N^n(|k/b_n|).$$

As $n \to \infty$,

$$\varphi_N^n(k/b_n) \sim \left\{1 - \frac{A\Gamma(N/2)\Gamma(1-\alpha/2)}{\Gamma((N+\alpha)/2)}|k/(2b_n)|^\alpha\right\}^n.$$

Setting

$$b_n = b_1 n^{1/\alpha}, \tag{7.5.2}$$

we obtain

$$f_N^{(n)}(k) \to g_N(k;\alpha) = e^{-|k|^\alpha}, \qquad n \to \infty. \tag{7.5.3}$$

with

$$b_1 = \frac{1}{2}\left[\frac{A\Gamma(N/2)\Gamma(1-\alpha/2)}{\Gamma((N+\alpha)/2)}\right]^{1/\alpha}. \tag{7.5.4}$$

In the univariate case, the coefficients (7.5.2), (7.5.3) coincide with those given in Table 2.1 ($A = 2c$).

Thus, the N-dimensional densities of spherically symmetric stable laws are represented as

$$q_N(x; \alpha) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{-i(k,x)-|k|^\alpha} dk,$$

while the corresponding radial functions, by virtue of (7.4.4), are of the form

$$\rho_N(r; \alpha) = (2\pi)^{-N/2} r^{1-N/2} \int_0^\infty e^{-t^\alpha} J_{N/2-1}(rt) t^{N/2} dt. \tag{7.5.5}$$

Expanding the exponential or the Bessel function into a series, we obtain two expansions of the radial functions of spherically symmetric stable densities

$$\rho_N(r; \alpha) = \frac{1}{\pi(r\sqrt{\pi})^N} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \Gamma((n\alpha + N)/2)\Gamma(n\alpha/2 + 1) \sin(\alpha n\pi/2)(r/2)^{-n\alpha}, \tag{7.5.6}$$

$$\rho_N(r; \alpha) = \frac{2}{\alpha(2\sqrt{\pi})^N} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma((2n + N)/\alpha)}{\Gamma(n + N/2)} (r/2)^{2n}. \tag{7.5.7}$$

Using the known properties of the gamma function, we easily see that, as $N = 1$, the expansions (7.5.6) and (7.5.7) transform to the expansions of symmetric univariate distributions (4.1.4) and (4.1.3) respectively. As in the univariate case, series (7.5.6) is convergent for $\alpha < 1$ and asymptotical for $\alpha \geq 1$, whereas series (7.5.7), on the other hand, converges for $\alpha \geq 1$ and is an asymptotic series for $\alpha < 1$.

Multiplying (7.5.5) by $r^s\,dr$ and integrating along the semiaxis, we obtain the Mellin transform of the radial function

$$\bar{\rho}_N(s; \alpha) = \frac{2^{1+s}}{\alpha(4\pi)^{N/2}} \frac{\Gamma((N - s - 1)/\alpha)\Gamma((1 + s)/2)}{\Gamma((N - s - 1)/2)} \tag{7.5.8}$$

Making use of this expression or of expansions (7.5.6)–(7.5.7), we are able to express the radial function in terms of the Fox function (see Section 6.8). Formula (7.5.8) allows us to represent the absolute moment of a random vector $Y(\alpha)$ with stable spherically symmetric distribution in an explicit form:

$$\mathsf{E}|Y(\alpha)|^s = \Omega_N \bar{\rho}_N(s + N - 1; \alpha) = 2^s \frac{\Gamma(1 - s/\alpha)\Gamma((s + N)/2)}{\Gamma(1 - s/2)\Gamma(N/2)}. \tag{7.5.9}$$

The obtained expression can be regarded as an analytical function in the s plane excluding the points of the form $s = k\alpha$ and $s = -N - k + 1$ ($k = 1, 2, \ldots$) where this function possesses simple poles. Hence it follows, in particular, that $\mathsf{E}|Y|^s$ admits the Taylor expansion in a power series of s, in the circle $|s| < \min(N, \alpha)$.

A more detailed discussion of multivariate symmetric distributions can be found in (Fang *et al.*, 1990).

8

Simulation

8.1. The inverse function method

Two problems will be considered in this and next chapter: simulation of stable random variables and estimation of their parameters. The first of them answers the question how to obtain a sequence of independent realizations $Y_1, Y_2, Y_3, \ldots$ of a random stable variable $Y(\Gamma)$ with given parameters $\Gamma = (\alpha, \beta, \gamma, \lambda)$. The second one is an inverse problem: how to deduce the parameters Γ of a random stable variable $Y(\Gamma)$ from the given sequence $Y_1, Y_2, Y_3, \ldots$ We start with the first of them.

There exists a great body of algorithms to simulate random variables of different kinds (Shreider, 1966; Sobol, 1973; Hamersley & Handscomb, 1964; Ermakov, 1975; Hengartner, 1978). Many of them use the inverse function method called also the direct method based on the following assertion.

THEOREM 8.1.1 (on inverse function). *Let U be a random variable distributed uniformly on the interval $(0,1)$, and let $F(x)$ be a monotonically increasing function on (a,b) possessing the derivative and the limits $F(x) \to 0$, $x \to a$ and $F(x) \to 1$, $x \to b$ (the cases $a = -\infty$ and (or) $b = \infty$ are allowed). Then the inverse function $F^{-1}(u)$, $u \in (0,1)$, exists, and the random variable*

$$X = F^{-1}(U) \tag{8.1.1}$$

is distributed on the interval (a,b) with the density

$$p_X(x) = F'(x). \tag{8.1.2}$$

PROOF. Since the function $F(x)$ is strictly increasing and $F_U(x) = x$, then

$$F_X(x) = \mathsf{P}\{X < x\} = \mathsf{P}\{F^{-1}(U) < x\} = \mathsf{P}\{U < F(x)\} = F_U(F(x)) = F(x),$$

and we arrive at (8.1.2).

To illustrate some applications of the theorem, we consider the following simple examples. Three of them relate to stable variables.

INVERSE POWER VARIABLE P. Its distribution function is of the form

$$F_P(x) = 1 - (x/\varepsilon)^{-\alpha}, \qquad 0 < \varepsilon \le x.$$

The theorem yields

$$P = \varepsilon U^{-1/\alpha}, \tag{8.1.3}$$

because the random variables U and $1 - U$ are of the same distribution:

$$U \stackrel{d}{=} 1 - U.$$

STANDARD EXPONENTIAL VARIABLE E. This variable has the distribution function

$$F_E(x) = 1 - e^{-x},$$

hence

$$E = -\ln U. \tag{8.1.4}$$

The following algorithm corresponds to some simple cases of stable variables (we use form B).

CAUCHY VARIABLE $Y_B(1, 0)$. Its distribution function is of the form

$$F_Y(x) = \frac{1}{2}\int_{-\infty}^{x} \frac{dy}{(\pi^2)^2 + y^2} = \frac{1}{\pi}\left[\arctan\left(\frac{2x}{\pi}\right) + \frac{\pi}{2}\right].$$

By virtue of the theorem on inverse function, we obtain

$$Y_B(1, 0) = (\pi/2)\tan\phi \tag{8.1.5}$$

where $\phi = \pi(U - 1/2)$ is uniform on $(-\pi/2, \pi/2)$.

GAUSSIAN VARIABLE $Y_B(2, 0)$. In this case, the distribution function is

$$F_{Y(2,0)}(x) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{x} e^{-y^2/4} dy$$

and its inversion function cannot be expressed in an elementary way. But, as follows from (2.3.12), the sum of two squared independent Gaussian variables

$$R^2 = Y_1^2(2, 0) + Y_2^2(2, 0)$$

is distributed with the density

$$p_{R^2}(x) = e^{-x^2/4}dx^2/4,$$

hence $R = 2E^{1/2}$, and

$$Y_1(2,0) = 2E^{1/2}\cos\phi, \tag{8.1.6}$$

$$Y_2(2,0) = 2E^{1/2}\sin\phi. \tag{8.1.7}$$

LÉVY VARIABLE $Y_B(1/2, 1)$. Using (2.3.5) we immediately obtain

$$Y(1/2,1) = [Y(2,0)]^{-2} = [4E\cos^2\phi]^{-1}. \tag{8.1.8}$$

OTHER ONE-SIDED STABLE VARIABLES ($\alpha < 1$, $\beta = 1$). We rewrite the corresponding distribution function (4.5.2)

$$G(x;\alpha,1) = (1/\pi)\int_{-\pi/2}^{\pi/2}\exp\{-x^{-\alpha/(\alpha-1)}U(\varphi;\alpha,1)\}\,d\varphi$$

with

$$U(\varphi;\alpha,1) = \frac{[\sin(\alpha\theta)]^{\alpha/(1-\alpha)}\sin((1-\alpha)\theta)}{(\sin\theta)^{1/(1-\alpha)}} \equiv U_\alpha(\theta), \qquad \theta = \pi/2 + \varphi,$$

as

$$G(x;\alpha,1) = \int_0^\pi F(x;\alpha,1 \mid \theta)p(\theta)\,d\theta. \tag{8.1.9}$$

Here

$$p(\theta) = 1/\pi, \qquad 0 < \theta < \pi,$$

and

$$F(x;\alpha,1 \mid \theta) \equiv \mathsf{P}\{Y_B(\alpha,1) < x \mid \theta\} = \exp\{-x^{-\alpha/(1-\alpha)}U_\alpha(\theta)\}$$

is a conditional distribution function. The right-hand side of this equation is the probability

$$\mathsf{P}\{E > x^{-\alpha/(1-\alpha)}U_\alpha(\theta)\} = \mathsf{P}\{E^{-1} < x^{\alpha/(1-\alpha)}/U_\alpha(\theta)\}$$
$$= \mathsf{P}\{[U_\alpha(\theta)/E]^{(1-\alpha)/\alpha} < x\}.$$

Therefore, in view of (8.1.9), the random variables $Y_B(\alpha, 1)$ and $[U_\alpha(\theta)/E]^{(1-\alpha)/\alpha}$ with θ uniformly distributed on $(0, \pi)$ and E derived from (8.1.4) possess the same distribution:

$$Y_B(\alpha,1) \stackrel{d}{=} [U_\alpha(\theta)/E]^{(1-\alpha)/\alpha}. \tag{8.1.10}$$

This result was obtained in (Kanter, 1975).

STABLE VARIABLES WITH $\alpha < 1$ AND ARBITRARY β. From (3.7.7) it immediately follows that

$$Y(\alpha,\beta) \stackrel{d}{=} [(1+\beta)/2]^{1/\alpha} Y_1(\alpha,1) - [(1-\beta)/2]^{1/\alpha} Y_2(\alpha,1),$$

where Y_1 and Y_2 are independent identically distributed one-sided stable variables (8.1.10).

8.2. The general formula

Kanter's formula was extended in (Chambers *et al.*, 1976) to the whole family of stable variables:

$$Y_B(\alpha,\beta) = \frac{\sin[\alpha(\phi+\varphi_0)]}{(\cos\phi)^{1/\alpha}} \left(\frac{\cos(\phi-\alpha(\phi+\varphi_0))}{E}\right)^{(1-\alpha)/\alpha}, \qquad \alpha \neq 1, \tag{8.2.1}$$

$$Y_A(1,\beta) = (2/\pi)\left[(\pi/2+\beta\phi)\tan\phi - \beta\ln\left(\frac{(\pi/2)E\cos\phi}{\pi/2+\beta\phi}\right)\right], \tag{8.2.2}$$

where E and ϕ are the same as before and

$$\varphi_0 = \beta\Phi(\alpha)/\alpha. \tag{8.2.3}$$

It is easily seen that (8.2.1) is reduced to (8.1.10) if $\alpha < 1$ and $\beta \to 1$ ($\varphi_0 \to \pi/2$, $\theta = \phi + \pi/2$). If $\alpha = 2$, $\beta = 0$, we obtain

$$Y_B(2,0) = E^{1/2}\sin(2\phi)/\cos\phi = 2E^{1/2}\sin\phi,$$

which coincides with (8.1.7). For $\alpha = 1$, $\beta = 0$, we obtain, in form A, the following relation similar to (8.1.5):

$$Y_A(1,0) = \tan\phi.$$

For form A, the similar result was obtained in (Weron & Weron, 1995) (see also (Janicki & Weron, 1994) and (Zolotarev, 1986)). For $\alpha \neq 1$,

$$Y_A(\alpha,\beta) = [1+\beta^2\tan^2(\alpha\pi/2)]^{1/(2\alpha)} \frac{\sin(\alpha(\phi+b))}{(\cos\phi)^{1/\alpha}} \left[\frac{\cos(\phi-\alpha(\phi+b))}{E}\right]^{(1-\alpha)/\alpha}, \tag{8.2.4}$$

where

$$b = \alpha^{-1}\arctan(\beta\tan(\alpha\pi/2)). \tag{8.2.5}$$

As follows from (3.6.6),

$$\arctan(\beta^A\tan(\alpha\pi/2)) = \beta^B\Phi(\alpha);$$

therefore,

$$b = \varphi_0.$$

Moreover,

$$[1 + \beta^{A^2} \tan^2(\alpha\pi/2)]^{1/(2\alpha)} = [1 + \tan^2(\beta^B \Phi(\alpha))]^{1/(2\alpha)} = [\cos(\beta^B \Phi(\alpha))]^{-1/\alpha}$$

which agrees with the relation

$$Y_A(\alpha, \beta) = [\cos(\beta^B \Phi(\alpha))]^{-1/\alpha} Y_B(\alpha, \beta^B)$$

given in Section 3.7. Thus, (8.2.4) is true.

As concerns the case where $\alpha = 1$, (8.2.5) in (Weron & Weron, 1995) becomes

$$Y_A(1, \beta) = (2/\pi) \left[(\pi/2 + \beta\phi) \tan\phi - \beta \ln\left(\frac{E\cos\phi}{\pi/2 + \beta\phi} \right) \right]. \tag{8.2.6}$$

It does not contain the factor $\pi/2$ in the argument of the logarithm in contradistinction to (8.2.2).

Let us consider this case in more details. The conditional probability

$$\begin{aligned} \mathsf{P}\{Y_B(1, \beta) < x \mid \varphi\} &= \exp\{-e^{-x/\beta} U(\varphi; 1, \beta)\} \\ &= \mathsf{P}\{E > e^{-x/\beta} U(\varphi; 1, \beta)\} \\ &= \mathsf{P}\{\beta \ln \left[U(\varphi; 1, \beta)/E\right] < x\} \end{aligned}$$

yields

$$Y_B(1, \beta) = \beta \ln[U(\varphi; 1, \beta)/E].$$

Substituting $U(\varphi; 1, \beta)$ from the end of Section 4.4, we obtain

$$Y_B(1, \beta) = \left(\pi/2 + \beta\phi\right) \tan\phi - \beta \ln\left(\frac{E\cos\phi}{\pi/2 + \beta\phi} \right).$$

Passing to form A by means of the equality

$$Y_B(1, \beta) = \beta \ln(\pi/2) + (\pi/2) Y_A(1, \beta)$$

given in Section 3.7, we arrive at (8.2.2), so (8.2.6) should be corrected.

To avoid the discontinuity in limit behavior of (8.2.2) as $\alpha \to 1$, in (Chambers *et al.*, 1976) form M was used:

$$Y_M(\alpha, \beta) = -\tan(\alpha\varphi_0) + (\cos\phi)^{-1}[\sin(\alpha\phi) + \tan(\alpha\varphi_0)\cos(\alpha\phi)] z^{(1-\alpha)/\alpha}, \tag{8.2.7}$$

where

$$z = [\cos((1-\alpha)\phi) + \tan(\alpha\varphi_0)\sin((1-\alpha)\phi)]/(E\cos\phi).$$

Because $(1-\alpha)\phi \to 0$ and $\tan(\alpha\varphi_0) \to \infty$ as $\alpha \to 1$, it is necessary to rearrange the calculations. At the same time one can reduce the trigonometric computations to the tangent function of halved angles. To retain accuracy, one should use the auxiliary functions

$$D_2(x) = (e^x - 1)/x, \tag{8.2.8}$$

$$\tan_2(x) = (\tan x)/x. \tag{8.2.9}$$

Although these are non-standard functions, approximations to them may be derived from existing approximations to $D(x) = e^x - 1$ and $\tan x$. The latter functions are often computed via functions of the form $xR(x)$, where $R(x)$ is a rational function of special form; thus, one can use $R(x)$ to approximate functions (8.2.8) and (8.2.9). Such approximations are used in the FORTRAN function RSTAB given in the Appendix to (Chambers *et al.*, 1976).

8.3. Approximate algorithm for one-dimensional symmetric stable variables

Despite the existence of the exact algorithm (8.1.6)–(8.1.7) to simulate Gaussian random variables, other approximate methods are often used in practice, because they turn out to be faster than exact ones. As follows from the central limit theorem, the random variable

$$N^{(n)} = \sqrt{12/n}\sum_{i=1}^{n}(U_i - 1/2) \tag{8.3.1}$$

is asymptotically normal with

$$\mathsf{E}N^{(n)} = 0, \qquad \operatorname{Var} N^{(n)} = 1.$$

In usual practice, (8.3.1) is considered as normal if $n \geq 10$ (Sobol, 1973). The value $n = 12$ is especially convenient because $N^{(12)} = \sum_{i=1}^{12} U_i - 6$. Of course, the distribution of (8.3.1) deviates from the normal one in the domain of large $|x|$.

Since the general formula for stable variables is more complicated, there exists a reason for searching for approximate algorithms. They always can be improved using the generalized limit theorem for stable laws.

To illustrate one of such algorithms presented in (Mantegna, 1994) for symmetric variables, we consider the ratio

$$V = N_1/|N_2|^{1/\alpha}, \tag{8.3.2}$$

where N_1 and N_2 are two normal random variables with standard deviations σ_1 and σ_2 respectively. Its probability density

$$p_V(x) = \frac{1}{\pi\sigma_1\sigma_2}\int_0^\infty y^{1/\alpha}\exp\left\{-\frac{y^2}{2\sigma_2^2} - \frac{x^2y^{2/\alpha}}{2\sigma_1^2}\right\} dy$$

is determined for large arguments by the asymptotic approximation

$$p_V(x) \sim C_V(\sigma_1, \sigma_2, \alpha)|x|^{-1-\alpha}, \qquad x \to \infty, \tag{8.3.3}$$
$$C_V(\sigma_1, \sigma_2, \alpha) = \alpha 2^{(\alpha-1)/2} \sigma_1^\alpha \Gamma((\alpha+1)/2)/(\pi\sigma_2),$$

whereas its value at the origin is

$$p_V(0) = 2^{(1-\alpha)/(2\alpha)} \sigma_2^{1/\alpha} \Gamma((\alpha+1)/(2\alpha))/(\pi\sigma_1). \tag{8.3.4}$$

Let us compare (8.3.3) and (8.3.4) with the corresponding formulae for symmetric stable densities (4.1.4) and (4.1.3):

$$q(x; \alpha, 0) \sim \pi^{-1}\Gamma(\alpha+1)\sin(\alpha\pi/2)|x|^{-\alpha-1} \equiv C(\alpha)|x|^{-\alpha-1} \tag{8.3.5}$$

$$q(0; \alpha, 0) = \Gamma(1/\alpha)/(\pi\alpha). \tag{8.3.6}$$

The conditions

$$C_V(\sigma_1, \sigma_2, \alpha) = C(\alpha), \qquad p_V(0) = q(0; \alpha, 0)$$

are simultaneously satisfied for $\alpha = 1$ only by $\sigma_1 = \sigma_2 = 1$. In this case, the distribution $p_V(x)$ coincides with the Cauchy distribution. As the standard deviations σ_1 and σ_2 cannot be chosen independently for an arbitrary value of α, we set $\sigma_2 = 1$. After this, we determine σ_1 by

$$C_V(\sigma_1, 1, \alpha) = C(\alpha).$$

By (8.3.3) and (8.3.5), we obtain

$$\sigma_1(\alpha) = \left[\frac{\Gamma(1+\alpha)\sin(\alpha\pi/2)}{\Gamma((1+\alpha)/2)\alpha 2^{(\alpha-1)/2}}\right]^{1/\alpha}. \tag{8.3.7}$$

After this choice, the asymptotic behavior of the distributions for large values is the same, but they have different values at the origin:

$$p_V(0) < q(0; \alpha, 0), \qquad \alpha < 1,$$
$$p_V(0) > q(0; \alpha, 0), \qquad \alpha > 1.$$

An attempt to improve the result with the use of the formula

$$Z_n = n^{-1/\alpha} \sum_{k=1}^{n} V_k$$

demonstrates that the convergence of this sum is quite slow. Finally, in (Mantegna, 1994) the nonlinear transformation

$$W = \{(A(\alpha) - 1)e^{-V/B(\alpha)} + 1\}V \tag{8.3.8}$$

Table 8.1.

α	$\sigma_1(\alpha)$	$A(\alpha)$	$B(\alpha)$
0.8	1.13999	0.795112	2.483
0.9	1.06618	0.899389	2.7675
1.1	0.938291	1.10063	2.945
1.2	0.878829	1.20519	2.941
1.3	0.819837	1.31836	2.9005
1.4	0.759679	1.44647	2.8315
1.5	0.696575	1.59922	2.737
1.6	0.628231	1.79361	2.6125
1.7	0.551126	2.06448	2.4465
1.8	0.458638	2.50147	2.206
1.9	0.333819	3.4615	1.7915
1.95	0.241176	4.80663	1.3925
1.99	0.110693	10.498	0.6089

was used. The value of $A(\alpha)$ is determined by

$$p_W(0) = q(0; \alpha, 0); \tag{8.3.9}$$

in a neighborhood of the origin

$$W \sim A(\alpha)V,$$

and then (8.3.9) is satisfied as soon as

$$A(\alpha) = \frac{p_V(0)}{q(0; \alpha, 0)} = \frac{\alpha\Gamma[(\alpha+1)/(2\alpha)]}{\Gamma(1/\alpha)}\left[\frac{\alpha\Gamma[(\alpha+1)/2]}{\Gamma(1+\alpha)\sin(\alpha\pi/2)}\right]^{1/\alpha}. \tag{8.3.10}$$

To determine $B(\alpha)$, in (Mantegna, 1994) the first derivative of (8.3.8) was analyzed, resulting in the following integral equation for $B(\alpha)$:

$$\frac{1}{\pi\sigma_1}\int_0^\infty y^{1/\alpha}\exp\left\{-\frac{y^2}{2} - \frac{y^{2/\alpha}B^2(\alpha)}{2\sigma_1^2(\alpha)}\right\}dy$$
$$= \frac{1}{\pi}\int_0^\infty \cos\left[\left(\frac{A(\alpha)-1}{e}+1\right)B(\alpha)\right]\exp(-y^\alpha)\,dy, \tag{8.3.11}$$

solved numerically. The results are summarized in Table 8.1. The control parameters used with the algorithm are determined by (8.3.2) and (8.3.8). The parameters $\sigma_1(\alpha)$ and $A(\alpha)$ are obtained by evaluating (8.3.7) and (8.3.9), respectively, whereas $B(\alpha)$ is obtained from (8.3.11).

8.4. Simulation of three-dimensional spherically symmetric stable vectors

There is no problem to simulate the multi-dimensional vector $Y(2,0)$ because all its components $Y_1(2,0), \dots, Y_N(2,0)$ are independent, and

$$f_{Y(2,0)}(k) = e^{-|k|^2} = e^{-k_1^2} \dots e^{-k_N^2} = f_{Y_1(2,0)}(k_1) \dots f_{Y_N(2,0)}(k_N).$$

We discuss here an approximate algorithm to simulate three-dimensional stable vectors with $\alpha < 2$ developed in (Uchaikin & Gusarov, 1999; Gusarov, 1998).

Why do we fail to obtain an exact algorithm? This question is quite reasonable because for one-dimensional case such an algorithm exists. Let us look at it some more. For a symmetric distribution, $\beta = 0$, and we obtain from (4.5.2) and (4.5.3)

$$G_1(r;\alpha,0) = \begin{cases} 1/2 + \pi^{-1} \int_0^{\pi/2} \exp\left\{-r^{\alpha/(\alpha-1)} U(\varphi;\alpha,0)\right\} d\varphi, & \alpha < 1, \\ 1 - \pi^{-1} \int_0^{\pi/2} \exp\left\{-r^{\alpha/(\alpha-1)} U(\varphi;\alpha,0)\right\} d\varphi, & \alpha > 1. \end{cases}$$

where $r > 0$ and

$$U(\varphi;\alpha,0) = \left(\frac{\sin(\alpha\varphi)}{\cos\varphi}\right)^{\alpha/(1-\alpha)} \frac{\cos((\alpha-1)\varphi)}{\cos\varphi}, \qquad \alpha \neq 1.$$

Due to symmetry, it suffices to simulate the absolute value $|Y|$ and then to perform the simple operation

$$Y = |Y| \operatorname{sign}(U - 1/2),$$

where U is uniform on $(0,1)$. The distribution function of $|Y|$ is

$$\begin{aligned} F_1(r) &\equiv F_{|Y|}(r) = 2G_1(r;\alpha,0) - 1 \\ &= \begin{cases} (2/\pi) \int_0^{\pi/2} \exp\{-r^{\alpha/(\alpha-1)} U(\varphi;\alpha,0)\} d\varphi, & \alpha < 1, \\ (2/\pi) \int_0^{\pi/2} [1 - \exp\{-r^{\alpha/(\alpha-1)} U(\varphi;\alpha,0)\}] d\varphi, & \alpha > 1. \end{cases} \end{aligned} \tag{8.4.1}$$

In either case, the distribution function can be represented as

$$F_1(r) = \int_0^{\pi/2} K_1(r,\varphi) p(\varphi)\, d\varphi, \tag{8.4.2}$$

where

$$p(\varphi) = 2/\pi, \qquad 0 < K_1(r,\varphi) < 1 \tag{8.4.3}$$

for all $0 < r < \infty$ and $0 < \varphi < \pi/2$. The last condition allows for interpretation of the function $K_1(r,\varphi)$ as a conditional distribution function $K_1(r,\varphi) = F_1(r \mid \varphi)$, and, as a consequence, to perform the simulation in two steps: first, to choose ϕ uniformly on $(0, \pi/2)$, and then, to find $|Y|$ from the distribution function $F_{|Y|}(r \mid \phi)$.

In the three-dimensional case the distribution function of the absolute value $R = |\mathbf{Y}|$ of the vector $\mathbf{Y}$ is of the form

$$F(r) \equiv F_R(r) = 4\pi \int_0^r \rho_3(r; \alpha) r^2 dr. \tag{8.4.4}$$

Making use of the relation (7.2.9) and integrating (8.4.4) by parts, we obtain

$$F(r) = F_1(r) - 2r\rho_1(r; \alpha) = F_1(r) - r dF_1(r)/dr. \tag{8.4.5}$$

Substituting (8.4.1) into (8.4.5) yields

$$F(r) = \begin{cases} (2/\pi) \int_0^{\pi/2} H(\varphi, \alpha) \exp\{-r^{\alpha/(\alpha-1)} U(\varphi; \alpha, 0)\} d\varphi, & \alpha < 1, \\ 1 - (2/\pi) \int_0^{\pi/2} H(\varphi, \alpha) \exp\{-r^{\alpha/(\alpha-1)} U(\varphi; \alpha, 0)\} d\varphi, & \alpha > 1. \end{cases} \tag{8.4.6}$$

where

$$H(\varphi, \alpha) = [1 - \alpha r^{\alpha/(\alpha-1)} U(\varphi; \alpha, 0)/(1 - \alpha)].$$

An attempt to represent it in the form (8.4.2) demonstrates that the function $K_3(r, \varphi)$ does not satisfy condition (8.4.3) for all r and φ, and therefore, it cannot be interpreted as a conditional probability.

The algorithm described below (Uchaikin & Gusarov, 1998; Uchaikin & Gusarov, 1999) is based on the numerical inversion of the function $F(r)$:

$$r = F^{-1}(u) \equiv r(u), \qquad 0 < u < 1.$$

To perform this transformation, one has to make use of the series expansions for $F(r)$ that follow from analogous ones for the one-dimensional case:

$$1 - F(r) = \frac{2}{\pi\alpha} \sum_{n=1}^{\infty} (-1)^{n+1} \Gamma(n\alpha + 2) \sin(n\alpha\pi/2) n^{-1} r^{-n\alpha}, \tag{8.4.7}$$

$$F(r) = \frac{4}{\pi\alpha} \sum_{n=1}^{\infty} (-1)^{n+1} \Gamma\left(\frac{2n+1}{\alpha}\right) \frac{n}{(2n+1)!} r^{2n+1}. \tag{8.4.8}$$

Fig. 8.1 shows the behavior of the leading terms of expansions (8.4.8) as $r \to 0$ and (8.4.7) as $r \to \infty$, and the influence of the second terms on the result as compared with the exact value. It is seen that the account of the second term yields a more exact result, especially in far tail domain. (The labels near the graphs say how many leading terms of the expansions are used.)

Let us investigate the behavior of the function $r(u)$ as $u \to 0$ and $u \to 1$.

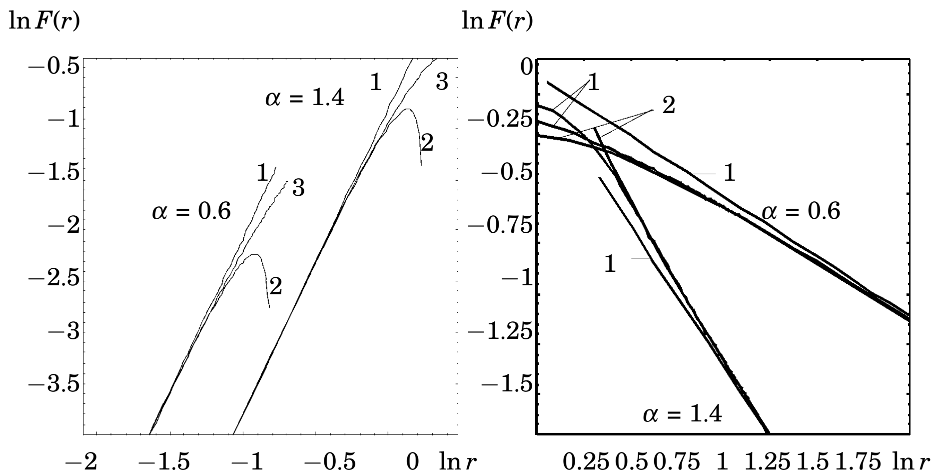


Figure 8.1.

In view of (8.4.8), we can write the following equation for $r(u)$ in a neighborhood of $u = 0$:

$$ar^3 - br^5 \approx u \tag{8.4.9}$$

where

$$a = 2\Gamma(3/\alpha)/(3\pi\alpha),$$
$$b = 8\Gamma(5/\alpha)/(5!\,\pi\alpha).$$

Setting

$$\theta_1(u) = (u/a)^{1/3} \tag{8.4.10}$$

as an initial approximation to the solution as $u \to 0$ and substituting $r(u) = \theta_1(u) + \Delta(u)$ into (8.4.9), we obtain

$$\Delta(u) = \frac{u - (a\theta_1^3 - b\theta_1^5)}{3a\theta_1^2 - 5b\theta_1^4}.$$

Hence the second approximation can be represented as

$$\tilde{\theta}_2(u) = (u/a)^{1/3} + \frac{b(u/a)}{3a - 5b(u/a)^{2/3}},$$

but divergence at the point where the denominator vanishes makes the approximation inconvenient for the use on $(0, 1)$ and the following asymptotically equivalent expression turns out to be more suitable:

$$\theta_2(u) = (u/a)^{1/3} + (b/a)(u/a)/3 + 5(b/a)^2(u/a)^{5/3}/9. \tag{8.4.11}$$

For large r, the leading term of series (8.4.7) provides us with an initial approximation to $r(u)$ in a neighborhood of 1:

$$\eta_1(u) = \left[B/(1-u)\right]^{1/\alpha},$$

where

$$B = 2\Gamma(\alpha+2)\sin(\alpha\pi/2)/(\pi\alpha).$$

The use of the second term in (8.4.7) yields the quadratic equation

$$Ar^{-2\alpha} - Br^{-\alpha} + 1 - u = 0$$

with the coefficient

$$A = \Gamma(2\alpha+2)\sin(\alpha\pi)/(2\pi\alpha).$$

The last is positive for $\alpha < 1$ but changes its sign while passing into the region $\alpha > 1$. In order to get a positive solution from

$$r^{-\alpha} = B/(2A) \pm \sqrt{B^2/(2A)^2 - (1-u)/A},$$

we should choose here the '$-$' sign for $\alpha < 1$ and the '+' sign for $\alpha > 1$. We thus obtain

$$\eta_2(u) = \begin{cases} \left[\sqrt{B^2/(2A)^2 + (1-u)/|A|} - B/|2A|\right]^{-1/\alpha}, & \alpha > 1. \\ \left[B/(2A) - \sqrt{B^2/(2A)^2 - (1-u)/A}\right]^{-1/\alpha}, & \alpha < 1. \end{cases}$$

Fig. 8.2 shows the contribution of above-described asymptotics as compared with the exact solution obtained by numerical inversion of (8.4.6). One can see that second approximation can be effectively applied on more longer interval than the first one. (The label 1 denotes the first approximation, 2, the second one, and 3 stands for the exact function.)

Now one can perform the approximation to $r(u)$ as a whole. Beginning with the case $\alpha > 1$, we write the representation for $r(u)$ in the form

$$r(u) = \theta_2(u) + \psi(u) + \eta_2(u),$$

where $\psi(u)$ is to be found by comparing the right-hand side of the equation with the exact solution obtained numerically. However, before doing this this it is necessary to note that the addend $\psi(u) + \eta_2(u)$ must not change the asymptotic behavior of $r(u)$ as $u \to 0$ which is given by $\theta_2(u)$ (see (8.4.11)) and conversely, the addend $\theta_2(u) + \psi(u)$ must not influence the asymptotics of $r(u)$ as $u \to 1$ given by $\eta_2(u)$. To satisfy these conditions, we introduce the auxiliary function

$$\psi(u) = -\eta_2(0) - \eta_2'(0)u - uP_n^*(u),$$

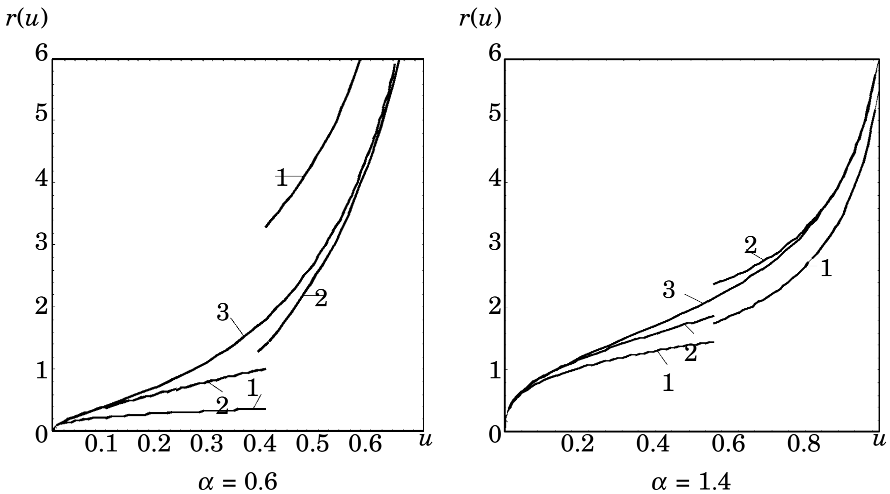


Figure 8.2.

where $P_n^*(u)$ is a polynomial of some degree n. Since the second term of the sum can be included in the first term of the polynomial (the asterisk is omitted in this case), we obtain the final approximation

$$r(u) \approx \theta_2(u) + [\eta_2(u) - \eta_2(0)] - uP_n^{(\alpha)}(u), \qquad \alpha > 1. \tag{8.4.12}$$

Calculating the coefficients $c_0, \dots, c_n$ in the expression

$$uP_n^{(\alpha)}(u) = c_0 u + c_1 u^2 + \dots + c_n u^{n+1}$$

by the least squares method we conclude that a quite acceptable accuracy can be achieved with polynomials of somewhat low degree.

This representation cannot be immediately extended to the region $\alpha < 1$, because $\eta_2(0)$ is complex-valued for all $\alpha < \alpha^* \approx 0.82$ but it turns out that the simpler approximation

$$r(u) \approx \theta_1(u) + u\eta_1(u)P_n^{(\alpha)}(u), \qquad \alpha < 1, \tag{8.4.13}$$

provides us with an acceptable accuracy in this region. It should be noted that the interval where this simple formula is usable also cannot be extended into $\alpha > 1$ without significant increasing the degree of the polynomial. Explicit forms of $P_n^{(\alpha)}(u)$ are cited in Appendix.

As we see in Fig. 8.3, the relative errors

$$\delta(r) = [\tilde{\rho}_3(r) - \rho_3(r)]/q_3(r)$$

in the three-dimensional density of the random vector obtained with the use of this algorithm do not exceed 1%.

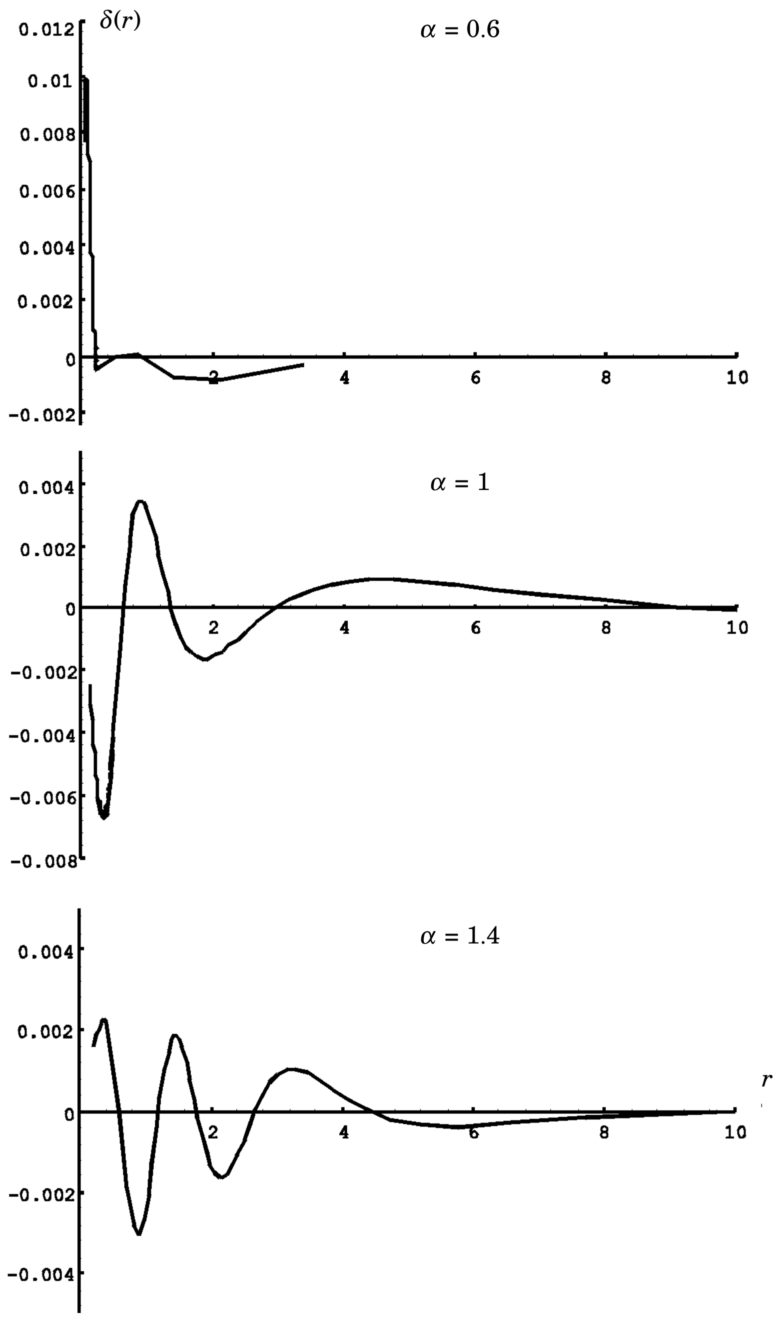

δ(r)
α = 0.6
0.012
0.01
0.008
0.006
0.004
0.002
0
-0.002
2
4
6
8
10
α = 1
0.004
0.002
0
-0.002
-0.004
-0.006
-0.008
α = 1.4
0.004
0.002
0
-0.002
-0.004
r

Figure 8.3.

Thus, the algorithm simulating three-dimensional random vectors with spherically symmetric stable distribution consists of two steps.

The first of them is to simulate the isotropic three-dimensional vector $\mathbf{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$.

To simulate an isotropic vector $\mathbf{\Omega}$, the well-known algorithms (Sobol, 1973; Ermakov & Mikhailov, 1976) can be used. We cite here two of them. The first uses the spherical coordinates with polar angle Θ and azimuth angle ϕ so that

$$\Omega_1 = \sqrt{1-\mu^2}\sin\phi,$$
$$\Omega_2 = \sqrt{1-\mu^2}\cos\phi,$$
$$\Omega_3 = \mu \equiv \cos\Theta,$$

where

$$\phi = 2\pi U_1, \qquad \mu = 2U_2 - 1.$$

The second algorithm uses the choice of a random point uniformly distributed within the three-dimensional cube with the edge length two,

$$X_i = 2U_i - 1$$

and, as soon as the point falls into a sphere inscribed into the cube,

$$X_1^2 + X_2^2 + X_3^2 \leq 1, \tag{8.4.14}$$

its Cartesian coordinates X_i are transformed into the Cartesian coordinates of the vector Ω_i sought for by the formula

$$\Omega_i = \frac{X_i}{\sqrt{X_1^2 + X_2^2 + X_3^2}}.$$

If the point (U_1, U_2, U_3) does not satisfy (8.4.14), then a new random point is chosen, and the procedure is repeated until (8.4.14) becomes true. The effectiveness of the method is determined by the ratio of the volumes of the sphere and the cube, and is equal to $\pi/6$. A higher effectiveness, $\pi/4$, can be achieved by combining these methods:

(1) $\Omega_1 = 2U_1 - 1$.

(2) $X_2 = 2U_2 - 1, X_3 = 2U_3 - 1$.

(3) If $D \equiv X_2^2 + X_3^2 > 1$, then go to 2; otherwise

$$\Omega_2 = X_2\sqrt{(1-\Omega_1^2)/D}, \qquad \Omega_3 = X_3\sqrt{(1-\Omega_1^2)/D}.$$

The second step of the algorithm is the sampling of the absolute value R with the use of the obtained approximations (8.4.12)–(8.4.13):

$$R = r(U).$$

The Cartesian components of the vector are

$$Y_i = \Omega_i R.$$

Theorem 7.4.1 allows us to use this algorithm to simulate two-dimensional axially symmetric stable vectors.

9

Estimation

9.1. Sample fractile technique

This chapter is devoted to the inverse problem: to the problem of parameter estimates for stable distributions. It can be said without exaggeration that the problem of constructing statistical estimators of stable laws entered mathematical statistics due to the works of Mandelbrot.The economic models considered in those works contained stable distributions whose parameters had to be determined empirically. Furthermore, it was discovered at once that mathematical statistics, while having a great body of methods at its disposal, can be of little help in this case, since these methods are based mainly on such assumptions as the availability of an explicit form for the density, the existence of some particular number of moments, and so on, which are certainly not satisfied for stable distributions. In the best case, they have only one moment of integer order (if $\alpha \neq 2$), and only in a very few cases there are explicit expressions for the density that would enable us to concretize the algorithms to estimate the parameters (say, by the maximum likelihood method).

However, the problem had emerged, and the search began of its solution. This search was conducted in various directions and led to estimators, as a rule $1/\sqrt{n}$-consistent estimators that satisfied the practical workers to some extent.

Modern mathematical statistics possesses several methods available for improving $1/\sqrt{n}$-consistent estimators, and they make it possible, in principle, to construct asymptotically efficient estimators of the parameters of stable laws, at least within groups given, on the basis of the estimators we obtained. These methods can be conventionally divided into the following two categories.

The first contains methods which do not use information about the analytic expression for the distribution density of sample elements. Here we cite (Beran, 1974), which contains, in particular, a brief survey of other publications of the same direction.

The second category includes methods in which some knowledge of the form of the distribution density of the sample elements is assumed. Among a lot of papers concerning the use of such methods, we mention (Dzhaparidze, 1974), whose results form a convenient basis for solving the general problem of asymptotically efficient estimators for stable distributions.

The preferability of the methods of the former category is obvious, especially because the regularity conditions they use for the distributions of the sample elements are weaker than the conditions that usually manifest themselves in the methods of the latter category. Unfortunately, they have been considerably less developed and have been related only to the problem of estimation of the scalar shift parameter of a distribution, which allows us to use them only for estimating the parameter γ and, in some cases, λ. Therefore, in solving the problem of construction of asymptotically efficient estimators in the general situation, we must resort to methods of the latter category. The first of them is based on the so-called sample fractile technique.

Fama and Roll provided estimators for symmetric stable laws with characteristic functions

$$g(k; \alpha, c, \delta) = \exp\left\{i\delta k - |ck|^{\alpha}\right\}, \qquad \alpha \geq 1. \tag{9.1.1}$$

The following results were obtained.

1. THE 50 PERCENT TRUNCATED SAMPLE MEAN AS AN ESTIMATE OF δ. For sample size $n = 100$, the 50 percent truncated mean performs about as well as both the 25 and 75 percent truncated means for symmetric stable distributions with α lying in the range $1.1 \leq \alpha \leq 1.7$. Computation of $\hat{\delta}$ for 301 independent samples of size 101 drawn from $G(x; 3/2, 0)$ (the reduced distribution function with $\beta = 0$) produced the estimate 0.144 for the standard deviation of $\hat{\delta}$.

2. ESTIMATE OF c BY MEANS OF SAMPLE FRACTILES. If the appropriate fractiles are chosen, the estimate will only slightly depend on the characteristic α.

In particular, the 0.72-fractile of a reduced (i.e., $\delta = 0$, $c = 1$) symmetric stable distribution is in the interval 0.827 ± 0.003 for $1 \leq \alpha \leq 2$. Thus, given a random sample of N observations, a reasonable estimator of c is

$$\hat{c} = (\hat{x}_{0.72} - \hat{x}_{0.28})/1.654$$

where $\hat{x}_f$ refers to the $(f)(N+1)$st order statistic, which is used to estimate $x_{0.28}$ and $x_{0.72}$, the 0.28 and 0.72 fractiles of the distribution of Y.

This estimator has an asymptotic bias of less than 0.4%. Being a linear combination of order statistics, it is asymptotically normally distributed with the standard deviation

$$\sigma_{\tau} = \sqrt{\operatorname{Var}\hat{c}} \approx \frac{0.300}{\sqrt{n}q(x_{0.72}; \alpha)} \tag{9.1.2}$$

where $q(x_{0.72}; \alpha)$ is the stable symmetric density of the distribution of Y evaluated at the sample 0.72-fractile. Since symmetry is assumed, the distribution of c does not depend on the location parameter δ of the underlying random variable Y. The scale of Y affects the asymptotic variance of $\hat{c}$ through the density $q(x_{0.72};\alpha)$ which appears in the denominator of (9.1.2). For a non-reduced symmetric stable distribution (i.e., $c \neq 1$), $\operatorname{Var}\hat{c}$ is, of course, c^2 times as large as $\operatorname{Var}\hat{c}$ for $c = 1$.

3. ESTIMATION OF α. As far as the exponent α is concerned, Fama and Roll assert that the smaller α, the more 'thick-tailed' stable distributions are. With standardized distributions, for example, the 0.95-fractile decreases monotonically from 6.31 for $\alpha = 1$ to 2.33 for $\alpha = 2$.

This behavior of higher fractiles suggests a simple estimator of α based on order statistics. For some large f (for example, $f = 0.95$) we first calculate

$$\hat{z}_f = \frac{\hat{x}_f - \hat{x}_{1-f}}{2\hat{c}} = 0.827\frac{\hat{x}_f - \hat{x}_{1-f}}{\hat{x}_{0.72} - \hat{x}_{0.28}} \tag{9.1.3}$$

from the sample. Given that Y is a symmetric stable variable with characteristic α and scale parameter c, $\hat{z}_f$ appears to be an estimator of the f-fractile of the standardized (reduced) symmetric stable distribution with characteristic α. Thus, an estimate of α can be obtained by searching through a 'table' of standardized symmetric stable cumulative distribution functions for the value, call it $\hat{\alpha}_f$, whose f-fractile matches $\hat{z}_f$ most closely. Formally,

$$\hat{\alpha}_f = H(f, \hat{z}_f)$$

where H is a function that uniquely maps the fractile z_f and the cumulative probability f onto α, and $\hat{z}_f$ is the sample fractile given by (9.1.3).

Simulation studies were performed using computer-generated random samples from stable distributions to measure the variance of the proposed estimators.

9.2. Method of characteristic functions

This method was suggested in (Paulson *et al.*, 1975), and makes use of the stable characteristic function in the form

$$g(k;\Gamma) = \exp\{i\delta k - \gamma|k|^\alpha + \psi(k)\}$$

with

$$\psi(k) = \psi(k;\alpha,\beta,\gamma) = -\gamma|k|^\alpha\beta(k/|k|)\omega(k;\alpha).$$

The key idea is as follows. Given a random sample $Y_j (j = 1,\dots,n)$ of stable variables, the empirical characteristic function $\hat{g}(k)$ is

$$\hat{g}(k) = n^{-1}\sum_{j=1}^{n}\exp\{ikY_j\}.$$

Now two distribution functions are equal if and only if their respective characteristic functions coincide on $-\infty < k < \infty$. Thus, $\hat{g}(k)$ contains useful information concerning $q(x; \alpha, \beta, \gamma, \delta)$ and is, moreover, a consistent estimator of $g(k)$. It should be possible to extract some information on α, β, γ, and δ from $\hat{g}(k)$ by finding those $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$, and $\hat{\delta}$ which make the integral

$$I(\alpha, \beta, \gamma, \delta) = \int_{-\infty}^{\infty} |\hat{g}(k) - g(k)|^2 e^{-k^2} dk \tag{9.2.1}$$

to attain its minimum. Elementary algebraic transformations yield

$$\begin{aligned} |\hat{g}(k) - g(k)|^2 &= \left[\Re(\hat{g}(k) - g(k))\right]^2 + \left[\Im(\hat{g}(k) - g(k))\right]^2 \\ &= [C(k) - e^{-\gamma|k|^\alpha} \cos(\delta k + \psi(k))]^2 + [S(k) - e^{-\gamma|k|^\alpha} \sin(\delta k + \psi(k))]^2, \end{aligned}$$

where

$$C(k) = n^{-1} \sum_{j=1}^{n} \cos(kY_i),$$

$$S(k) = n^{-1} \sum_{j=1}^{n} \sin(kY_j).$$

We set

$$|\hat{g}(k) - g(k)|^2 = \mu(k).$$

The integration in (9.2.1) was carried out numerically by 20 point Hermite's quadrature as

$$\int_{-\infty}^{\infty} \mu(k) e^{-k^2} dk = \sum_{i=1}^{20} w_i \mu(k_i),$$

where k_i are the zeros of the Hermite polynomial of degree 20 and w_i are the weights associated with these zeros (Abramowitz & Stegun, 1964, p.924). To find $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$, $\hat{\delta}$, one assigns some initial values $\hat{\alpha}_0$, $\hat{\beta}_0$, $\hat{\gamma}_0$, and $\hat{\delta}_0$ to α, β, γ, and δ, and then performs the unconstrained optimization of (9.2.1) by some gradient projection procedure. As asserted in (Paulson *et al.*, 1975), the procedure worked well for samples from $q(x; \alpha, \beta, 1, 0)$, $0 < \alpha \le 2$, $|\beta| \le 1$, but failed to give reasonable results for γ and δ much different from one and zero, respectively, even though the proper orders of magnitude relative to one and zero were obtained. To improve the method, some iterative procedure was elaborated.

9.3. Method of characteristic transforms: estimators of ν, θ and τ

One more approach to the problem of estimation of the parameters of stable laws, based on the use of explicit expressions for the corresponding characteristic transforms and the method of sample logarithmic moments was developed in (Zolotarev, 1978; Zolotarev, 1980; Zolotarev, 1981b; Zolotarev, 1986).

Let $V_1, \ldots, V_n$ be independent random variables distributed as $Y = Y_E(\nu, \theta, \tau)$, i.e., with the distribution $G(x; \nu, \theta, \tau)$, which is known only to belong to the class of strictly stable laws. The problem is to carry out the statistical estimation of the parameters of G.

From this sample we construct two collections of independent (within each collection) random variables

$$U_1 = \operatorname{sign} Y_1, \ldots, U_n = \operatorname{sign} Y_n,$$
$$V_1 = \ln |Y_1|, \ldots, V_n = \ln |Y_n|,$$

which are distributed as $U = \operatorname{sign} Y_E(\nu, \theta, \tau)$ and $V = \ln |Y_E(\nu, \theta, \tau)|$ respectively.

The idea for the construction of estimators for the parameters ν, θ, and τ is based on the following three equalities:

$$\nu = (6/\pi^2) \operatorname{Var} V - (3/2) \operatorname{Var} U + 1, \quad \theta = \mathsf{E}U, \quad \tau = \mathsf{E}V. \tag{9.3.1}$$

The last two are given in (A.11.3), and it is not hard to validate the first by computing the variances of U and V. By (A.11.3)–(A.11.5),

$$\operatorname{Var} U = \mathsf{E}U^2 - (\mathsf{E}U)^2 = 1 - \theta^2,$$
$$\operatorname{Var} V = \mathsf{E}V^2 - (\mathsf{E}V)^2 = \pi^2(2\nu - 3\theta^2 + 1)/12.$$

The required relation is obtained by eliminating θ^2 from these equalities.

The idea itself is simple and not new in mathematical statistics. As an illustrative example, we recall the classical problem of estimation of the parameters of the normal distribution with density

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-a}{\sigma}\right)^2\right\}. \tag{9.3.2}$$

Let $X_1, \ldots, X_n$ be independent random variables distributed with density (9.3.2). Since $a = \mathsf{E}X_1$ and $\sigma^2 = \operatorname{Var} X_1$, the fact that the convergence in probability

$$\bar{X} \xrightarrow{\mathsf{P}} a, \qquad S_X^2 \xrightarrow{\mathsf{P}} \sigma^2$$

holds as the sample size goes to infinity allows us to choose $\tilde{a} = \bar{X}$ and $\tilde{\sigma}^2 = S_X^2$ as the estimators for the parameters a and σ^2 respectively.

The method of sample moments is not very favored in mathematical statistics. It is regarded, not without reason, as an estimation method that is far from economical. However, in a number of cases where the distribution possesses sufficiently good analytic properties (for example, the existence of moments of any order, etc.) the method of moments is capable of giving parameter estimators meeting modern demands. And this category includes the case under consideration of the distributions of the random variables U and V which have finite moments of any order.

Based on the collections $U_1, \dots, U_n$ and $V_1, \dots, V_n$ generated by the independent sample $Y_1, \dots, Y_n$ we form the sample means A_U and A_V and take them as estimators of the parameters θ and τ, i.e.,

$$\tilde{\theta} = \bar{U}, \qquad \tilde{\tau} = \bar{V}. \tag{9.3.3}$$

LEMMA 9.3.1. *The statistics $\tilde{\theta}$ and $\tilde{\tau}$ are unbiased consistent estimators of the parameters θ and τ with variances*

$$\sigma_\theta^2 = \operatorname{Var} \tilde{\theta} = (1 - \theta^2)/n, \tag{9.3.4}$$

$$\sigma_\tau^2 = \operatorname{Var} \tilde{\tau} = \pi^2(2\nu - 3\theta^2 + 1)/(12n). \tag{9.3.5}$$

PROOF. The fact that estimators (9.3.3) are unbiased follows from (9.3.1). The form of the variances is obtained from (A.11.9), (A.11.3), (A.11.5). The consistency of the estimators follows from the fact that variances (9.3.4) and (9.3.5) of the estimators vanish as $n \to \infty$.

Since the random variable U takes only two values +1 and -1 with respective probabilities equal to

$$p = \mathsf{P}\{U = +1\} = (1 + \theta)/2, \qquad 1 - p = \mathsf{P}\{U = -1\} = (1 - \theta)/2,$$

estimation of θ is equivalent to that of p, which is a well-known problem in statistics. It is known that $\tilde{p} = (1 + \tilde{\theta})/2$ is an efficient estimator for the parameter p (see, e.g. (van der Waerden, 1957)).

In contrast to estimators (9.3.3), construction of an estimator for the parameter ν appears to be more complex. It might seem that the statistic

$$\hat{\nu} = (6/\pi^2)S_V^2 - (3/2)S_U^2 + 1 \tag{9.3.6}$$

could serve as such an estimator (consistent and unbiased), because, on the one hand,

$$S_U^2 = \frac{n}{n-1}\left[\overline{U^2} - \bar{U}^2\right] \xrightarrow{\mathsf{P}} \mathsf{E}U^2 - (\mathsf{E}U)^2 = \operatorname{Var} U$$

as $n \to \infty$, and, similarly,

$$S_V^2 \xrightarrow{\mathsf{P}} \operatorname{Var} V,$$

which yields

$$\hat{\nu} \xrightarrow{\mathsf{P}} \nu.$$

On the other hand,

$$\mathsf{E}\hat{\nu} = \nu$$

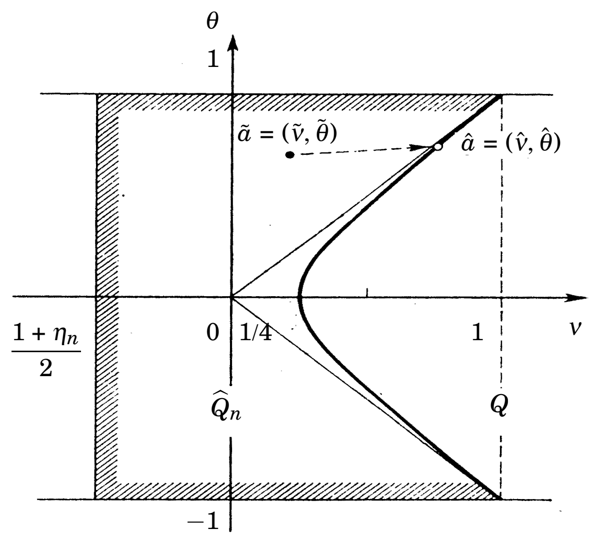


Figure 9.1.

by (A.11.10).

However, the use of $\hat{\nu}$ as an estimator of ν is hindered by the fact that the domain

$$Q = \{(\nu, \theta): |\theta| \leq \min(1, 2\sqrt{\nu} - 1)\}$$

of variation of the parameters ν and θ does not coincide with the domain $\hat{Q}_n$ of variation of the values of the pair $\hat{\nu}$, $\tilde{\theta}$, which is of the form (see Fig. 9.1)

$$\hat{Q}_n = \{(\hat{\nu}, \hat{\theta}): |\tilde{\theta}| \leq 1, \ \hat{\nu} \geq -(1 + \eta_n)/2\},$$

where $\eta_n = 3/(n - 1)$ for even n, and $\eta_n = 3/n$ for odd n. But this means that there can appear pairs of values of $\hat{\nu}$, $\tilde{\theta}$ which do not correspond to any stable distributions. Consequently, we must alter the estimators $\tilde{\nu}$ and $\tilde{\theta}$ in such a way that their new domain of variation coincides with Q. This can be done in different ways, for example, by drawing the normal from the point $(\hat{\nu}, \tilde{\theta})$ to the boundary of Q (if, of course, the point is outside Q) and taking the coordinates of the points of the normal on the boundary of Q as new estimators.

But we choose a simpler method when $\tilde{\theta}$ does not vary at all, but only $\hat{\nu}$

does. Namely, let

$$\tilde{v} = \max\left\{\hat{v}, (1+|\tilde{\theta}|)^2/4\right\} = \max\left\{\frac{6}{\pi^2}S_V^2 - \frac{3}{2}S_U^2 + 1, \frac{(1+|\tilde{\theta}|)^2}{4}\right\}. \tag{9.3.7}$$

With this definition of $\tilde{v}$, the domain of variation of of the pair $(\tilde{v}, \tilde{\theta})$ coincides with Q.

LEMMA 9.3.2. *For any $n \geq 2$,*

$$\sigma_v^2 = \operatorname{Var}\hat{v} = \left[\frac{22}{5}(v-1)^2 + \frac{6}{5}(9-5\theta^2)(v-1) + 3(1-\theta^2)(3+\theta^2)\right]/n + [2(v-1)^2 + 6(1-\theta^2)(v-1) + 9(1-\theta^2)^2]/(n(n-1)). \tag{9.3.8}$$

PROOF. By virtue of (9.3.6),

$$\sigma_v^2 = \operatorname{Var}\left[(6/\pi^2)S_V^2 - (3/2)S_U^2\right] = \frac{36}{\pi^4}\operatorname{Var}S_V^2 + \frac{9}{4}\operatorname{Var}S_U^2 - \frac{18}{\pi^2}\operatorname{cov}(S_U^2, S_V^2).$$

Let us transform the right-hand side of this equality using the known property of the sample variance (A.11.10) and (A.11.11):

$$\sigma_v^2 = \left\{\frac{36}{\pi^4}[\mathsf{E}V_0^4 - (\operatorname{Var}V)^2] + \frac{9}{4}[\mathsf{E}U_0^4 - (\operatorname{Var}U)^2] - \frac{18}{\pi^2}\operatorname{cov}(U_0^2, V_0^2)\right\}\frac{1}{n} + \left[\frac{72}{\pi^4}(\operatorname{Var}V)^2 + \frac{9}{2}(\operatorname{Var}U)^2\right]\frac{1}{n(n-1)}.$$

We know explicit expressions for the mixed central moments. With the use of (A.11.6), (A.11.7), and (A.11.8), along with the expression for $\operatorname{Var}U$ and $\operatorname{Var}V$, after simple transformations we obtain

$$\mathsf{E}U_0^4 = 1 + 2\theta^2 - 3\theta^4 = 4(1-\operatorname{Var}U)\operatorname{Var}U + (\operatorname{Var}U)^2,$$

$$\mathsf{E}U_0^2V_0^2 = (\operatorname{Var}U)\operatorname{Var}V + \frac{\pi^2}{3}(1-\operatorname{Var}U)\operatorname{Var}U,$$

$$\mathsf{E}V_0^4 = (\operatorname{Var}V)^2 + 2\left[(\operatorname{Var}V)^2 - \frac{\pi^4}{16}(\operatorname{Var}U)^2\right] + \frac{12}{5}(\operatorname{Var}V - \frac{\pi^2}{4}\operatorname{Var}U)^2 + \frac{4\pi^2}{5}(\operatorname{Var}V - \frac{\pi^2}{4}\operatorname{Var}U) + \frac{\pi^4}{4}\operatorname{Var}U.$$

Hence the first three terms in σ_ν^2 can be represented as

$$\begin{aligned}
\frac{36}{\pi^4}[\mathsf{E}V_0^4 - (\operatorname{Var} V)^2] &= \frac{22}{5}W^2 + 6\,(\operatorname{Var} U + 4/5)\,W + 9\operatorname{Var} U,\\
\frac{9}{4}[\mathsf{E}U_0^4 - (\operatorname{Var} U)^2] &= 9(1 - \operatorname{Var} U)\operatorname{Var} U,\\
\frac{18}{\pi^2}\operatorname{cov}(U_0^2, V_0^2) &= 6(1 - \operatorname{Var} U)\operatorname{Var} U,
\end{aligned}$$

where

$$W = \frac{6}{\pi^2}\operatorname{Var} V - \frac{3}{2}\operatorname{Var} U.$$

Therefore,

$$\begin{aligned}
\sigma_\nu^2 = \left[\frac{22}{5}W^2 + 6\left(\operatorname{Var} U + \frac{4}{5}\right)W + 3(4 - \operatorname{Var} U)\operatorname{Var} U\right]/n&\\
+ [2W^2 + 6W\operatorname{Var} U + 9(\operatorname{Var} U)^2]/(n(n-1)).&
\end{aligned}$$

To obtain (9.3.8), it remains to substitute $W = \nu - 1$ and $\operatorname{Var} U = 1 - \theta^2$.

LEMMA 9.3.3. *The inequalities*

$$(\mathsf{E}\tilde\nu - \nu)^2 \le \mathsf{E}(\tilde\nu - \nu)^2 \le \sigma_\nu^2 + \sigma_\theta^2 \tag{9.3.9}$$

are true.

PROOF. Since

$$|\theta| \le \min\left\{1, 2\sqrt{\nu} - 1\right\},$$

we obtain

$$\nu = \max\left\{\nu, (1 + |\theta|)^2/4\right\}.$$

Moreover, since

$$\max\{a, b\} - \max\{a', b'\} \le \max\{|a - a'|, |b - b'|\}$$

for any real a, a', b, and b', and because $|\tilde\theta| \le 1$ and $|\theta| \le 1$, we obtain

$$\begin{aligned}
\tilde\nu - \nu &= \max\left\{\hat\nu, (1 + |\tilde\theta|)^2/4\right\} - \max\left\{\nu, (1 + |\theta|)^2/4\right\}\\
&\le \max\{|\hat\nu - \nu|, |\tilde\theta - \theta|\}.
\end{aligned}$$

The inequality

$$\nu - \tilde\nu \le \max\{|\hat\nu - \nu|, |\tilde\theta - \theta|\}$$

is derived quite similarly, that is,

$$|\hat\nu - \nu| \le \max\{|\hat\nu - \nu|, |\tilde\theta - \theta|\}.$$

Hence,

$$(\tilde{\nu} - \nu)^2 \le \max\left\{(\hat{\nu} - \nu)^2, (\tilde{\theta} - \theta)^2\right\}$$
$$\le (\hat{\nu} - \nu)^2 + (\tilde{\theta} - \theta)^2.$$

Therefore,

$$\mathsf{E}(\tilde{\nu} - \nu)^2 \le \mathsf{E}(\hat{\nu} - \nu)^2 + \mathsf{E}(\tilde{\theta} - \theta)^2.$$

Now we formulate the assertion which follows from Lemmas 9.3.2 and 9.3.3.

THEOREM 9.3.1. *The statistic $\tilde{\nu}$ given by* (9.3.7) *is an asymptotically unbiased and consistent estimator of the parameter ν, and the square of the bias and the square deviation of $\tilde{\nu}$ from the true value of ν does not exceed the sum $\sigma_\nu^2 + \sigma_\theta^2$, with order of magnitude $O(1/n)$ as $n \to \infty$. The exact values of the terms in this sum are determined by* (9.3.4) *and* (9.3.8).

In the case of a symmetric stable variable, estimators (9.3.3) and (9.3.7) take a simpler form

$$\tilde{\tau} = \bar{V}, \qquad \nu = \max\left\{\frac{1}{4}, \frac{6}{\pi^2}S_V^2 - \frac{1}{2}\right\},$$

and can be extended to the multidimensional case. It was done in (Zolotarev, 1978; Zolotarev, 1981b).

The results can be formulated as follows.

Let $Y_1, \dots, Y_n$ be independent realizations of a random vector $Y \in \mathbb{R}^N$ obeying spherically symmetric stable law with parameters ν and τ, which have to be estimated. Introducing the variables

$$V_1 = \ln|Y_1|, \dots, V_n = \ln|Y_n|,$$

we denote

$$\bar{V} = n^{-1}\sum_{j=1}^{n} V_j,$$
$$S_V^2 = (n-1)^{-1}\sum_{j=1}^{n}(V_j - \bar{V})^2.$$

We introduce the sequences

$$A_1 = 0, \qquad A_2 = \ln 2, \quad A_N = \begin{cases} \sum_{m=1}^{r} (2m-1)^{-1}, & N = 2r+1, \\ \ln 2 + \sum_{m=1}^{r-1} (2m)^{-1}, & N = 2r, \end{cases}$$

$$B_1 = -1/2, \qquad B_N = \begin{cases} -1/2 + \frac{6}{\pi^2} \sum_{m=1}^{r} (2m-1)^{-2}, & N = 2r+1, \\ \frac{6}{\pi^2} \sum_{m=1}^{r-1} (2m)^{-2}, & N = 2r, \end{cases}$$

where $r = 1, 2, 3, \dots$

THEOREM 9.3.2. *The parameters τ and ν are estimated by*

$$\tilde{\tau} = \bar{V} - A_N, \qquad \tilde{\nu} = \max\left\{\frac{1}{4}, \frac{6}{\pi^2} S_V^2 + B_N\right\},$$

where $\tilde{\tau}$ is an unbiased estimator, and $\tilde{\nu}$ is an asymptotically unbiased one with the bias of order of magnitude $O(n^{-1/2})$. Both of the estimators are consistent.

In this section, we consider strictly stable random variables. As concerns arbitrary stable variables with distribution $G(x; \alpha, \beta, \gamma, \lambda)$, the key idea in the construction of estimators for the parameters α, β and λ consists in the transformation of the initial independent sample $Y_1, \dots, Y_n$ in such a way that the result is another set $Y_1', \dots, Y_m'$ of strictly stable variables with parameters related in a one-to-one way to the parameters α, β and λ. The size m of the new sample will be essentially smaller than that of the original one; however, this should be taken as the necessary cost for the distributions of the random variables Y_i' to possess the property we need.

Let $Y_1, \dots, Y_{6n}$ be an independent sample, and

$$Y_j^0 = Y_{2j-1} - Y_{2j}, \qquad j = 1, 2, \dots, 3n,$$

(the size $6n$ of the original sample is chosen so that while partitioning into thirds and halves we do not need to concern ourselves with remainders). We cite results for a simpler case involving estimators of the parameters α and λ on the basis of the transformed sample $Y_1^0, \dots, Y_{3n}^0$. If

$$V_j = \ln |Y_j^0|, \quad j = 1, \dots, 3n,$$

$\bar{V}$ is the sample mean and S_V^2 is the sample variance, then the statistics

$$\tilde{\alpha} = \left[\max\left\{\frac{1}{4}, \frac{6}{\pi^2}S_V^2 - \frac{1}{2}\right\}\right]^{-1/2},$$
$$\tilde{\lambda} = \frac{1}{2}\left\{(\bar{V} + C)\tilde{\alpha} - C\right\} \tag{9.3.10}$$

are consistent and asymptotically unbiased estimators of the parameters α and λ, with the bias and the mean square deviation of the estimated parameters of order of magnitude $O(n^{-1/2})$.

The details can be found in (Zolotarev, 1986).

9.4. Invariant estimation of α

An interesting approach to estimation of the parameter α invariant with respect to other parameters γ, β, and λ, was suggested in (Nagaev & Shkolnik, 1985).

Let $Y = Y(\alpha, \beta, \gamma, \lambda)$ and

$$\tau = (Y_1 - Y_2)/(Y_3 - Y_4), \qquad \bar{\tau} = \min(|\tau|, |\tau|^{-1}). \tag{9.4.1}$$

According to (Nagaev, 1979), we obtain

$$F_\alpha(x) = \mathsf{P}\{\tau < x\} = \frac{1}{2} + \frac{1}{\alpha\pi^2}\int_0^\infty \ln\frac{1 + |x+y|^\alpha}{1 + |x-y|^\alpha}\frac{dy}{y}, \qquad |x| \le \infty. \tag{9.4.2}$$

It is easy to show that

$$\bar{F}_\alpha(x) = \mathsf{P}\{\bar{\tau} < x\} = 4(F_\alpha(x) - 1/2), \qquad 0 \le x \le 1. \tag{9.4.3}$$

Using formulae (9.4.1)–(9.4.3) and the moment method, one can derive a number of different estimates of the parameter α that are invariant relative to the parameters β, γ, and λ. Many of them are no worse asymptotically than the corresponding estimates in (Zolotarev, 1980). These include, for example, an estimate based on the values of the sample mean of the logarithm of $\bar{\tau}$. The practical implementation of such estimation procedures is accomplished by means of tables of the corresponding functional of distribution (9.4.3).

Let $p_\alpha(x)$ be the density of family (9.4.3).

PROPERTY A. Families (9.4.2) and (9.4.3) are extendable in α from the interval $(0, 2]$ to the half-line $(0, \infty)$:

(1) for $\alpha = 2n$, with integer n, F_α is a mixture of Cauchy distributions; in particular,

$$p_4(x) = 2\left(\frac{1}{\sigma}q\left(\frac{x-a}{\sigma}; 1\right) + \frac{1}{\sigma}q\left(\frac{x+a}{\sigma}; 1\right)\right), \qquad 0 \le x \le 1,$$

where $a = \cos \pi/4$, $\sigma = \sin \pi/4$,

$$q(x; 1) = \frac{1}{\pi(1 + x^2)};$$

(2) for $\alpha = 2n - 1$, with integer n, F_α is a slightly different mixture; in particular,

$$p_3(x) = \frac{8}{\pi^2} \frac{x^2 \ln(1/x)}{1 - x^6} + \frac{16}{9} \left(\frac{1}{\sigma} q \left(\frac{x - \alpha}{\sigma}; 1 \right) + \frac{1}{\sigma} q \left(\frac{x + a}{\sigma}; 1 \right) \right),$$

where $a = \cos \pi/3$, $\sigma = \sin \pi/3$;

(3) as $\alpha \to \infty$,

$$p_\infty(x) = \frac{4}{\pi^2 x} \ln \frac{1 + x}{1 + x}, \qquad 0 \le x < 1.$$

For non-integer α, expansions have been derived for $p_\alpha(x)$ in the form of convergent and asymptotic power series in x.

PROPERTY B. Families (9.4.2) and (9.4.3) do not uniquely determine the class of stable distributions.

9.5. Estimators of parameter γ

The parameter γ plays a special part in the above approach to solving the general problem of statistical estimation of parameters of stable laws. The strategy used to construct estimators of the parameters α, β, and λ consists in the transformation of the original sample into a new sample of smaller size which is associated in turn with distributions of the class of strictly stable laws, and within this class the logarithmic moments possess comparatively simple expressions in terms of the parameters under estimation. A similar universal transformation of the original sample which would allow us to estimate the parameter γ in the same way has not been found yet, and it seems likely that such a transformation does not exist at all. Therefore, new ways should be opened to constructing estimators of γ.

In the case where $\alpha > 1$, the distribution $G^A(x; \alpha, \beta, \gamma, \lambda)$ has finite mean equal to $\lambda\gamma$, and this provides us with the opportunity to use the sample mean as an estimator. However, these distributions do not possess finite variances for $\alpha < 2$ but only finite moments of order smaller than α. The dissipation of the values of the sample mean in the variable $\lambda\gamma$ being estimated thus turns out to be large, and it is all the layer, the closer α is to one.

The basic idea for constructing an estimator for the parameter γ from an independent sample $Y_1, \dots, Y_n$ of sufficiently large size is as follows. Let α, β, γ, and λ be the parameters of the stable law (in form A) which is the distribution of the random variables Y_j. We consider a simplified version of the problem, assuming that we know the parameters α, β, and λ, and that only the parameter γ should be estimated.

It is assumed hereafter that $Y'_1, Y'_2, \ldots$ is a given sequence whose random elements are distributed by the stable law $G^A(x; \alpha, \beta, 0, \lambda)$ and that the values of the parameters α, β, and γ are known. The initial sample $Y_1, \ldots, Y_n$ is transformed with the use of random variables $Y'_1, \ldots, Y'_n$ into a new collection of variables $\tilde{Y}_1, \ldots, \tilde{Y}_n$ by the rule

$$\tilde{Y}_j = \lambda^{-1}(Y_j - Y'_j), \qquad j = 1, 2, \ldots, n.$$

The variables $\tilde{Y}_j$ can be interpreted as an independent sample from a collection which is distributed by a stable law. According to (3.7.1) and (3.7.8),

$$\tilde{Y}_j \stackrel{d}{=} \lambda^{-1} Y_A(\alpha, 0, \gamma/2, 2\lambda) \stackrel{d}{=} \lambda^{-1} Y_A(\alpha, 0, 0, 2\lambda) + \gamma,$$

which makes it clear that $\tilde{Y}_j$ obey a symmetric stable law biased by γ. More rigorously,

$$\tilde{F}(x - \gamma) = \mathsf{P}\left\{\tilde{Y}_j < x\right\} = G^A(\lambda(x - \gamma); \alpha, 0, 0, 2\lambda).$$

Consequently, γ coincides with the median of the distribution $\tilde{F}(x-\gamma)$, and this, in turn, allows us to make use of the sample median method for estimating γ well known in statistics (Zolotarev, 1986).

Let us arrange the observations $\tilde{Y}_j$ in the increasing order, and denote the terms of the resulting sequence by w_i: $w_1 < w_2 < \ldots < w_n$. The sample median μ_n is defined by

$$\mu_n = \begin{cases} w_{(n+1)/2} & \text{if } n \text{ is odd,} \\ [w_{(n+2)/2} + w_{n/2}]/2 & \text{if } n \text{ is even.} \end{cases}$$

We take this statistic as an estimator of the unknown parameter γ, i.e., we set

$$\tilde{\gamma} = \mu_n. \tag{9.5.1}$$

There are no essential differences between the cases of even and odd n, and the corresponding asymptotic analysis (as $n \to \infty$) of the properties of the estimator $\tilde{\gamma}$ leads in both cases to identical conclusions, but in the technical respect, the case of odd n is somewhat simpler, and for this reason we restrict the discussion to analyzing the case where $n = 2m + 1$.

Let

$$F_n(x) = \mathsf{P}\left\{\tilde{\gamma} - \gamma < x\right\}.$$

It is not hard to see (see, e.g. (van der Waerden, 1957, §17)) that

$$F_n(x) = \sum_{k=m+1}^{n} \binom{n}{k} \tilde{F}^k(x)[1 - \tilde{F}(x)]^{n-k}.$$

Since the distribution function $\tilde{F}(x)$ possesses the density

$$\tilde{p}(x) = \lambda q^A(x\lambda; \alpha, 0, 0, 2\lambda),$$

so does $F_n(x)$, and

$$\begin{aligned} p_n(x) = F_n'(x) &= n \binom{2m}{m} [\tilde{F}(x) - \tilde{F}^2(x))]^m \tilde{p}(x) \\ &= a_n \exp(-m\psi(x))\tilde{p}(x), \end{aligned} \qquad (9.5.2)$$

where

$$\begin{aligned} \psi(x) &= -\ln \tilde{F}(x) - \ln(1 - \tilde{F}(x)), \\ a_n &= n\binom{2m}{m}. \end{aligned}$$

It follows from (9.5.2) that the bias $\mathsf{E}(\tilde{\gamma} - \gamma)$ and the mean square error $\mathsf{E}(\tilde{\gamma} - \gamma)^2$ of the estimator $\tilde{\gamma}$ can be expressed as the integrals

$$\mathsf{E}(\tilde{\gamma} - \gamma) = a_n \int x\tilde{p}(x) \exp\{-m\psi(x)\}\, dx, \qquad (9.5.3)$$

$$\mathsf{E}(\tilde{\gamma} - \gamma)^2 = a_n \int x^2\tilde{p}(x) \exp\{-m\psi(x)\}\, dx. \qquad (9.5.4)$$

The distribution $\tilde{F}(x)$ is symmetric, i.e.,

$$1 - \tilde{F}(x) = \tilde{F}(-x), \qquad \tilde{p}(x) = \tilde{p}(-x)$$

for all x. This implies that the functions $\psi(x)$ and $p_n(x)$ are even, and that

$$p_n(x) \sim \operatorname{const} x^{-\alpha(m+1)-1}, \qquad x \to \infty.$$

Therefore, integrals (9.5.3) and (9.5.4) exist, provided that $n > 4/\alpha - 1$ (the former integral exists for $n > 2/\alpha - 1$), and, moreover, integral (9.5.3) vanishes, i.e., the estimator $\tilde{\gamma}$ is unbiased.

Let us now consider the asymptotic behavior of integral (9.5.4). Since the distribution $\tilde{F}(x)$ is symmetric, we obtain

$$\begin{aligned} \psi(0) = 2\ln 2, \qquad \psi'(0) = 0, \\ \psi''(0) = 8\tilde{p}^2(0) = 8\lambda^2[q^A(0; \alpha, 0, 0, 2\lambda)]^2. \end{aligned} \qquad (9.5.5)$$

The value of the density of a symmetric stable distribution at zero is known (see Section 4.9). Therefore, by virtue of (3.7.2)

$$\psi''(0) = 2[\pi^{-1}\Gamma(1 + 1/\alpha)(2\lambda)^{1-1/\alpha}]^2 > 0. \qquad (9.5.6)$$

Properties (4.5.5) and (4.5.6) allow us to use the Laplace method to obtain an asymptotic representation of integral (4.5.4) as $n \to \infty$; we have already used this method under similar circumstances in Section 4.7. By this method, we obtain

$$I_n = a_n \int x^2 \tilde{p}(x) \exp\{-m\psi(x)\}\, dx$$
$$= a_n \exp(-m\psi(0)) \int x^2 \tilde{p}(x) \exp\left\{-m\psi''(0)\frac{x^2}{2} + \dots\right\} dx.$$

Since $\sigma_n^2 = m\psi''(0) \to \infty$ as $n \to \infty$, we arrive at the following asymptotic formula after the change of variable $\sum_n x = t$ and the appropriate estimation of the 'tails' of the integral:

$$I_n \sim a_n \sigma_n^{-3} \tilde{p}(0) \exp\{-m\psi(0)\} \int x^2 \exp\left\{-x^2/2\right\} dx$$
$$= (2m+1)\binom{2m}{m} 2^{-2m} (8m)^{-3/2} \sqrt{2\pi} (\tilde{p}(0))^{-3} \sim cn^{-1},$$

where

$$c = [\pi(2\lambda)^{(1-\alpha)/\alpha} / \Gamma(1 + 1/\alpha)]^2. \tag{9.5.7}$$

The above reasoning can be summarized as follows.

THEOREM 9.5.1. *The statistic $\tilde{\gamma} = \mu_n$ is a consistent unbiased estimator of the parameter γ for all odd $n > (2-\alpha)/\alpha$, with mean square error*

$$\mathsf{E}(\tilde{\gamma} - \gamma)^2 \sim cn^{-1}, \qquad n \to \infty,$$

where the constant c is given by (9.5.7).

REMARK 9.5.1. For even n, the statistic $\tilde{\gamma} = \mu_n$ is a consistent and asymptotically unbiased estimator of γ.

REMARK 9.5.2. We indicate one more variant of an estimator for γ under the assumption that the values of the remaining parameters are known. Formally, this variant has more advantages than that considered above, because it does not require a transformation of the original sample. The following considerations are taken as a basis for the estimator. By virtue of (3.7.2),

$$G^A(x; \alpha, \beta, \gamma, \lambda) = G((x - l)\lambda^{-1/\alpha}, \alpha, \beta, 0, 1),$$

where $l = \lambda(\gamma + b_0)$, and b_0 is uniquely determined by the parameters α, β, and λ. Hence we conclude that the median $m(\alpha, \beta, \gamma, \lambda)$ of the distribution $G(x; \alpha, \beta, \gamma, \lambda)$ is related to the median $m(\alpha, \beta) = m(\alpha, \beta, 0, 1)$ by the equality

$$\lambda^{-1} m(\alpha, \beta, \gamma, \lambda) = \lambda^{1/\alpha - 1} m(\alpha, \beta) + b_0 + \gamma. \tag{9.5.8}$$

We then consider the sample median $\tilde{\mu}$ derived from the original sample $Y_1, \dots, Y_n$ with distribution $G(x; \alpha, \beta, \gamma, \lambda)$. Because of good analytic properties of stable laws, the statistic μ_n turns out to be asymptotically normally distributed with mean $\mu = m(\alpha, \beta, \gamma, \lambda)$ and variance

$$\sigma^2 \sim q^{-2}(\mu; \alpha, \beta, \gamma, \lambda)(4n)^{-1}, \qquad n \to \infty$$

(see, e.g. (Zolotarev, 1986)).

Therefore, replacing the median μ in (9.5.8) by the sample median $\tilde{\mu}$, we obtain the estimator of γ

$$\tilde{\gamma} = \lambda^{-1}\tilde{\mu} - b_0 - c\lambda^{1/\alpha - 1}, \tag{9.5.9}$$

where $c = m(\alpha, \beta)$ is the unique solution of the equation $G(x; \alpha, \beta) = 1/2$.

Like the estimator in Theorem 9.5.1, $\tilde{\gamma}$ is asymptotically unbiased and consistent (more precisely, $1/\sqrt{n}$-consistent).

9.6. Maximum likelihood estimators

Let $p(x, \mu)$ be the distribution density of independent random variables $X_1, \dots, X_n$ forming a sample which serves as the base to construct an asymptotically efficient estimator of the parameter μ, given a $1/\sqrt{n}$-consistent estimator $\tilde{\mu}$ of this parameter. Let

$$L(X \mid \mu) = \sum_{j=1}^{n} \ln p(X_j, \mu)$$

be the likelihood function, and let $L'(X \mid \mu)$ and $L''(X \mid \mu)$ be its first and second derivatives with respect to μ.

It turns out that the statistic

$$\hat{\mu} = \tilde{\mu} - L'(X \mid \tilde{\mu})/L''(X \mid \tilde{\mu}) \tag{9.6.1}$$

is an asymptotically efficient estimator of μ under certain regularity conditions imposed on the density $p(x, \mu)$, for example, the existence and continuity of the second derivative of $p(x, \mu)$ with respect to μ, etc. In the cases where the Fisher information

$$I(\mu) = \int \left(\frac{\partial}{\partial \mu} \ln p(x, \mu)\right)^2 p(x, \mu)\, dx$$

associated with the distribution corresponding to $p(x, \mu)$ can be computed, it is possible to use the statistic

$$\mu^* = \tilde{\mu} + L'(X \mid \tilde{\mu})/(nI(\mu)) \tag{9.6.2}$$

instead of (9.6.1).

The fact that an explicit expression for the density is used in construction of the statistic $\hat{\mu}$ and μ^* presents some challenge in the case of estimating the parameters of stable laws, because, with some exceptions, we are aware of only somewhat complicated forms of expressions for the corresponding densities as series or integrals. However, these are not the problems that arise in constructing the maximum likelihood estimator μ_0, where one must solve the transcendental equation $L'(X \mid \mu) = 0$ for the variable μ (μ_0 is the solution of this equation).

We assume, for example, that we are looking for the value of only the single parameter μ of a stable law under the condition that we know the remaining parameters. To use estimator (9.6.1), we obviously have, at least, to know some tables of values of the functions

$$\frac{\partial}{\partial\mu}\ln p(x,\mu), \qquad \frac{\partial^2}{\partial\mu^2}\ln p(x,\mu).$$

In constructing similar estimators for several parameters (i.e., for estimators of vectors), the problem becomes significantly more difficult from the computational viewpoint, because it is required to tabulate all the first and second mixed derivatives of the logarithm of the density with respect to the parameters to be estimated.

As we know, among the stable laws there are those whose densities can be expressed in terms of elementary functions. The parameter sets ($\alpha = 1/2$, $\beta = 1$, γ, λ), ($\alpha = 1$, $\beta = 0$, γ, λ), and ($\alpha = 2$, β, γ, λ) correspond to them. The last set, which corresponds to the normal distributions, is well known, so there is no need to comment on the associated problem of estimation of the parameters. The remaining two cases in this scheme are less known. It is thus useful, in our opinion, to analyze these cases to illustrate the considerations given above. We consider the simplest problem of estimation of one of the parameters γ or λ under the condition that the value of the second parameter is known.

In the first case,

$$q^B(x;1/2,1,\gamma,\lambda) = \frac{\lambda}{2\sqrt{\pi}}(x-\gamma\lambda)^{-3/2}\exp\left(-\frac{\lambda^2}{4(x-\gamma\lambda)}\right), \quad x > \gamma\lambda. \tag{9.6.3}$$

We assume that the value of λ is known, while the value of γ is unknown and has to be estimated. We introduce the likelihood function $L(Y \mid \gamma)$. Its derivative with respect to γ is of the form

$$L'(Y \mid \gamma) = \frac{3}{2}\lambda\sum_j (Y_j-\gamma\lambda)^{-1} - \frac{\lambda^3}{4}\sum_j (Y_j-\gamma\lambda)^{-2}.$$

The likelihood equation $L'(Y \mid \gamma) = 0$, reduced to an algebraic equation, does not allow us to write out the maximum likelihood estimator, though. Therefore, we look for an asymptotically efficient estimator $\hat{\gamma}$ of the parameter γ by following the hints given above. First of all, the explicit form (9.6.3) of the density allows us to compute the corresponding Fisher information $I(\gamma)$. Easy calculations yield $I(\gamma) = 42/\lambda^2$, which allows us to construct the estimator $\hat{\gamma}$ of form (9.6.2).

At our hands, we have two $1/\sqrt{n}$-consistent estimators of the parameter γ. One was given in Theorem 9.5.1, and the second, in Remark 9.5.1. The most convenient estimator in this case turns out to be (9.5.9). Indeed,

$$\tilde{\gamma} = \lambda^{-1}\tilde{\mu} - c\lambda,$$

where $\tilde{\mu}$ is the sample median and $c = 1.08\ldots$ is the solution of the equation

$$G(x; 1/2, 1) = 2[1 - \Phi(1/\sqrt{2x})] = 1/2,$$

where Φ is the distribution function of the standard normal law.

Finally, with the use of (9.6.2) we obtain the statistic

$$\hat{\gamma} = \tilde{\gamma} + \frac{\lambda^2}{42} L'(Y \mid \tilde{\gamma}) n^{-1} \tag{9.6.4}$$

as an asymptotically efficient estimator of γ.

We assume now that the value of γ is known, but the value of λ has to be estimated. The case is treated as before: we construct the likelihood function $L(Y \mid \lambda)$ and then find its derivatives with respect to λ:

$$L'(Y \mid \lambda) = n\lambda^{-1} + \left(\frac{3}{2}\gamma - \lambda\right) \sum_j (Y_j - \gamma\lambda)^{-1} - \frac{\lambda^2\gamma}{4} \sum_j (Y_j - \gamma\lambda)^{-2}.$$

Of course, we should not try to solve the likelihood equation $L'(Y \mid \lambda) = 0$, and we have to construct an asymptotically efficient estimator $\hat{\lambda}$ of λ by the rule pointed out. However, in contrast to the preceding case, the Fisher information $I(\lambda)$, though computable, turns out to depend on λ. Therefore, $\hat{\lambda}$ should be constructed by (9.6.1). The $1/\sqrt{n}$-consistent estimator $\tilde{\lambda}$ of λ sought for can be found in Section 9.3 (equality (9.3.10)). Let us compute the second derivative of $L(Y \mid \lambda)$ with respect to λ:

$$\begin{aligned} L''(Y \mid \lambda) = n\lambda^{-2} - \sum_j (Y_j - \gamma\lambda)^{-1} \\ + \tfrac{1}{2}\gamma(3\gamma - \lambda) \sum_j (Y_j - \gamma\lambda)^{-2} - \tfrac{1}{2}\lambda^2\gamma^2 \sum_j (Y_j - \gamma\lambda)^{-3}. \end{aligned}$$

According to (9.6.1), the asymptotically efficient estimator $\hat{\lambda}$ is of the form

$$\hat{\lambda} = \tilde{\lambda} - L'(Y \mid \tilde{\lambda})/L''(Y \mid \hat{\lambda}).$$

As we see, the estimator of λ occurs to be more complicated than the estimator of γ. This is due, in the first glance, to the choice of parameterization. If, instead of the system $(\alpha, \beta, \gamma, \lambda)$, we considered the system $(\alpha, \beta, \gamma', \lambda)$, where $\gamma' = \gamma\lambda$, then the complexity of the estimator of γ' in the case where $\alpha = 1/2$ and $\beta = 1$ should stay the same, while the estimator of λ, provided that γ' is known, should be essentially simpler to calculate, since the likelihood equation is fairly simple to solve. It turns out that the resulting maximum likelihood estimator is a sufficient statistic (see (Januškevičiene, 1981)).

Asymptotically efficient estimators of one of the parameters γ or λ in the case $\alpha = 1$ and $\beta = 0$ can be constructed in the same way with the use of the explicit expression

$$q(x; 1, 0, \gamma, \lambda) = \frac{\lambda}{\pi}[\lambda^2 + (x - \gamma\lambda)^2]^{-1}$$

for the density.

We give only a sketch of the basic idea of (Dzhaparidze, 1974) in the simplest situation where the only parameter μ has to be estimated.

The part of μ can be played by any parameter of a stable law for which we know a $1/\sqrt{n}$-consistent estimator. At the same time, it should be mentioned that the possibilities of this method are much wider, and it allows us to construct asymptotically efficient estimators for the parameter collection $\mu = (\alpha, \beta, \lambda)$, because, as we have seen, there are $1/\sqrt{n}$-consistent estimators for the vector-valued parameter μ.

In (Worsdale, 1976), the parameters α, δ, λ of the symmetric stable distribution with characteristic function

$$g(k; \alpha, \delta, \lambda) = \exp\{i\delta k - (1/\alpha)|\lambda k|^\alpha\}$$

were estimated using a modification of the method of maximum likelihood.

A procedure for estimating all three parameters α, δ, and λ was given in (Worsdale, 1976).

DuMouchel (DuMouchel, 1975) analyzed Fama and Roll's results using the Fisher information matrices $I(\Gamma)$ for the parameters $\Gamma \equiv \{\alpha, \beta, c, \delta\}$ of stable distributions

$$q(x; \Gamma) = q^A((x - \delta)/c; \alpha, \beta)/c.$$

The elements of $I(\Gamma)$ are determined by

$$I_{\Gamma'\Gamma''} = I_{\Gamma'\Gamma''}(\Gamma) = \int_{-\infty}^{\infty} \left(\frac{\partial q}{\partial \Gamma'}\right)\left(\frac{\partial q}{\partial \Gamma''}\right)\frac{1}{q}dx, \tag{9.6.5}$$

where Γ' and Γ'' each represent α, β, c and δ.

Two kinds of approximations were used in the computations. The first is less critical and involves replacing derivatives with respect to α, β, c and δ by the corresponding differences. The second approximation used is more delicate; it involves the replacement of the continuous density $q^A(x;\alpha,\beta)$ by a discrete approximation. We partition the real axis into m intervals whose end points are $-\infty = y_0 < y_1 < \ldots < y_{m-1} < y_m = \infty$. Then, let

$$p_k(\Gamma) = \int_{y_{k-1}}^{y_k} q(x;\Gamma)\,dx, \qquad k = 1,2,\ldots,m.$$

Then the definition of $I(\Gamma)$ given by (9.6.5) is replaced by

$$I_{\Gamma'\Gamma''}(\Gamma) = \sum_{k=1}^{m} \frac{\partial p_k(\Gamma)}{\partial \Gamma'} \frac{\partial p_k(\Gamma)}{\partial \Gamma''} \Big/ p_k(\Gamma).$$

This definition of I leans upon the assumption that the data are grouped into m classes defined by the points $y_1,\ldots,y_{m-1}$, and that only n_k, the number of observations fallen into the kth class, $k = 1,\ldots,m$, is recorded. Such a reduction of continuous data always involves some loss of information.

These concepts were applied to the stable distribution case. The asymptotic standard deviations and correlations of the maximum likelihood estimators (MLEs) of the index, skewness, scale, and location parameters were computed and tabularized, and used to compute the relative asymptotic efficiency of other proposed estimators. It is shown that if the true stable distribution is symmetric, the MLEs of the index and the scale parameters are asymptotically independent of those of skewness and location. The effect on the available information of grouping the data is investigated both analytically and numerically, and the most serious loss of information is shown to occur if extreme observations are grouped while estimating α. In particular, it was found that an estimator of δ proposed by Fama and Roll (Fama & Roll, 1968; Fama & Roll, 1971) is very efficient and that their estimators for α and c, while not quite so efficient, are easily computed and thus might be profitably be used as initial values for computing the maximum likelihood estimate by an iterative process, if greater efficiency were needed.

9.7. Fisher's information for α close to 2

With the use of (4.8.10), in (Nagaev & Shkolnik, 1988) the asymptotic behavior of Fisher's information

$$I(\alpha) = \int_{-\infty}^{\infty} \left(\frac{\partial \ln q(x;\alpha,0)}{\partial \alpha} \right)^2 q(x;\alpha,0)\,dx \tag{9.7.1}$$

as a function of α was investigated for $\alpha \to 2$.

THEOREM 9.7.1. *As $\Delta = 2 - \alpha \to 0$,*

$$I(\alpha) \sim [4\Delta|\ln\Delta|]^{-1}. \tag{9.7.2}$$

The proof of the theorem is based on the following lemma.

LEMMA 9.7.1. *As* $\Delta = 2 - \alpha \to 0$,

$$\frac{\partial q(x;\alpha,0)}{\partial\alpha} = -x^{\Delta-3}(1 + \theta c\Delta + \theta c\Delta\ln x + \theta c x^{\Delta-2}\ln x) \tag{9.7.3}$$

for all $x \geq x_0 > 0$, *where* c *is a positive constant and* $\theta \in (-1, 1)$.

After proving the lemma, (9.7.1) is rewritten as the sum of integrals

$$I(\alpha) = \sum_{k=1}^{5} I_k, \tag{9.7.4}$$

decomposing the domain of integration into the intervals $[0, T)$, $[T, x_1(\Delta))$, $[x_1(\Delta), x_2(\Delta))$, $[x_2(\Delta), x_3(\Delta))$, and $x_3(\Delta), \infty)$, where T is some positive constant and

$$x_1(\Delta) = (2-\varepsilon)|\ln\Delta|^{1/2}, \quad x_2(\Delta) = (2+\varepsilon)|\ln\Delta|^{1/2}, \quad x_3(\Delta) = \exp(\Delta^{-1/2}),$$

with some small positive ε (see (4.8.10)). Using (4.8.10) and (9.7.3), one can show the following.

(1) As $\Delta \to 0$,

$$I_1 < \infty. \tag{9.7.5}$$

(2) If $T \leq x \leq x_1(\Delta)$ and $\Delta \to 0$, then

$$q(x;\alpha,0) = q(x;2,0)(1 + o(1)),$$
$$\frac{\partial q(x;\alpha,0)}{\partial\alpha} = \theta c x^{-3},$$

and therefore,

$$I_2 = 2\theta c\int_T^{x_1(\Delta)} x^{-6}\exp(x^2/4)\,dx = \frac{2\theta c}{\Delta^{1-\omega(\varepsilon)}|\ln\Delta|^{7/2}}, \tag{9.7.6}$$

where $\omega(t)$ stands for any positive-valued function possessing the property

$$\lim_{t\to 0}\omega(t) = 0.$$

(3) If $x_1(\Delta) \leq x \leq x_2(\Delta)$, then

$$q(x;\alpha,0) \geq c\Delta x^{-3}$$

and

$$\frac{\partial q(x;\alpha,0)}{\partial\alpha} = \theta c x^{-3},$$

hence

$$I_3 = \frac{2\theta c}{\Delta}\int_{x_1(\Delta)}^{x_2(\Delta)} x^{-3}\,dx = \frac{2\theta c\omega(\varepsilon)}{\Delta|\ln\Delta|}. \tag{9.7.7}$$

(4) Further, if $x_2(\Delta) \le x \le x_3(\Delta)$, then

$$q(x;\alpha,0) = \Delta x^{\Delta-3}(1+o(1)),$$
$$\frac{\partial q(x;\alpha,0)}{\partial\alpha} = -x^{-3}(1+o(1))$$

as $\Delta \to 0$, $x \to \infty$; therefore,

$$I_4 = \frac{2}{\Delta}\int_{x_2(\Delta)}^{x_3(\Delta)} x^{-3}\,dx\,(1+o(1)) = \frac{1+o(1)}{\Delta(2+\varepsilon)^2|\ln\Delta|}. \tag{9.7.8}$$

(5) Finally, for $x \ge x_3(\Delta)$

$$q(x;\alpha,0) = \Delta x^{\Delta-3}(1+o(1)),$$
$$\frac{\partial q(x;\alpha,0)}{\partial\alpha} = -x^{-3}(1+\theta c\Delta\ln x)(1+o(1))$$

as $\Delta \to 0$, $x \to \infty$. Thus,

$$I_5 = \frac{2\theta c}{\Delta}\int_{x_3(\Delta)}^{\infty} x^{\Delta-3}(\max\{1,\Delta\ln x\})^2\,dx = \omega(\Delta). \tag{9.7.9}$$

Since ε is arbitrary, (9.7.4)–(9.7.9) immediately imply the desired assertion.

9.8. Concluding remarks

The estimators given in Sections 9.3–9.6 for the parameters of stable laws are not best possible estimators even in the asymptotic sense as the sample size n increases to infinity. This is due, first of all, to the fact that all the estimators were constructed from the sample moments and the sample median which are of a simple structure but, as a rule, of poor efficiency. Also, in constructing estimators in the general situation we use a transformation of the original sample which leads either to a reduction in its size by a factor of two or three, or to an increase of the dissipation of the random variables making up the sample in isolated cases. At the same time, the estimators we found possess a number of merits that not only allow us to take these estimators as a convenient tool for solving practical problems, but also to make them a good basis for the construction of best possible (in a certain sense) estimators of the corresponding parameters.

Let $\tilde{\mu}$ stand for an estimator of the parameter μ (which is allowed to be any of the parameters considered in Sections 9.3–9.6); then with regard to its properties we can assume that

(1) it is of somewhat algorithmically simple structure (this applies, first of all, to the estimators of the parameters α, θ, and τ corresponding to form E);

(2) it is asymptotically normal; and

(3) it possesses the mean square deviation $\mathsf{E}(\tilde{\mu} - \mu)^2$ of order $1/n$.

Property 2 has not been proved for any of the estimators discussed,but it can be established without difficulty with the use of the central limit theorem by recalling the 'locally additive' structure of the estimators by virtue of the obvious inequalities

$$(\mathsf{E}\tilde{\mu} - \mu)^2 \le \mathsf{E}(\tilde{\mu} - \mu)^2, \qquad \operatorname{Var}\tilde{\mu} \le 4\mathsf{E}(\tilde{\mu} - \mu)^2,$$
$$\mathsf{P}(\sqrt{n}|\tilde{\mu} - \mu| \ge T) \le n\mathsf{E}(\tilde{\mu} - \mu)^2/T^2, \qquad T > 0.$$

Property 3 implies that $\tilde{\mu}$ is asymptotically unbiased with bias of order at most $1/\sqrt{n}$, that the variance of the estimator is of order at most $1/n$, and, finally, that $\tilde{\mu}$ is a $1/\sqrt{n}$-consistent estimator of the parameter μ.

It is worthwhile to mention another fact. Constructions of the estimators for the parameters of stable distributions were carried out within two groups. One included the parameters α, β, and λ, and the other, the parameter γ. These groups differ, first of all, in the form of the transformations of the original sample in constructing of the estimators. The estimators of the first group can be regarded in total as the coordinates of a three-dimensional random vector which is obviously a $1/\sqrt{n}$-consistent estimator of the corresponding triple of parameters. It can be improved as was said above.

To conclude the section, we remark that it is well known how important in practice is to find confidence intervals for parameters being estimated. In general, we do not dwell upon this question, although the estimators given for parameters ν, θ and τ, due to their simple structure, allow us to construct confidence intervals for them in a fairly simple manner. For example, the estimator for θ is reduced to the estimator for the parameter $p = (1 + \theta)/2$ in the binomial scheme (see the reasoning after Lemma 9.3.1), and the problem of constructing confidence intervals for this estimator has become classical (see (van der Waerden, 1957; Kendall & Stuart, 1967)).

A method of estimation of bivariate stable distributions applied to financial problems will be discussed in Chapter 17.

Part II

Applications

10

Some probabilistic models

10.1. Generating functions

In the present chapter, we consider some probabilistic models giving rise to stable distributions. We present this material in a separate chapter for the reason that these models are too schematic constructions deprived of those particular details which could unambiguously orient readers on particular applications. However, there is an advantage in this way: with a suitable modification they can be applied to solving various, occasionally very far from each other, problems. As a matter of fact, we have already worked enough with such a model: it is the summation of random variables. Moreover, we touched also the case where the number of terms is random and is distributed by the Poisson law (Section 3.5). In essence, the models considered below are based on the idea of summation as well, in various interpretations.

Before going to these models, we consider one more characteristic of random variables used in the case where they take non-negative integer values. Basically, it plays the same role as the characteristic function, but is more convenient while working with integer-valued random variables.

DEFINITION 10.1.1. Let N be a random variable taking the values $0, 1, 2, \dots$ with probabilities $p_0, p_1, p_2, \dots$ The function

$$\varphi_N(u) = \mathsf{E}u^N = \sum_{n=0}^{\infty} u^n p_n \qquad (10.1.1)$$

is called the generating function (g.f.) of the r.v. N.

Let us give some properties of g.f.'s.

(a) A g.f. $\varphi(u)$ is defined for all u from the unit circle $|u| \le 1$ of a complex plane; it is analytic for $|u| < 1$.

(b) The equality

$$\varphi(1) = 1 \tag{10.1.2}$$

is true.

(c) One and only one generating function $\varphi(u)$ corresponds to each probability distribution $\{p_0, p_1, p_2, \dots\}$; one and only one probability distribution $\{p_0, p_1, p_2, \dots\}$ corresponds to each function $\varphi(u)$ which is analytic in the circle $|u| < 1$, possesses non-negative coefficients when expanded in a power series, satisfies (10.1.2), and

$$p_n \equiv \mathsf{P}\{N = n\} = \frac{1}{n!}\varphi^{(n)}(0), \qquad n = 0, 1, 2, \dots \tag{10.1.3}$$

(d) A g.f. $\varphi(u)$ and all its derivatives $\varphi^{(n)}(u)$, $n = 1, 2, 3, \dots$, are non-negative, non-decreasing, and convex on $0 \le u \le 1$.

(e) For any natural n,

$$\varphi^{(n)}(1) = \mathsf{E}N^{[n]} \equiv \mathsf{E}N(N-1)\dots(N-n+1). \tag{10.1.4}$$

The value on the right-hand side is called the nth factorial moment of a r.v. N. For $n = 1$, it coincides with the expectation

$$\mathsf{E}N = \varphi'(1), \tag{10.1.5}$$

and for $n = 2$, leads to the following expression for the variance:

$$\operatorname{Var} N = \varphi_N''(1) + \varphi_N'(1) - \left[\varphi_N'(1)\right]^2. \tag{10.1.6}$$

Since for $u > 1$ the function $\varphi_N(u)$ may not be defined, its derivatives at the point 1 are understood as the left derivatives.

(f) Let the nth factorial moment N be finite. Then

$$\varphi(u) = \sum_{k=0}^{n-1} \varphi^{(k)}(1)\frac{(u-1)^k}{k!} + R_n(u)\frac{(u-1)^n}{n!}; \tag{10.1.7}$$

$R_n(u)$ does not decrease for real $u \in [0, 1]$, $0 \le R_n(u) \le \varphi^{(n)}(1)$, $|R_n(u)| \le \varphi^{(n)}(1)$ for complex $|u| \le 1$, and $R_n(u) \to \varphi^{(n)}(1)$ as $s \to 1$.

(g) If $N_1, \dots, N_n$ are independent r.v.'s, then

$$\varphi_{N_1+\dots+N_n}(u) = \varphi_{N_1}(u)\dots\varphi_{N_n}(u). \tag{10.1.8}$$

(h) Let $K, N_1, \dots, N_n$ be independent integer-valued random variables, where $N_1, \dots, N_n, \dots$ are identically distributed. Then the g.f. of the sum $S_K = N_1 + \dots + N_K$ of the random number K of the terms N_i is related with the g.f.'s $\varphi_N(u)$ and $\varphi_K(u)$ by

$$\varphi_{S_K}(u) = \varphi_K(\varphi_N(u)). \tag{10.1.9}$$

This property is validated by replacing n by K in (10.1.8) followed by averaging over K:

$$\varphi_{S_K}(u) = \mathsf{E}\mathsf{E}(u^{S_K} \mid K) = \mathsf{E}[\varphi_N(u)]^K = \varphi_K(\varphi_N(u)). \tag{10.1.10}$$

A more detailed consideration of generating functions and their applications can be found in (Sevastyanov, 1974; Uchaikin & Ryzhov, 1988), and others.

Here $\mathsf{E}\{X \mid k\}$ stands for the conditional mathematical expectation of X under the condition that $K = k$.

10.2. Stable laws in games

Let us consider the so-called fair game. Two players participate in the game. The initial capital of the first player is equal to an integer m, the second one is infinitely rich. Each of them wins or loses in each game with probability 1/2, irrespective of outcomes of previous games, and his capital increases or decreases accordingly by one each time. Let $m + S_k$ be the capital of the first player after the kth game, and $N(m)$ be the number of steps until he is ruined, i.e.,

$$N(m) = \min\{k : m + S_k = 0\}, \qquad N(0) = 0.$$

First of all, we show that $N(m)$ is a proper random variable, i.e., a r.v. taking a finite value with probability one. This means that with probability one the first player will be ruined irrespective of what capital he began to play with (the second player is infinitely rich and not threatened with ruin). We assume that

$$p(m) = \mathsf{P}\{N(m) < \infty\}, \qquad p(0) = 1.$$

and denote by B_1 the event 'the first player wins a game' and by B_2, the event 'the first player loses a game'. The law of total probability yields

$$p(m) = \mathsf{P}\{N(m) < \infty \mid B_1\}\mathsf{P}\{B_1\} + \mathsf{P}\{N(m) < \infty \mid B_2\}\mathsf{P}\{B_2\}. \tag{10.2.1}$$

Because B_1 implies the increase of the capital by one, and B_2, its decrease by one,

$$\mathsf{P}\{N(m) < \infty \mid B_1\} = \mathsf{P}\{N(m+1) < \infty\} = p(m+1),$$
$$\mathsf{P}\{N(m) < \infty \mid B_2\} = \mathsf{P}\{N(m-1) < \infty\} = p(m-1),$$

and, due to the fair character of the game,

$$\mathsf{P}\{B_1\} = \mathsf{P}\{B_2\} = 1/2,$$

from (10.2.1) we obtain

$$p(m) = \tfrac{1}{2}[p(m+1) + p(m-1)]. \tag{10.2.2}$$

Assuming that

$$\Delta(m) = p(m+1) - p(m), \qquad z \geq 0,$$

from (10.2.2) we obtain

$$\Delta(m) - \Delta(m-1) = 0, \qquad \Delta(m) = \text{const} \equiv \delta.$$

Since

$$p(m+1) = p(0) + \sum_{k=1}^{m} \Delta(k) = p(0) + m\delta$$

for all m, it is clear that δ can be nothing but zero. Hence

$$p(m) \equiv 1 \text{ for all } m.$$

and therefore, the first player is ruined with probability one.

We consider now the distribution of the r.v. $N \equiv N(1)$. The player starts the game with one dollar, and each time he wins or loses one dollar with equal probabilities. It is easy to realize that

$$N(2) = N_1 + N_2 + 1, \tag{10.2.3}$$

where N_1 and N_2 are independent random variables distributed as $N(1)$. Since each of these events occurs after a unit of time, it is possible to write the following stochastic relation:

$$N = \begin{cases} 1 & \text{with probability } 1/2. \\ N_1 + N_2 + 1 & \text{with probability } 1/2. \end{cases}$$

Using it while evaluating the corresponding g.f.'s, we arrive at the algebraic equation

$$\varphi_N(u) = \mathsf{E}u^N = \frac{1}{2}\left(u + \mathsf{E}u^{N_1+N_2+1}\right) = \frac{u}{2}\left[1 + \varphi_N^2(u)\right], \tag{10.2.4}$$

whose solution is of the form

$$\varphi_N(u) = \left[1 - \sqrt{1-u^2}\right]/u. \tag{10.2.5}$$

Here the sign before the square root is chosen so that function (10.2.5) satisfies Property (c). It is easy to see that $\varphi_N(1) = 1$, but $\varphi'_N(1) = \infty$. The latter means that the mathematical expectation of N is infinite:

$$\mathsf{E}N = \infty. \tag{10.2.6}$$

In order to get the probability distribution

$$p_n = \mathsf{P}\{N(1) = n\},$$

one expands the square root in (10.2.5) in terms of u^2:

$$\varphi_N(u) = \sum_{k=1}^{\infty} \frac{(2k-3)!!}{(2k)!!} u^{2k-1}.$$

Comparing this expression with the general formula

$$\varphi_N(u) = \sum_{n=0}^{\infty} \frac{1}{n!} \varphi_N^{(n)}(0) u^n$$

and taking (10.1.3) into account, we obtain

$$p_n = \frac{(2k-3)!!}{(2k)!!} = \frac{(n-2)!!}{(n+1)!!}, \quad n = 2k-1, \quad k = 1, 2, \ldots, \tag{10.2.7}$$

and

$$p_n = 0, \qquad n \text{ is even.}$$

In the same way one can also find the distribution of the time of gambler's ruin for any initial capital m:

$$p_n(m) = \mathsf{P}\{N(m) = n\} = 2^{-n} \frac{m}{n} \binom{n}{(n-m)/2} \tag{10.2.8}$$

provided that n and m are simultaneously even or odd, and

$$p_n(m) = 0$$

if n and m are of different parity. For large m, however, evaluations can be avoided if we notice that (10.2.3) also remains true for $m > 2$:

$$N(m) = N_1 + \ldots + N_m. \tag{10.2.9}$$

Thus, the problem is reduced to the determination of the limiting distribution for the sum of independent r.v.'s with infinite mathematical expectation (10.2.6). Using (10.2.7), we immediately obtain

$$\mathsf{P}\{N(1) > n\} = \sum_{k>(x+1)/2} p_{2k-1} \sim \sqrt{2/\pi}\, n^{-1/2}, \qquad x \to \infty. \tag{10.2.10}$$

Therefore, the distribution of the r.v. $N(1)$ belongs to the domain of attraction of the stable law with density $q(x; 1/2, 1)$ (Lévy law), and according to the generalized limit theorem (Section 2.5)

$$\mathsf{P}\{N(m) < n\} = \mathsf{P}\{N(m)/b_m < n/b_m\} \sim G^A(n/b_m; 1/2, 1), \qquad m \to \infty, \tag{10.2.11}$$

where

$$b_m = b_1(1/2)m^2 = m^2.$$

Another stable law manifests itself in the so-called StPetersburg game. The game is to flip a coin until the head appears. The coin is symmetric, so the random number N of the flip when the head appears the first time has the probability distribution

$$\mathsf{P}\{N = n\} = 2^{-n}, \qquad n = 1, 2, \dots$$

As the head occurs, the player wins $X = 2^N$ dollars. This game was introduced by Nicholas Bernoulli in the early 1700s, and is known as the StPetersburg paradox, because Daniel Bernoulli wrote about it in the Reviews of the StPetersburg Academy. The question is how many dollars a player has to risk (the ante) to play. In a fair game, the ante should be equal the expected winning, but it is infinite:

$$\mathsf{E}X = \sum_{n=1}^{\infty} 2^n 2^{-n} = \infty.$$

Feller found (see (Székely, 1986)) how to define the ante R in order that the game would be fair. Supposing that the game is fair, where

$$\mathsf{P}\{|X_n/R_n - 1| < \varepsilon\} \to 1$$

as the number of games of the player $n \to \infty$, he obtained that for this case one should set

$$R_n = n \ln_2 n. \tag{10.2.12}$$

Such unusual non-linear law for the ante stems from the theory of stable laws Indeed,

$$\mathsf{P}\{X > x\} = \mathsf{P}\{2^N > x\} = \mathsf{P}\{N > \ln_2 x\}$$
$$= \sum_{n \ge \ln_2 x} 2^{-n} \approx 2x^{-1}$$

and according to the generalized limit theorem the random variable X belongs to the domain of attraction of the stable law with parameters $\alpha = 1$ and β, which yields (10.2.12) (see also (Shlesinger *et al.*, 1993)).

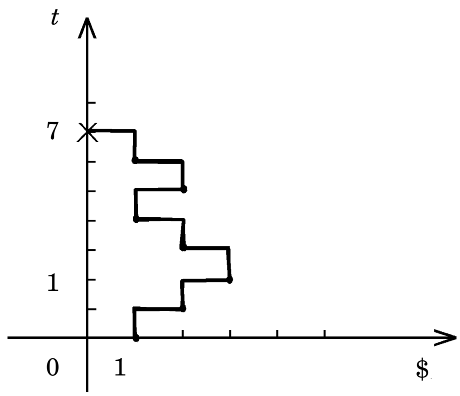


Figure 10.1.

10.3. Random walks and diffusion

The player's fate in the above situation can be considered as a random function of the integer variable, the number of the game, which will be called time. The graph of a realization of such a function is presented in Fig. 10.1. Looking at this, we see that the player who begins playing with one dollar wins one dollar in the first game and has three dollars after the second game. Further, alas, he was not lucky and in the seventh game he was ruined. However, he cannot be considered as an unlucky person: the probability to hold out up to the 7th game is vanishingly small, indeed.

Particular content of the given problem apart, we consider the graph given in Fig. 10.1 as a trajectory of some particle making random unit jumps along x-axis in every unit of time: +1 with probability 1/2 and -1 with the same probability 1/2. Let the particle continue its movement in a predetermined fashion also to the left-hand half of the axis. The event 'the player is ruined' corresponds, in this terminology, to the event 'the particle passes the point $x = 0$ for the first time'.

By permitting the particle to move also along the negative semi-axis, we obtain a symmetric picture, and now can (for the sake of convenience) locate it at initial time $t = 0$ at the origin $x = 0$. Assume now that time intervals between jumps are equal to τ and the amplitude of each jump is equal to ξ. Let $p(x, t)$ stand for the probability to be at location $x = \xi k$ ($k = 0, \pm 1, \pm 2, \dots$) at time $t = \tau n$ ($n = 0, 1, 2, \dots$). It is clear that

$$p(x, t + \tau) = \tfrac{1}{2}[p(x - \xi, t) + p(x + \xi, t)], \qquad (10.3.1)$$

$$p(x, 0) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0, \end{cases}$$

For τ and ξ small enough, we can expand the probabilities in terms of τ and ξ

$$p(x,t) + \frac{\partial p(x,t)}{\partial t}\tau = p(x,t) + \frac{1}{2}\frac{\partial^2 p(x,t)}{\partial x^2}\xi^2.$$

If $\tau \to 0$ and $\xi \to 0$ so that

$$\xi^2/(2\tau) \to D \neq 0, \tag{10.3.2}$$

we arrive at the well-known diffusion equation

$$\frac{\partial p(x,t)}{\partial t} = D\frac{\partial^2 p(x,t)}{\partial x^2}, \tag{10.3.3}$$

describing the Brownian motion with the initial condition $p(x,0) = \delta(x)$. It can be decomposed into two equations: the continuity equation

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial j(x,t)}{\partial x} \tag{10.3.4}$$

and the Fick law

$$j(x,t) = -D\partial p(x,t)/\partial x. \tag{10.3.5}$$

If we consider the first passage through the point $a > 0$, then $a/\xi \to \infty$ as $\xi \to 0$, and for the first passage time $T = N\tau$ we obtain:

$$\lim_{\substack{\tau\to 0\\ \xi\to 0}} \mathsf{P}\left\{N/b_m < n/b_m\right\} = \mathsf{P}\{T < t\} = G^A(2Dt/a^2; 1/2, 1).$$

Differentiating this relation with respect to time, one obtain the density function of the random variable T

$$p_T(t) = (2D/a^2)q^A(2Dt/a^2; 1/2, 1) = \frac{a}{\sqrt{4\pi D t^3}}e^{-a^2/(4Dt)}. \tag{10.3.6}$$

This result can easily be obtained by solving diffusion equation (10.3.3) with the zero boundary condition at the point $x = a$ as well:

$$p(a,t) = 0. \tag{10.3.7}$$

The zero condition arises due to the restriction of the process by the first passage; to suppress other visits of this point, we assume that the particle dies after the first visit. One can imagine an absorbing screen at the point $x = a$, and the unknown distribution follows from (10.3.5):

$$p_T(t) = -D\left.\frac{\partial p(x,t)}{\partial x}\right|_{x=a}. \tag{10.3.8}$$

Instead of that, one can place the same additional source at the point $x = 2a$ and evaluate the difference between the two solutions:

$$p(x,t) = \frac{1}{\sqrt{4\pi Dt}}\left[e^{-x^2/(4Dt)} - e^{-(x-2a)^2/(4Dt)}\right]. \qquad (10.3.9)$$

Substituting this into (10.3.8), we obtain (10.3.7).

Let Θ be the moment when the particle leaves the interval $[-a,a]$ for the first time. In this case, the solution of equation (10.3.2) with boundary conditions

$$p(a,t) = p(-a,t) = 0$$

is obtained by separating the variables and is of the form

$$p(x,t) = \frac{1}{a}\sum_{l=0}^{\infty}\cos\left[\frac{(2l+1)\pi x}{2a}\right]\exp\left\{-\left[\frac{(2l+1)\pi}{2a}\right]^2 Dt\right\}. \qquad (10.3.10)$$

The distribution of Θ is given by the relation similar to (10.3.8)

$$\begin{aligned} p_\Theta(t) &= D\left\{\left.\frac{\partial p(x,t)}{\partial x}\right|_{x=-a} - \left.\frac{\partial p(x,t)}{\partial x}\right|_{x=a}\right\} \\ &= \frac{\pi D}{a^2}\sum_{l=0}^{\infty}(-1)^l(2l+1)\exp\left\{-D\left[\frac{(2l+1)\pi}{2a}\right]^2 t\right\} \end{aligned}$$

It is easy to see that the density satisfies the normalization

$$\int_0^\infty p_\Theta(t)\,dt = \frac{4}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1} = 1.$$

The mathematical expectation of Θ is now finite:

$$\bar{\Theta}(a) \equiv \mathsf{E}\Theta = \int_0^\infty t p_\Theta(t)\,dt = \frac{16a^2}{\pi^3 D}\sum_{l=0}^{\infty}\frac{(-1)^l}{(2l+1)^3} = \frac{a^2}{2D}. \qquad (10.3.11)$$

It is useful to obtain this result without the use of the Fick law. The integral

$$I = \int_{-a}^{a} p(x,t)\,dx$$

differs from one, and the difference $1 - I$ is equal to the probability $\mathsf{P}\{\Theta < t\}$, which is the distribution function $F_\Theta(t)$ of the random time Θ:

$$\int_{-a}^{a} p(x,t)\,dx = 1 - F_\Theta(t) \equiv \bar{F}_\Theta(t).$$

The mean time can be expressed in terms of $\bar{F}_\Theta(t)$ as follows:

$$\bar{\Theta}(a) = -\int_0^\infty t(d\bar{F}_\Theta/dt)\,dt = \int_0^\infty \bar{F}_\Theta(t)\,dt.$$

Therefore,

$$\bar{\Theta}(a) = \int_0^\infty dt \int_{-a}^{a} dx\, p(x,t). \tag{10.3.12}$$

Inserting (10.3.10) into this relation and integrating, we arrive at (10.3.11) again.

Consider now the two-dimensional diffusion on a plane. The diffusion equation in this case is of the form

$$\frac{\partial p(\mathbf{r},t)}{\partial t} = D\Delta p(\mathbf{r},t),$$

where $\mathbf{r}$ is the two-dimensional vector with coordinates (x,y), and Δ is the two-dimensional Laplacian. Its solution for a particle starting the walk from the origin at $t = 0$ is

$$p(r,t) = \frac{1}{4\pi Dt}\exp\left\{-\frac{r^2}{4Dt}\right\}, \qquad r^2 = x^2 + y^2. \tag{10.3.13}$$

It is a mere product of two one-dimensional solutions

$$\begin{aligned} p(x,t) &= \frac{1}{\sqrt{4\pi Dt}}\exp\left\{-\frac{x^2}{4Dt}\right\}, \\ p(y,t) &= \frac{1}{\sqrt{4\pi Dt}}\exp\left\{-\frac{y^2}{4Dt}\right\}, \end{aligned} \tag{10.3.14}$$

which reflects the independence of Cartesian coordinates of the particle while walking.

Let us draw the straight line $x = a$ on this plane and look for the distribution of the first passage coordinate Y_a, i.e., the coordinate of the point where the particle passes the line for the first time (Fig. 10.2). The distribution can be represented as

$$p_{Y_a}(y) = \int_0^\infty p(y,t)p_T(t)\,dt, \tag{10.3.15}$$

where $p_T(t)$ was obtained above. Substituting (10.3.6) and (10.3.14) into (10.3.15), we arrive at the Cauchy distribution

$$p_{Y_a}(y) = \frac{a}{4\pi D}\int_0^\infty \exp\left\{-\frac{(a^2+y^2)}{4Dt}\right\} t^{-2}dt = \frac{a}{\pi(a^2+y^2)}.$$

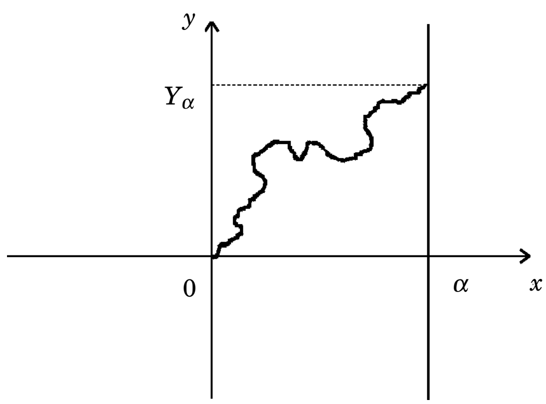


Figure 10.2.

Dealing with the three-dimensional diffusion, one can find the distribution of the random coordinates Y_a, Z_a of the point where the particle passes the plane $x = a$ for the first time:

$$p_{Y_aZ_a}(y,z) = \frac{a}{(4\pi D)^{3/2}} \int_0^\infty \exp\left\{-\frac{(a^2+y^2+z^2)}{4Dt}\right\} t^{-5/2}dt$$
$$= \frac{a}{2\pi(a^2+y^2+z^2)^{3/2}}, \qquad (10.3.16)$$

which is the two-dimensional Cauchy distribution. A similar result also takes place in the case of N-dimensional diffusion: the distribution of the $(N-1)$-dimensional vector $(X_1 \ldots X_{N-1})$ corresponding to the first passage of the point a by the component X_N is given by the $(N-1)$-dimensional Cauchy distribution (see (7.4.8))

$$p(x_1 \ldots x_{N-1}) = \frac{a}{(4\pi D)^{N/2}} \int_0^\infty \exp\left\{-\left(a^2 + \sum_{i=1}^{N-1} x_i^2\right) \Big/ (4Dt)\right\} t^{-N/2-1}dt$$
$$= \frac{a\Gamma(N/2)}{\pi^{N/2}(a^2+x_1^2+\ldots+x_{N-1}^2)^{N/2}}. \qquad (10.3.17)$$

10.4. Stable processes

The Brownian motion of a particle considered above belongs to the well-known class of random processes with independent increments.

Recall that a random process $X = X(t)$ on the real line is said to have independent increments if for any values $t_0 \le t_1 \le t_2 \le \ldots$ the increments $X(t_{n+1}) - X(t_n)$, $n = 0, 1, 2, \ldots$, are independent random variables.

The process $X(t)$ with independent increments is called homogeneous if the probability distribution of increments $X(t)-X(s)$ depends only on the difference $t-s$.

DEFINITION 10.4.1 ((Prokhorov & Rozanov, 1969)). A homogeneous process $X(t)$ with independent increments is called stable if its increments are distributed by the stable law of the same type.

The Brownian motion is a Gaussian process, i.e., a stable process with $\alpha = 2$. Its characteristic function $f(k,t)$ satisfies the differential equation

$$\frac{\partial f(k,t)}{\partial t} = -k^2 D f(k,t)$$

with the initial condition

$$f(k,0) = 1.$$

Its solution is of the form of the characteristic function of the normal law with the product Dt playing the part of the parameter λ:

$$f^{(2)}(k,t) = e^{-Dt|k|^2}.$$

A homogeneous random process $X(t;\alpha,\beta)$ with independent increments is called the Lévy stable process if its increments are distributed according to the Lévy stable law (Feller, 1966). We consider the strictly stable processes satisfying the condition

$$X(t;\alpha,\beta) \stackrel{d}{=} t^{1/\alpha} Y(\alpha,\beta), \qquad t > 0, \tag{10.4.1}$$

where $Y(\alpha,\beta)$ is the standardized strictly stable variable with characteristic exponent α and skewness β.

In terms of distribution densities, relation (10.4.1) is expressed as follows:

$$p(x,t;\alpha,\beta) = t^{-1/\alpha} q(xt^{-1/\alpha};\alpha,\beta), \tag{10.4.2}$$

where p and q stand for the probability densities of random variables X and Y respectively.

If $\alpha = 2$, then process (10.4.1) has the characteristic function

$$f(k,t;2,0) = e^{-tk^2},$$

and is merely the Wiener process considered above. If $\alpha < 2$ and $\beta = 0$, we have a symmetric Lévy process with the characteristic function

$$f(k,t;\alpha) = e^{-t|k|^\alpha}. \tag{10.4.3}$$

This process was considered in (Seshadri & West, 1982). We cite here some results.

Let α be a rational number equal to m/n, where m and n are integers. When m is even, we differentiate (10.4.3) successively n times with respect to time and apply the inverse Fourier transformation to the resulting equation to obtain

$$\frac{\partial^n p(x,t)}{\partial t^n} = (-1)^{n+m/2} \frac{\partial^m p(x,t)}{\partial x^m}. \tag{10.4.4}$$

To ensure that the solutions of (10.4.4) are real and positive, they must satisfy n initial conditions

$$p(x,0), \partial p(x,t)/\partial t\big|_{t=0}, \dots, \partial^{n-1} p(x,t)/\partial t^{n-1}\Big|_{t=0},$$

Though these sufficient initial conditions may be formally obtained from (10.4.3), they are, in general, as hard to evaluate as the inverse Fourier transform of the function $f(k,t;\alpha)$ itself.

When m is odd, we differentiate (10.4.3) successively $2n$ times and, after applying the inverse Fourier transformation

$$p(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} f(k,t;m/n)dk,$$

obtain

$$\frac{\partial^{2n} p(x,t)}{\partial t^{2n}} = (-1)^m \frac{\partial^{2m} p(x,t)}{\partial x^{2m}}.$$

The exact first passage time to reach the point $x = \pm a$, starting from the origin, is computed for the symmetric Lévy-process with $\alpha = 2/n$ in (Seshadri & West, 1982). In this case, $p(x,t)$ satisfies the equation

$$\frac{\partial^n p(x,t)}{\partial t^n} = (-1)^{n+1} \frac{\partial^2 p(x,t)}{\partial x},$$

with the boundary conditions

$$p(\pm a, t) = 0$$

and the initial condition

$$p(x,0) = \delta(x).$$

In addition, those n initial conditions on the derivatives of $p(x,t)$ must be given to guarantee that $p(x,t)$ is real and positive. The solution $p(x,t)$ is of the form similar to (10.3.10):

$$p(x,t) = \frac{1}{a} \sum_{l=0}^{\infty} \cos\left[\frac{(2l+1)\pi x}{2a}\right] \exp\left\{-\left[\frac{(2l+1)\pi}{2a}\right]^{2/n} t\right\}.$$

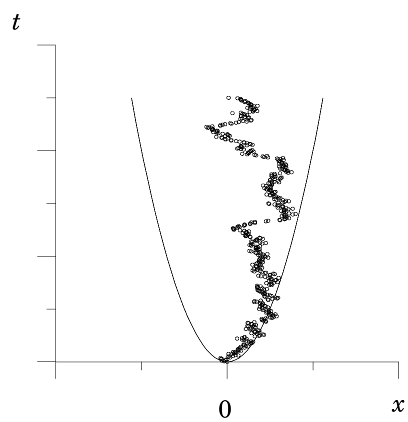


Figure 10.3. Trajectory of the Gauss process ($\alpha = 2$), $\Delta(t) \propto t^{1/2}$

Now the Fick law cannot be expected to hold, and we have to go by the second way in (10.3.12):

$$\bar{\Theta}(a) = \frac{4(2a)^{2/n}}{\pi^{1+2/n}} \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l+1)^{1+2/n}}.$$

The most important feature of the behavior of the first passage time is resides in the first factor $a^{2/n}$, $2/n = \alpha$. The second factor is a monotonic increasing factor of n. When $n = 1$, we arrive at the result for the normal diffusion process (10.3.11). When $n = 2$, we obtain the result for the Cauchy process,

$$\bar{\Theta}(a) = a \left(\frac{8G}{\pi^2} \right),$$

where G is Catalan's constant (0.915956…).

The scaling behavior, established above for one dimension, can be extended to higher dimensions. For example, in two dimensions the mean first passage time to a circle of radius a, and in the three dimensions to a sphere of radius a, can be computed from the radially and spherically symmetric distribution function respectively (Seshadri & West, 1982). The result is

$$\bar{\Theta}(a) \sim a^{\alpha}, \quad \alpha = 2/n,$$

as obtained above.

Some random realizations of the trajectories of stable processes are plotted in Figures 10.3–10.5 (the solid line shows the width of the diffusion packet $\Delta(t)$).

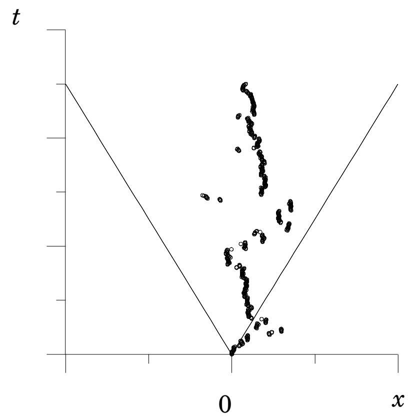


Figure 10.4. Trajectory of the Cauchy process ($\alpha = 1$), $\Delta(t) \propto t$

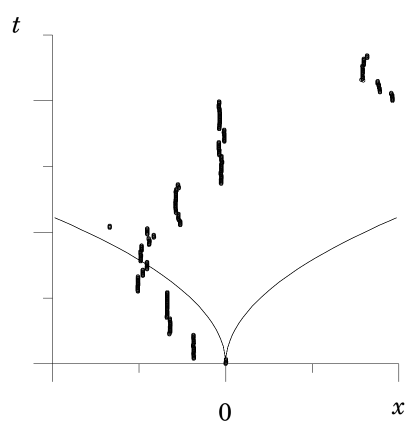


Figure 10.5. Trajectory of the symmetrized Lévy process ($\alpha = 1/2$), $\Delta(t) \propto t^2$

Now we consider the process with an arbitrary $\alpha \in (0,2]$ using two forms A and C. The characteristic functions

$$f^A(k,t;\alpha,\beta) = \begin{cases} \exp\{-t|k|^\alpha[1 - i\beta\tan(\alpha\pi/2)\operatorname{sign} k]\}, & \alpha \neq 1,\ |\beta| \leq 1; \\ \exp\{-t|k|\}, & \alpha = 1,\ \beta = 0, \end{cases}$$
$$f^C(k,t;\alpha,\delta) = \exp\{-t|k|^\alpha e^{-i\delta\pi/2}\operatorname{sign} k\}, \qquad |\delta| \leq \min\{\alpha, 2-\alpha\},$$

satisfy the evolution equations

$$\partial f^A(k,t;\alpha,\beta)/\partial t = -|k|^\alpha[1 - i\beta\tan(\alpha\pi/2)\operatorname{sign}k]f^A(k,t;\alpha,\beta), \quad \alpha \neq 1, \tag{10.4.5}$$

$$\partial f^A(k,t;1,0)/\partial t = -|k|f^A(k,t;1,0),$$

$$\partial f^C(k,t;\alpha,\delta)/\partial t = -|k|^\alpha e^{-i\delta\pi/2\operatorname{sign}k} f^C(k,t;\alpha,\delta) \tag{10.4.6}$$

with the initial condition

$$f^A(k,0;\alpha,\beta) = f^C(k,0;\alpha,\delta) = 1.$$

We rewrite (10.4.5) in the form

$$\frac{1 + i\beta\tan(\alpha\pi/2)\operatorname{sign}k}{|k|^\alpha[1 + \beta^2 tg^2(\alpha\pi/2)]}\frac{\partial f^A}{\partial t} = -f^A.$$

Assuming

$$\Omega^2 = [1 + \beta^2\tan^2(\alpha\pi/2)]\cos(\alpha\pi/2)$$

and writing $\hat{F}p^A$ for f^A, we obtain

$$\frac{\cos(\alpha\pi/2) + i\beta\sin(\alpha\pi/2)\operatorname{sign}k}{|k|^\alpha\Omega^2}\hat{F}\frac{\partial p^A}{\partial t} = -\hat{F}p^A.$$

Comparing the left-hand side of this equality with the Fourier transform of Feller's potential (A.8.16) and inverting the transform, we arrive at the equation

$$M^\alpha_{u,v}\frac{\partial p^A}{\partial t} = -p^A(x,t;\alpha,\beta)$$

or

$$\frac{\partial p^A}{\partial t} = -(M^\alpha_{u,v})^{-1}p^A(x,t;\alpha,\beta) \tag{10.4.7}$$

with

$$u = \frac{1+\beta}{2\Omega^2},$$
$$v = \frac{1-\beta}{2\Omega^2}.$$

We use the symbols for fractional integro-differentiation operator according to (Samko *et al.*, 1993).

According to (A.8.11), evolution equation (10.4.7) can be rewritten in the following explicit forms:

$$\frac{\partial p^A(x,t;\alpha,\beta)}{\partial t} = -\frac{\alpha}{A\Gamma(1-\alpha)} \int_{-\infty}^{\infty} \frac{1+\beta\,\mathrm{sign}(x-\xi)}{|x-\xi|^{1+\alpha}} \times [p^A(x,t;\alpha,\beta) - p^A(\xi,t;\alpha,\beta)]\,d\xi, \qquad (10.4.8)$$

$$\frac{\partial p^A(x,t;\alpha,\beta)}{\partial t} = -\frac{\alpha}{A\Gamma(1-\alpha)} \int_0^{\infty} [2p^A(x,t;\alpha,\beta) - (1+\beta)p^A(x-\xi,t;\alpha,\beta) - (1-\beta)p^A(x+\xi,t;\alpha,\beta)]\xi^{-1-\alpha}d\xi \qquad (10.4.9)$$

where

$$A = [1+\beta^2\tan(\alpha\pi/2)]^{-1}.$$

In the case of a symmetric process ($\beta = 0$), the operator in the right-hand side of (10.4.9) coincides with the Riesz derivative (A.8.9)

$$\frac{\partial p^A(x,t;\alpha,0)}{\partial t} = -D^\alpha p^A(x,t;\alpha,0).$$

For $\beta = 1$, we have the one-sided stable process with the evolution equation

$$\frac{\partial p^A(x,t;\alpha,1)}{\partial t} = -[\cos(\alpha\pi/2)]^{-1}\mathbf{D}_+^\alpha p^A(x,t,\alpha,1)$$

where $\mathbf{D}_+^\alpha p^A$ is the fractional Marchoud derivative (A.8.6).

To transform equation (10.4.6) for characteristic function to the corresponding equation for the density $p^C(x,t;\alpha,\delta)$, we rewrite it in the form

$$|k|^{-\alpha-\delta}\hat{F}\frac{\partial p^C(x,t;\alpha,\delta)}{\partial t} = |k|^\delta e^{-i\delta\pi/2\,\mathrm{sign}\,k}\hat{F}p^C(x,t;\alpha,\delta)$$

and use (A.8.15) and (A.8.18); we obtain

$$I^{\alpha-\delta}\frac{\partial p^C(x,t;\alpha,\delta)}{\partial t} = -D_+^\delta p^C(x,t;\alpha,\delta),$$

or

$$\frac{\partial p^C(x,t;\alpha,\delta)}{\partial t} = -D^{\alpha-\delta}D_+^\delta p^C(x,t;\alpha,\delta) \qquad (10.4.10)$$

In the symmetric case ($\delta = 0$),

$$\frac{\partial p^C(x,t;\alpha,0)}{\partial t} = -D^\alpha p^C(x,t,\alpha,0).$$

In the extremely asymmetric case ($\delta = \alpha$), $X(t;\alpha,1) > 0$ for $\alpha < 1$, and (10.4.10) takes the form

$$\frac{\partial p^C(x,t;\alpha,\alpha)}{\partial t} = -D^\alpha_{0+}p^C(x,t;\alpha,\alpha), \qquad (10.4.11)$$

where D^α_{0+} is given by (A.8.5). Performing the Laplace transformations, we obtain for

$$\tilde{p}^C(\lambda,t;\alpha,\alpha) = \int_0^\infty e^{-\lambda x}p^C(x,t;\alpha,\alpha)\,dx$$

the equation

$$\frac{\partial \tilde{p}^C(\lambda,t;\alpha,\alpha)}{\partial t} = -\lambda^\alpha \tilde{p}^C(\lambda,t;\alpha,\alpha).$$

Under the initial condition

$$\tilde{p}^C(\lambda,0;\alpha,\alpha) = 1$$

this yields

$$\tilde{p}^C(\lambda,t;\alpha,\alpha) = e^{-\lambda^\alpha t}.$$

The cases considered above concern the domain $\alpha < 1$. The transition to the domain $\alpha > 1$ can be performed with the help of the duality law which for the case of stable processes takes the form

$$\alpha\mathsf{P}\{X(t;\alpha,\delta) > x\} = \mathsf{P}\{0 < X(t';\alpha',\delta') < (t'/x)^\alpha t\}$$

where $\alpha' = 1/\alpha$, $\delta' = 1 + (\delta-1)/\alpha$, and t' is an arbitrary positive number. Regarding t' as a function of x and t and passing from probabilities to probability densities, we obtain the expression

$$\alpha\int_x^\infty p(x',t;\alpha,\delta)dx' = \int_0^{[t'(x,t)/x]^\alpha t} p(x',t'(x,t);\alpha',\delta')dx'$$

which after differentiating with respect to x takes the form

$$p(x,t;\alpha,\delta) = (t'/x)^\alpha[x^{-1} - (\partial t'/\partial x)/t']tp((t'/x)^\alpha t, t';\alpha',\delta') - \alpha^{-1}\int_0^{(t'/x)^\alpha t} \frac{\partial p(x',t';\alpha',\delta')}{\partial t'}\frac{\partial t'(x,t)}{\partial x}dx'. \qquad (10.4.12)$$

In the case where t' does not depend on x, we obtain

$$p(x,t;\alpha,\delta) = (t'/x)^\alpha(t/x)p((t'/x)^\alpha t, t';\alpha',\delta').$$

If

$$t'(x,t) = x,$$

then

$$p(x,t;\alpha,\delta) = -\alpha^{-1}\int_0^t \frac{\partial p(x',x;\alpha',\delta')}{\partial x}\,dx'. \qquad (10.4.13)$$

Substitution of (10.4.12) into the right-hand side of (10.4.13) yields

$$\frac{\partial p(x',x;\alpha',\delta')}{\partial x} = -\alpha x^{-\alpha-1}\{g'(x'x^{-\alpha};\alpha',\delta')x'x^{-\alpha} + g(x'x^{-\alpha};\alpha',\delta')\}$$

and

$$\int_0^t \frac{\partial p(x',x;\alpha',\delta')}{\partial x}dx' = -\alpha x^{-1}\left\{\int_0^{tx^{-\alpha}} g'(z;\alpha',\delta')z\,dz + \int_0^{tx^{-\alpha}} g(z;\alpha',\delta')\,dz\right\},$$

where $g'(z;\alpha',\delta')$ is the derivative of $g(z;\alpha',\delta')$ with respect to z. After computing the first integral in the right side by parts, we obtain

$$\int_0^t \frac{\partial p(x',x;\alpha',\delta')}{\partial x}dx' = -\alpha x^{-1-\alpha}tq(tx^{-\alpha},\alpha',\delta') = -\alpha x^{-1}tp(t,x;\alpha',\delta'). \qquad (10.4.14)$$

Finally, the following duality relation results from (10.4.13) and (10.4.14):

$$xp(x,t;\alpha,\delta) = tp(t,x;1/\alpha, 1+(\delta-1)/\alpha), \qquad \alpha \geq 1. \qquad (10.4.15)$$

The use of this relation allows us to pass from evolution equations for $\alpha < 1$ derived above to the equations for $\alpha > 1$. We will illustrate it by means of the following example.

Let us consider the Lévy process whose density $p(x,t;1/2,1/2)$ satisfies (10.4.11) with $\alpha = 1/2$:

$$\frac{\partial p(x,t;1/2,1/2)}{\partial t} = -D_{0+}^{1/2}p(x,t;1/2,1/2).$$

It satisfies the equation

$$\frac{\partial^2 p(x,t;1/2,1/2)}{\partial t^2} = \frac{\partial p(x,t;1/2,1/2)}{\partial x} \qquad (10.4.16)$$

as well. Duality relation (10.4.15) in this case is of the form

$$xp(x,t;2,0) = tp(t,x;1/2,1/2).$$

Inserting this into (10.4.16) and changing the variable x for t, we obtain

$$\partial[(x/t)p(x,t;2,0)]/\partial t = \partial^2[(x/t)p(x,t;2,0)]/\partial x^2.$$

Simple relations

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)p(x,t;2,0)=\left(\frac{1}{t}-4\frac{\partial}{\partial(x^2)}\right)p(x,t;2,0)=0$$

lead us to the Einstein equation, as was to be shown.

However, evolution equations for the symmetric stable processes $X(t;\alpha,0)\equiv X(t;\alpha)$ with an arbitrary α can be derived immediately from the corresponding equations for characteristic functions and are of the form

$$\partial p(x,t;\alpha)/\partial t=-D^{\alpha}p(x,t;\alpha),\qquad p(x,t;\alpha)\equiv p(x,t;\alpha,0),$$

which remain valid in the m-dimensional case, too:

$$\partial p_m(x,t;\alpha)/\partial t=-(-\Delta_n)^{\alpha/2}p_m(x,t;\alpha),\qquad x\in\mathbb{R}^n. \tag{10.4.17}$$

In conclusion, we touch on (Seshadri & West, 1982; Allegrini *et al.*, 1995; Allegrini *et al.*, 1996; West, 1990; West, 1994; West & Grigolini, 1997; West & Seshadri, 1982), where the evolution equation is given as follows:

$$\partial p(x,t)/\partial t=\text{const}\int_{-\infty}^{\infty}[1+c\,\text{sign}(\xi-x)]\frac{p(\xi,t)}{|x-\xi|^{\alpha+1}}d\xi.$$

This equation is evidently incorrect for all positive α because of the explicit divergence of the integral. Just the presence of the difference $p(x,t)-p(\xi,t)$ under the integral in the correct equation (10.4.8) provides for its convergence for $\alpha<1$.

10.5. Branching processes

Processes with independent increments possess an important property which in terms of characteristic functions looks as follows: for any $t_0<t'<t$

$$f(k;t,t_0)=f(k;t,t')f(k;t',t_0).$$

After inverse Fourier transformation, we obtain

$$p(x;t,t_0)=\int p(x-x';t,t')p(x';t',t_0)dx'.$$

A generalization of the process leads us to the class of Markov processes determined by the Chapman–Kolmogorov equation:

$$P(A,t;x_0,t_0)=\int P(A,t;x',t')P(dx',t';x_0,t_0),\qquad t_0<t'<t, \tag{10.5.1}$$

where $P(A, t; x_0, t_0)$ is the probability for a particle located at x_0 at the moment t_0 to be found in the domain A at the moment t. Equation (10.5.1) asserts that if the probability for the walking particle to be at the point x' at the moment $t > t'$ depends only on the variables (x', t') and does not depend on prehistory.

If the random coordinate X of the walking particle can take only discrete values x_i, for example $x_i = \pm i$, $i = 0, 1, 2, \ldots$, then such a process is called the Markov chain.

If $t = 0, 1, 2, \ldots$, then the homogeneous process is completely determined by the initial distribution

$$p_i(0) = \mathsf{P}\left\{X(0) = i\right\} \tag{10.5.2}$$

and the one-step transition probabilities:

$$p_{ij} \equiv \mathsf{P}\left\{X(t+1) = i \middle| X(t) = j\right\}.$$

Generally speaking, the probabilities p_{ij} can be arbitrary positive numbers which obey the normalization

$$\sum_{i=-\infty}^{\infty} p_{ij} = 1, \tag{10.5.3}$$

and the Chapman–Kolmogorov equation takes the form of the set of simultaneous algebraic equations

$$p_i(t) = \sum_{j=-\infty}^{\infty} p_{ij} p_j(t-1). \tag{10.5.4}$$

If we assume

$$p_{ij} \equiv p_{j-i} = \begin{cases} 1/2, & i - j = 1, \\ 1/2, & i - j = -1, \\ 0 & \text{otherwise}, \end{cases}$$

then we obtain equation (10.3.9) describing a one-dimensional jump-like walking of a particle. Certainly, the structure of the matrix p_{ij} is very simple here.

Let us construct a little more complicated matrix p_{ij}. As the set of states, we take only non-negative i, $i = 0, 1, 2, \ldots$ Moreover, we assume that $p_{i0} = 0$ for all $i \neq 0$, so $p_{00} = 1$, i.e., the trajectory of the process, once appeared at the point $i = 0$, will never leave it. To construct the remaining part of the matrix p_{ij}, we take some set of non-negative numbers q_i, $i = 0, 1, 2, \ldots$, which satisfy the condition

$$\sum_{i=0}^{\infty} q_i = 1, \tag{10.5.5}$$

and set

$$p_{ij} = \begin{cases} \delta_{i0}, & j = 0, \\ \sum \dots \sum_{\{i_1 \dots i_j\}} q_{i_1} \dots q_{i_j}, & j \geq 1, \end{cases} \tag{10.5.6}$$

where $\{i_1, \dots, i_j\}$ is the set of collections of non-negative integers obeying the condition $i_1 + \dots + i_j = i$. It is easy to see that the matrix satisfies (10.5.3), and therefore, can be taken as a transition matrix.

Despite of an apparent artificiality of matrix (10.5.6), the process defined by this matrix admits a simple interpretation. Notice that the indices i and j themselves can be thought of as the numbers of particles, so p_{ij} can be interpreted as the probability that j particles of any generations produce i particles in the next generation. Let $j = 1$, then it follows from (10.5.6) that $p_{i1} = q_i$, i.e., q_i determine the probabilities that one particle produces i particles in the next generation. If $i = 0$, then the particle dies; if $i = 1$, then it survives, if $i = 2$, then it generates one more particle, etc. It is more convenient, however, to consider that the particle dies in each generation and produces i descendants of the same type. Let now $j = 2$; then

$$p_{i2} = \sum_{i_1=0}^{i} q_{i_1} q_{i-i_1} = q_0 q_i + q_1 q_{i-1} + \dots + q_i q_0. \tag{10.5.7}$$

If there are two particles in a state given, then the number i of offspring of these particles can be obtained by various (but inconsistent) ways $(0, i)$, $(1, i-1), \dots, (i, 0)$, so the presence of a sum in (10.5.7) is quite clear. But the fact that each term of the sum is a product of probabilities shows that the particles generate their descendants independently of each other. It is an intrinsic property of branching processes.

A branching process with discrete time is called the Galton–Watson process, after the famous English scientists Sir Francis Galton, an explorer and anthropologist, and a mathematician Henry Watson, who considered the problem of extinction of families in the end of XIX century. Galton gave a precise mathematical formulation to the problem of extinction of families and persuaded Watson to take up the matter; so, their joint work *On the Probability of Extinction of Families* came into light. This problem is formulated as follows. In some society the number of sons of one father (who will have his surname in future) is a random variable with distribution $\{q_i\}$. It is assumed that the numbers of sons of different fathers are independent. The sons, who become fathers in further, will have their own sons with the same probabilities q_i, etc. Let there exist one ancestor who is the zero generation ($t = 0$). The first generation ($t = 1$) consists of the sons, the second one includes all his grandsons inheriting his surname (i.e., the sons of his sons), etc. In this case, $N(t)$, $t = 0, 1, 2$ is the random size of the tth generation whose distribution is given by probabilities

$$p_i(t) = \mathsf{P}\{N(t) = i\}.$$

The limit $p_0(t)$ as $t \to \infty$ is called the extinction probability, and if this probability is equal to one, the process is called extinct. The process becomes extinct if and only if the average number of sons of one father sons is no greater than one and $q_1 < 1$.

Thus, over unit time, each of alive particles produces in the next generation, independently of each other, a random number N of particles with probability $\mathsf{P}\{N = i\} = q_i$, $i = 0, 1, 2, \ldots$ Substituting (10.5.6) into (10.5.4), we obtain the equation

$$p_i(t) = \sum_{j=0}^{\infty} \sum_{\{i_1,\ldots,i_j\}} q_{i_1} \ldots q_{i_j} p_j(t-1),$$

which should be supplemented by the initial condition

$$p_i(0) = Q_i \ge 0, \qquad \sum_{i=1}^{\infty} Q_i = 1. \tag{10.5.8}$$

However, in view of the independent behavior of particles, it is enough to consider the case with a single particle in the initial state

$$Q_i = \delta_{i1},$$

which will be assumed in what follows. Sometimes they call a branching process with one particle at $t = 0$ a cascade process, or simply a cascade.

Multiplying (10.5.8) by u^i and summing over $0 \le i < \infty$, we obtain the equation for the generating function

$$\varphi(u,t) = \sum_{j=0}^{\infty} [\varphi(u,1)]^j p_j(t-1) \equiv \varphi(\varphi(u,1), t-1). \tag{10.5.9}$$

It is not hard to see that it can be represented in the equivalent form

$$\varphi(u,t) = \varphi(\varphi(u,t-1), 1). \tag{10.5.10}$$

It is worthwhile to notice that the equations for generating functions are of more compact form than those for for probabilities (10.5.8), but they are nonlinear, so their analysis is not a simple problem.

We presented equations (10.5.9)–(10.5.10) for one particular characteristic of the branching process, namely for the number of particles in the generation at time t. To consider other characteristics, we show that there is no necessity to use (10.5.9) to derive (10.5.10). One may only use the properties of branching processes, as we found the generating function for the time of gambler's ruin. Indeed, $N(t)$ relates to $N(1)$ as follows:

$$N(t) = \sum_{i=1}^{N(1)} N_i(t-1).$$

We obtain herefrom

$$\varphi(u,t) = \mathsf{E}u^{N(t)} = \mathsf{E}u^{\sum_{i=1}^{N(1)} N_i(t-1)} = \mathsf{E}\left\{\mathsf{E}u^{N_1(t-1)}\dots u^{N_k(t-1)} \mid N(1) = k\right\}$$
$$= \mathsf{E}[\varphi(u,t-1)]^{N(1)} = \varphi(\varphi(u,t-1),1),$$

which coincides with (10.5.10).

By using these equations we are able, in particular, to demonstrate that if

$$\varphi'(1,1) = \sum_{n=1}^{\infty} nq_n \le 1, \tag{10.5.11}$$

in other words, if the mean number of offsprings of a single particle does not exceed one, the process becomes extinct with probability one. In the case of equality in (10.5.11) the process is called critical; in the case of inequality, subcritical, and in the case of an inverse inequality, it is called supercritical.

We consider the number M_0 of particles falling down into the state 0 ('final' particles) during the evolution of a critical cascade. It is clear that this problem is solved more easily, since the generating function is of only one variable u; in this case

$$M_0 = \begin{cases} 1 & \text{with probability } q_0, \\ M_{01} + \ldots + M_{0k} & \text{with probability } q_k. \end{cases}$$

and we obtain

$$\varphi_0(u) = q_0 u + \sum_{n=1}^{\infty} [\varphi_0(u)]^k q_k. \tag{10.5.12}$$

Let all q_i but q_0 and q_2 be equal to zero; then

$$q_0 = q_2 = 1/2,$$

and (10.5.12) reduces to

$$\varphi_0(u) = \tfrac{1}{2}[u + \varphi_0^2(u)]. \tag{10.5.13}$$

Since

$$\mathsf{P}\left\{M_0 = 0\right\} = 0,$$

$\varphi_0(0) = 0$, and equation (10.5.13) possesses the solution

$$\varphi_0(u) = 1 - \sqrt{1-u}. \tag{10.5.14}$$

Expanding the square root in terms of u, we obtain

$$p_n \equiv \mathsf{P}\left\{M_0 = n\right\} = \frac{1}{n!}\varphi_0^{(n)}(0) = \frac{(2n-3)!!}{2^n n!}, \qquad n = 1,2,3,\dots \tag{10.5.15}$$

Before dwelling upon interrelations between the obtained distribution and stable laws, we turn to solving one more problem close to that mentioned above. We consider the progeny M, assuming that the parent particle and its descendants are different.

This model can be thought of as a rough scheme of development of a neutron cascade in a homogeneous multiplying medium; then q_0 is the capture probability, q_1 is the scattering probability and q_2 is the fission probability with production of two secondary neutrons. In this case M_0 is the number of captures in the random cascade and M is the number of all collisions including captures.

The corresponding generating function satisfies the equation

$$\varphi(u) = u[q\varphi(u) + (1-q)(1+\varphi^2(u))/2],$$

whose solution is of the form

$$\varphi(u) = \frac{1 - qu - \sqrt{1 - 2qu - (1-2q)u^2}}{(1-q)u} \tag{10.5.16}$$

under condition

$$\lim_{u \to 0} \varphi(u) = 0.$$

Expanding the square root into a series and using property (10.1.3), we obtain

$$p_m \equiv \mathsf{P}\{M = m\}$$
$$= \frac{(1/(2q)-1)^{m+1}}{1-q} \sum_{k=k_m}^{m+1} \frac{(2k-3)!!}{(2k)!!} \binom{k}{m-k+1} \left(\frac{(2q)^2}{1-2q}\right)^k, \tag{10.5.17}$$

where

$$k_m = \begin{cases} (m+1)/2 & \text{if } m \text{ is odd,} \\ m/2 + 1, & \text{if } m \text{ is even.} \end{cases}$$

It is easy to see that

$$\lim_{n \to \infty} \sum_{m=1}^{n} p_m = 1,$$

i.e., the progeny M of the critical cascade under consideration is finite with probability one.

We put stress on the fact that, as $q \to 0$, probabilities (10.5.15) vanish for even m, whereas for odd $m = 2n - 1$, $n = 1, 2, 3, \dots$, they are of the form

$$p_m = \frac{(2n-3)!!}{2^n n!},$$

which coincides with (10.5.15).

Having obtained explicit expressions for distributions (10.5.15) and (10.5.17), we are able to determine the asymptotic behavior of the probabilities $\mathsf{P}\{M_0 > m\}$ and $\mathsf{P}\{M > m\}$ as $m \to \infty$. However, we can proceed in a simpler way, recalling expressions for generating functions (10.5.14) and (10.5.15) with the account of the fact that

$$\varphi(e^{-\lambda}) = \mathsf{E}e^{-\lambda M} = \sum_{m=1}^{\infty} e^{-\lambda m} p_m$$

is the Laplace transform of a discrete distribution p_m. Then we obtain

$$1 - \varphi_0(e^{-\lambda}) = \sqrt{1 - e^{-\lambda}} \sim \sqrt{\lambda},$$

$$1 - \varphi(e^{-\lambda}) \sim \sqrt{\frac{2\lambda}{1-q}}$$

as $\lambda \to 0$. By virtue of the Tauberian theorems (see Section 5.3),

$$\mathsf{P}\{M_0 > x\} \sim \frac{1}{\sqrt{\pi}} x^{-1/2}$$

in the first case. and

$$\mathsf{P}\{M > x\} \sim \sqrt{\frac{2/\pi}{1-q}} x^{-1/2}$$

in the second one.

Now the connection of these models with stable distributions becomes clear. Indeed, if we consider a population of a large number n of independent cascades, then the progenies

$$M_0 = \sum_{i=1}^{n} M_{0i},$$

and

$$M = \sum_{i=1}^{n} M_i,$$

under a suitable normalization, have the Lévy distribution

$$\mathsf{P}\left\{\frac{4M_0}{n^2} < x\right\} \to G^A(x; 1/2, 1), \tag{10.5.18}$$

$$\mathsf{P}\left\{\frac{2(1-q)M}{n^2} < x\right\} \to G^A(x; 1/2, 1). \tag{10.5.19}$$

It is shown in (Sevastyanov, 1957) that in the case of s different types of particles one-sided stable distributions arise with $\alpha = 1/2^r$, $1 \le r \le s$.

Sevastyanov commented (10.5.18) in (Sevastyanov, 1974) as follows. Let us imagine the branching process as some, say, chemical reaction, in which there are active non-final particles and final particles being the terminal product of the reaction. As far as in our model the particles produce other particles independently of each other, it is natural to expect that the amount of the terminal product of the reaction is proportional to the amount of active particles. This is so indeed in the case where the process is non-critical. If the process is critical, then, as we see from (10.5.18), a somewhat surprising phenomenon arises: the amount of the final product of the reaction grows proportionally to the second power of the amount of initial active particles, but the proportionality coefficient is random and distributed by law (10.5.18).

10.6. The model of point sources: two-dimensional case

In this and the following sections we consider a model originated from (Holtsmark, 1919), used by Chandrasekhar and von Neumann (Chandrasekhar & Neumann, 1941; Chandrasekhar & Neumann, 1943; Chandrasekhar, 1944a; Chandrasekhar, 1944b), and then generalized by Zolotarev (Zolotarev, 1986). We begin with a simple version of this model.

Let random points $\mathbf{X}_i(X_i, Y_i)$ be distributed on the plane (x, y) as the homogeneous Poisson ensemble, i.e., let the following conditions be satisfied.

(1) For any domain U of the plane of a finite area $S(U)$, the random number of points $N = N(U)$ falling into it has the Poisson distribution with mean

$$a = \mathsf{E}N(U) = \rho S(U), \qquad \rho = \text{const}, \tag{10.6.1}$$

i.e., for any non-negative integer n

$$\mathsf{P}\{N = n\} = \frac{a^n}{n!}e^{-a}. \tag{10.6.2}$$

(2) For any pair of non-intersecting domains U_1 and U_2 of the plane, the numbers $N_1 = N(U_1)$ and $N_2 = N(U_2)$ of points falling into them are independent random variables.

The above properties, as a matter of fact, mean that the position of any finite group of particles (in particular, one particle) on the plane does not influence the positions of the remaining particles, and the conditional distribution of a particle, provided that it has fallen in U, is uniform in U, i.e., has a density equal to $[S(U)]^{-1}$ inside U and equal to zero outside of U.

Let, further, $\Theta_1, \Theta_2, \dots$ be independent random variables not depending on $\mathbf{X}_i$ with distribution $F(\theta)$. A pair of random variables (X_i, Θ_i) is called a random point source. Each source produces a field satisfying the principle of superposition: the field created by several sources is equal to the sum of fields created by each of them separately.

We start with a scalar field and choose the function determining the field created at the origin of coordinates by a source with parameters $\mathbf{X} = \mathbf{r}$ and $\Theta = \theta$ as

$$v(\mathbf{r}, \theta) = \theta r^{-\mu}, \quad r = |\mathbf{r}|, \quad \mu > 1. \tag{10.6.3}$$

The field created by all sources appeared in a circle U_R of the radius R with center at the origin is equal to the sum of the random number $N(U_R)$ of independent random summands

$$W = \sum_{i=1}^{N(U_R)} V_i = \sum_{i=1}^{N(U_R)} v(\mathbf{X}_i, \Theta_i). \tag{10.6.4}$$

Before proceeding further, we make several remarks.

The distribution function of a single summand is of the form

$$F_V(v) = \frac{2}{R^2} \int_0^R F(vr^\mu) r\, dr,$$

so the density is

$$p_V(v) = \frac{2}{R^2} \int_0^R p(vr^\mu) r^{\mu+1} dr, \qquad p(\theta) = F'(\theta). \tag{10.6.5}$$

In particular, if $\Theta = c_1 > 0$ is non-random, then

$$p_V(v) = \frac{2c_1^{2/\mu}}{\mu R^2} v^{-2/\mu-1}, \qquad v > c_1/R^\mu. \tag{10.6.6}$$

For $\Theta = -c_2$, $c_2 > 0$, instead of (10.6.5) we obtain

$$p_V(v) = \frac{2c_2^{2/\mu}}{\mu R^2} |v|^{-2/\mu-1}, \qquad v < -c_2/R^\mu. \tag{10.6.7}$$

If the mean number of sources in the circle $\bar{N}$ is large, then the fluctuations in Poisson distribution can be neglected, and one can consider the sum of a non-random number of terms instead of (10.6.4). Then it is clear that, after appropriate centering and normalizing the sum (10.6.4), we arrive at the extreme stable laws $q(x; 2/\mu, 1)$ in the case (10.6.6) and $q(x; 2/\mu, -1)$ in the case (10.6.7), as $\bar{N} \to \infty$ (i.e., as $\rho \to \infty$ and R is fixed). It is also obvious that if

$$\Theta = \begin{cases} c_1 & \text{with probability } 1/2, \\ -c_2 & \text{with probability } 1/2, \end{cases}$$

then we obtain $q(x;\alpha,\beta)$, where

$$\beta = \frac{c_1^{2/\mu} - c_2^{2/\mu}}{c_1^{2/\mu} + c_2^{2/\mu}},$$

and the characteristic index remains the same: $\alpha = 2/\mu$. It is determined by the exponent μ of the denominator of the source function (10.6.3) and can vary only in the case where the distribution of the random variable Θ has tails of power type.

Let us choose the distribution $p(\theta)$ in the Zipf–Pareto form (3.3.10):

$$p(\theta) = \begin{cases} \nu c\theta^{-\nu-1}, & \theta > \varepsilon, \\ 0, & -\varepsilon < \theta < \varepsilon, \\ \nu d|\theta|^{-\nu-1}, & \theta < -\varepsilon. \end{cases} \qquad (10.6.8)$$

Substituting this into (10.6.5), we obtain

$$p_V(v) = \begin{cases} \dfrac{2\nu c}{\mu\nu-2}\left\{R^{-2}\varepsilon^{2/\mu-\nu}v^{-2/\mu-1} - R^{-\mu\nu}v^{-\nu-1}\right\}, & v > \varepsilon R^{-\mu}, \\ \dfrac{2\nu d}{\mu\nu-2}\left\{R^{-2}\varepsilon^{2/\mu-\nu}|v|^{-2/\mu-1} - R^{-\mu\nu}|v|^{-\nu-1}\right\}, & v < -\varepsilon R^{-\mu}. \end{cases}$$

If $\nu > 2/\mu$, then

$$p_V(v) \sim \begin{cases} \dfrac{2\nu cR^{-2}\varepsilon^{2/\mu-\nu}}{\mu\nu-2}v^{-2/\mu-1}, & v \to \infty, \\ \dfrac{2\nu dR^{-2}\varepsilon^{2/\mu-\nu}}{\mu\nu-2}|v|^{-2/\mu-1}, & v \to -\infty, \end{cases} \qquad (10.6.9)$$

and for W we obtain the stable distribution with $\alpha = 2/\mu$ again. But if $\nu < 2/\mu$, then in the asymptotics of large v in (10.6.8) other terms are leading as $|v| \to \infty$:

$$p_V(v) \sim \begin{cases} \dfrac{2\nu cR^{-\mu\nu}}{2-\mu\nu}v^{-\nu-1}, & v \to \infty, \\ \dfrac{2\nu dR^{\mu\nu}}{2-\mu\nu}|v|^{-\mu-1}, & v \to -\infty, \end{cases} \qquad (10.6.10)$$

and we arrive at the stable law with $\alpha = \nu$ determined by (10.6.8).

The scheme considered here is nothing but summation of random variables considered in Section 3.5.

The model of sources differs from this scheme in that the limit is considered as $R \to \infty$ and $\rho = \text{const}$, instead of the limit as $R = \text{const}$ and $\rho \to \infty$, which the above notes concern.

Let us turn back to random variable (10.6.4). Its characteristic function is given by (3.5.5):

$$f_W(k;R) = \exp\{\pi\rho R^2[f_V(k;R) - 1]\} \qquad (10.6.11)$$

where

$$f_V(k;R) = \frac{2}{R^2}\int_{-\infty}^{\infty} d\theta\, p(\theta) \int_0^R dr\, r\, e^{ikv(r,\theta)} \qquad (10.6.12)$$

is the characteristic function of a single term V. Substituting it into (10.6.11), we obtain

$$\ln f_W(k;R) = 2\pi\rho \int_{-\infty}^{\infty} d\theta\, p(\theta) \int_0^R dr\, r\,[e^{ikv(r,\theta)} - 1]. \qquad (10.6.13)$$

Splitting the outer integral into two integrals over the positive and negative semi-axes, respectively, and passing to the new integration variable v, with the use of relations (10.6.3) we obtain

$$\ln f_W(k;R) = \frac{2\pi\rho}{\mu}\left\{\int_0^{\infty} d\theta p(\theta)\theta^{2/\mu} I^{(-2/\mu)}(k;\theta R^{-\mu}) + \int_0^{\infty} d\theta p(-\theta)\theta^{2/\mu} I^{(-2/\mu)}(-k;\theta R^{-\mu})\right\}, \qquad (10.6.14)$$

where

$$I^{(-\alpha)}(k;\varepsilon) = \int_{\varepsilon}^{\infty} [e^{ikv} - 1] v^{-\alpha-1} dv.$$

Using the obvious connection with the integrals $I_s^{(-\alpha)}$ and $I_c^{(-\alpha)}$ introduced in Section 3.3, it is possible to make sure that the integral $I^{(-\alpha)}(k,\varepsilon)$ converges to

$$\lim_{\varepsilon\to 0} I^{(-\alpha)}(k;\varepsilon) \equiv I^{(-\alpha)}(k;0) = \Gamma(-\alpha)|k|^{\alpha} e^{i\alpha\pi/2\,\mathrm{sign}\,k},$$

as $\varepsilon \to 0$, provided that $\alpha \equiv 2/\mu < 1$. If, moreover, there exists the absolute moment

$$\int_{-\infty}^{\infty} |\theta|^{2/\mu} p(\theta)\, d\theta = \langle|\theta|^{2/\mu}\rangle,$$

then the limit of function (10.6.14) as $R \to \infty$ exists, and is of the form of the one-dimensional stable characteristic function

$$\ln f_W(k;\infty) = -2\pi p \Gamma(1-\alpha)\cos(\alpha\pi/2)|k|^{\alpha}[1 - i\beta\tan(\alpha\pi/2)\,\mathrm{sign}\,k], \qquad (10.6.15)$$

with parameters

$$\alpha = 2/\mu < 1, \qquad \beta = \frac{\int_0^{\infty}\theta^{2/\mu}p(\theta)d\theta - \int_0^{\infty}\theta^{2/\mu}p(-\theta)d\theta}{\int_{-\infty}^{\infty}|\theta|^{2/\mu}p(\theta)d\theta}.$$

To see what happens in the case where $\langle|\theta|^{2/\mu}\rangle = \infty$, we turn back to (10.6.8) again. Inserting it into (10.6.14) and changing the order of integration, we obtain

$$\ln f_W(k;R) = \frac{2\pi\rho\nu}{2-\mu\nu}\{cR^{2-\mu\nu}I^{(-\nu)}(k;\varepsilon R^{-\mu}) - c\varepsilon^{2/\mu-\nu}I^{(-2/\mu)}(k;\varepsilon R^{-\mu}) + dR^{2-\mu\nu}I^{(-\nu)}(-k;\varepsilon R^{-\mu}) - d\varepsilon^{2/\mu-\nu}I^{(-2/\mu)}(-k;\varepsilon R^{-\mu})\}.$$

If $\nu > 2/\mu$ and $R \to \infty$, the main contribution is due to the terms containing the integrals $I^{(-2/\mu)}$, and we obtain the well-known result (10.6.15). The moment $\langle|\theta|^{2/\mu}\rangle$ should exists in this case. However, if $\nu < 2/\mu$, the other integrals are in the lead:

$$\ln f_W(k;R) \sim \frac{2\pi\rho\nu R^{2-\mu\nu}}{2-\mu\nu}\left\{cI^{(-\nu)}(k;\varepsilon R^{-\mu}) + dI^{(-\nu)}(-k;\varepsilon R^{-\mu})\right\}, \qquad R \to \infty.$$

The content of the brackets can be transformed in the same way as we derived (10.6.15), but the presence of the factor $R^{2-\mu\nu}$ at $|k|^\nu$ results in the need for re-normalization:

$$W_R' = W_R R^{2/\nu-\mu}. \tag{10.6.16}$$

In the limit as $R \to \infty$, the random variable obeys the stable law with $\alpha = \nu$. Thus, in the scheme under consideration the index α is the least of ν and $2/\mu$. This conclusion is true also in the case where $\min\{\nu, 2/\mu\}$ exceeds one. Then the mathematical expectation $\langle W_R\rangle = \mathsf{E}W_R$ exists and should be used for centering the random variable W_R; the rest of calculations practically remains the same and leads to a stable distribution again.

Let us now pass from a scalar field to a vector one created by the same ensemble of sources with the source function

$$\mathbf{v}(\mathbf{r},\theta) = \theta\mathbf{r}r^{-\mu-1}, \qquad \mu > 1,$$

where both v and $\mathbf{r}$ are two-dimensional vectors. The analogue of (10.6.13) in this case is

$$\ln f_W(\mathbf{k};R) = 2\pi\rho\int_{-\infty}^{\infty} d\theta\, p(\theta)\int_0^R dr\, r[J_0(k\theta r^{-\mu}) - 1].$$

The Bessel function is even; therefore, instead of (10.6.14), we obtain

$$\ln f_W(\mathbf{k};R) = -\frac{2\pi\rho k^{2/\mu}}{\mu}\int_0^{\infty} d\theta\,\hat p(\theta)\theta^{2/\mu}\int_{k\theta R^{-\mu}}^{\infty}[1 - J_0(v)]v^{-2/\mu-1}dv,$$

where

$$\hat p(\theta) = p(\theta) + p(-\theta), \qquad \theta > 0. \tag{10.6.17}$$

Because

$$1 - J_0(v)) \sim v^2/4, \qquad v \to 0,$$

the inner integral converges for all $\mu > 1$, and we obtain (in the case where $\langle |\theta|^{2/\mu} \rangle < \infty$)

$$\ln f_W(k; \infty) = -Ck^{2/\mu}, \tag{10.6.18}$$

where

$$C = \frac{2\pi\rho}{\mu} \langle |\theta|^{2/\mu} \rangle \int_0^\infty [1 - J_0(v)] v^{-2/\mu - 1} dv.$$

Thus, for any distribution $p(\theta)$ with a finite absolute moment of order $2/\mu$ we obtain the axially symmetric two-dimensional stable distribution with characteristic function (10.6.18). Of course, it follows from the vector nature of the field and from the homogeneity of the Poisson ensemble: the source with parameter θ located at the point $\mathbf{r}$ creates the same field at the origin as the source with parameter $-\theta$ located at the point $-\mathbf{r}$. There will be no changes if we replace negative values of parameters θ by positive ones, which is exactly the sense of (10.6.17).

It is not hard to see that, as well as in the scalar case, the use of the power law of distribution of parameter (10.6.8) with $\nu < 2/\mu$ results in the corresponding change of the characteristic exponent of the stable law, but it still remains symmetric. Due to the symmetry, the necessity of centering of $\mathbf{W}_R$ disappears in the case $\alpha > 1$, whereas for $\alpha = \nu$, as above, the re-normalization of $\mathbf{W}_R$ by formula (10.6.16) is required.

The extension to the three-dimensional case seems to be rather evident, and we proceed to expanding the general model of sources.

10.7. Point sources: multidimensional case

Now we consider some domain U of finite volume in the n-dimensional Euclidean space $\mathbb{R}^n$; in particular problems, its part can be played by a domain of physical space, space-time, phase space, etc. There is a countable set of random point sources $\{X_i\}$ in this space, and each of the sources generates a m-dimensional vector field

$$V_i(x^0) = v(x^0; X_i, \Theta_i), \qquad v \in \mathbb{R}^m \tag{10.7.1}$$

at a point $x^0 \in \mathbb{R}^n$. Here $v(x^0; x, \theta)$ is a deterministic (non-random) function of $x^0, x \in \mathbb{R}^n$, $\theta \in T$, and X_i and Θ_i are random coordinates and parameter of the ith source respectively.

We pose the following four assumptions concerning this system.

(a) The set of random points $\{X_i\}$ constitutes a homogeneous Poisson ensemble.

(b) For any domain $U \subset \mathbb{R}^n$ of finite volume u, the number of sources lying in U, the positions $X_1, X_2, \dots$ of these sources, and the values $\Theta_1, \Theta_2, \dots$ of the parameter θ characterizing them are independent random variables.

(c) The random variables $\Theta_1, \Theta_2, \dots$ are distributed by one and the same law $P(d\theta)$.

(d) For each domain U of finite volume, the random field generated at x^0 by the sources lying in U is the sum of random terms

$$W(x^0, U) = \sum_{i=1}^{\infty} v(x^0; X_i, \Theta_i)\mathbf{1}(X_i; U) \tag{10.7.2}$$

where

$$\mathbf{1}(x; U) = \begin{cases} 1, & x \in U, \\ 0, & \text{otherwise}, \end{cases}$$

is the indicator function of the set U; the function $v(x^0; x, \theta)$ takes values lying in some subset C of the m-dimensional Euclidean space $\mathbb{R}^m$.

The field generated by the whole system of sources is treated as the weak limit of the field

$$W_R(x^0) \equiv W(x^0; U_R),$$

where U_R is the intersection of the domain U with the sphere of radius R centered at x^0. To determine conditions for the existence of this limit and to find its analytic description, we consider the question of the limit value, as $R \to \infty$, of the characteristic function $f_R(k)$ of the variable $W_R(x^0)$:

$$f_R(k) = \sum_{n=0}^{\infty} \exp(-\rho u_R)\frac{(\rho u_R)^n}{n!}\varphi_R^n(k) = \exp\left\{\rho u_R[\varphi_R(k) - 1]\right\}. \tag{10.7.3}$$

Here ρ is the mean density of the concentration of sources, u_R is the volume of the domain U_R, and $\varphi_R(k)$ is the characteristic function of a single random term (10.7.1).

We rewrite the argument of the exponential of the last equality as

$$\ln f_R(k) = \rho u_R[\varphi_R(k) - 1] = \rho \int_{U_R \times T} [\exp\{i(k, v)\} - 1]dx\, P(d\theta). \tag{10.7.4}$$

It can be transformed further by taking the function $v = v(x^0; x, \theta)$ as a new integration variable and introducing the measure

$$\mu(B) = \rho \int_{B_*} dx\, P(d\theta) \tag{10.7.5}$$

on $\mathbb{R}^m$, where B is a Borel subset of $C \subseteq \mathbb{R}^m$ and $B_* = \{(x, \Theta) : v(x^0; x, \theta) \in B\}$. The use of this measure, obviously, presupposes the measurability of the function v with respect to the pair (x, θ) for each $x_0 \in U$.

From (10.7.3)–(10.7.5) we obtain

$$\ln f_R(k) = \int_{C_R} [\exp\{i(k, v)\} - 1]\mu(dv), \qquad (10.7.6)$$

where

$$C_R = \{(x, z) \colon v(x^0; x, \theta) \in U_R \times T\}.$$

Thus, the function $f_R(k)$ possesses the canonical Lévy representation of characteristic functions of infinitely divisible distributions on $\mathbb{R}^m$. The question of the convergence of the integral in (10.7.6) does not arise, because $\mu(C_R) = \rho u_R$, where u_R is the volume of U_R. Since the set C_R, obviously, does not shrink as R grows, the question of existence of the limit of $f_R(k)$ as $R \to \infty$ is reduced to the question of existence of the limit of the integral

$$J_R = \int_{C_R} [\exp\{i(k, v)\} - 1]\mu(dv). \qquad (10.7.7)$$

Several essentially differing situations can occur. Let

$$L_R = \int_{C_R} v\, \mathbf{1}(v; |v| < 1)\mu(dv).$$

Obviously, this integral (vector) exists for all $R \to \infty$. We rewrite integral (10.7.7) as

$$J_R = i(k, L_R) + \int_{C_R} \left[e^{i(k,v)} - 1 - i(k, v)\mathbf{1}(v; |v| < 1)\right] \mu(dv).$$

In the second term, the integration over C_R can be replaced by integration over C if we simultaneously replace the measure μ by the measure μ_R coinciding with μ on C_R and equal to zero outside this set. Furthermore, $\mu_R \to \mu$ as $R \to \infty$, because C_R does not shrink as R increases.

Since $\ln f_R(k) = J_R$, $f_R(k)$ converges to some characteristic function $f(k)$ as $R \to \infty$ if and only if (this follows from the general theory of infinitely divisible laws in $\mathbb{R}^m$)

(1) $L_R \to L$ as $R \to \infty$, where L is a vector in $\mathbb{R}^m$;

(2) $D = \int_C \min\{1, |v|^2\}\mu(dv) < \infty$.

Under these conditions, the characteristic function $f(k)$ of the limit distribution is of the form

$$\ln f(k) = i(k, L) + \int_C \left[e^{i(k,v)} - 1 - i(k, v)\mathbf{1}(v; |v| < 1)\right] \mu(dv). \qquad (10.7.8)$$

We indicate two cases where (10.7.8) takes a simpler form.

1. If

$$D_1 = \int_C \min\{1, |v|\}\mu(dv) < \infty,$$

then conditions 1 and 2 are satisfied, and

$$\ln f(k) = \int_C [\exp(i(k,v)) - 1]\mu(dv). \tag{10.7.9}$$

2. If condition 1 holds along with the condition

$$D_2 = \int_C \min\{1, |v|\}|v|\mu(dv) < \infty,$$

which immediately implies validity of condition 2, then

$$\ln f(k) = i(k,M) + \int_C \left[e^{i(k,v)} - 1 - i(k,v)\right]\mu(dv), \tag{10.7.10}$$

where

$$M = L + \int_{|v|\geq 1} v\mu(dv). \tag{10.7.11}$$

In the case under consideration, the expression for the limit value L is not necessarily of integral form required to interpret M as the mean value of the limit distribution. If, nevertheless, such a representation of L exists, then it is easy to transform (10.7.10).

What actually happens is that the violation of condition 2 breaks the existence of the limit distribution of $W_R(x^0)$, even after centering. Therefore, we consider the situation where condition 2 holds but condition 1 does not. In this case, it makes sense to study not the random variable W_R itself but its shift $\tilde{W}_R = W_R - L_R$. The corresponding weak limit $\tilde{W}$ of this variable as $R \to \infty$ exists if and only if condition 2 holds, and the characteristic function $\tilde{f}$ of the random variable $\tilde{W}$ under condition 2 is of the form

$$\ln \tilde{f}(k) = \int_C \left[e^{i(k,v)} - 1 - i(k,v)\mathbf{1}(v; |v| < 1)\right]\mu(dv). \tag{10.7.12}$$

If the condition $D_2 < \infty$ also holds, then W_R can be shifted by the mathematical expectation $M_R = \mathsf{E}W_R$, which is finite in our situation. With this shift, the random variable $\hat{W}_R = W_R - M_R$ converges weakly to the random variable $\hat{W}$ with characteristic function $\hat{f}$ of the form

$$\ln \hat{f}(k) = \int \left[e^{i(k,v)} - 1 - i(k,v)\right]\mu(dv). \tag{10.7.13}$$

Various shifts of W_R are carried out in the cases where W_R itself has no weak limit, i.e., in the cases where the field generated by the whole system of sources merely does not exist. Nevertheless, a practical meaning can be assigned to the above construction of limit approximations of the distributions of the random variables $\tilde{W}_R$ and W_R. The point is that the appearance of a domain U of infinite volume in the model is nothing but a mathematical idealization. In actual practice, the domain U has a large but finite volume. In this case we study random fluctuations of the field generated by the particles in U with a certain constant component singled out in it. After separating the constant component, we can simplify the computations by an appropriate expansion of U, provided, of course, that this does not introduce essential distortions into our calculations.

The limit distributions (10.7.8)–(10.7.10) and (10.7.12)–(10.7.13) considered above are infinitely divisible. It is natural to expect that in certain cases we come against m-dimensional stable laws; this is completely determined by the form of the measure defined by (10.7.5), which, in turn, depends on the properties of the source functions $v(x^0, x, \theta)$ and, to a lesser degree, on the properties of the distribution $P(d\theta)$.

10.8. A class of sources generating stable distributions

In this section we consider a class of source functions generating stable distributions. It is connected with transformations of formula (10.7.9) accompanied by the condition $D_1 < \infty$, and of formulae (10.7.10) and (10.7.13) obtained under the condition $D_2 < \infty$.

A semi-cone (with vertex at the origin) in the subspace $\mathbb{R}^{n_1}$ of $\mathbb{R}^n$, $0 < n_1 \le n$, is defined to be a set U_1 such that if $u_1 \in U_1$ and $c > 0$, then $cu_1 \in U_1$ and $-cu_1 \notin U_1$. With this definition in mind, the following assumptions are made about the set U and the points x_0 at which the distributions of the random variables $W(x^0)$, $\tilde{W}(x^0)$, and $\hat{W}(x^0)$ are to be computed.

(i) The set $U_{x^0} = \{y : y = x - x^0, x \in U\}$ is either a semi-cone in $\mathbb{R}^n$ itself or a direct product $U_1 \times U_2$ of semi-cones U_1 and U_2 in orthogonal subspaces $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, $n_1 + n_2 = n$.

The first of these two cases is obviously reduced to the second if the dimension n_2 is allowed to take the value $n_2 = 0$. This convention will be followed below.

The decomposition of $\mathbb{R}^n$ into orthogonal subspaces corresponds to the notation $x = (x_1, x_2)$ for vectors $x \in \mathbb{R}^n$ where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$.

(ii) The source function $v(x^0; x, \theta)$ is zero for all $x \notin U$, is continuous in x at all interior points of U, and is of the following structure there:

$$v(x^0; x, \theta) = |x_1 - x_1^0|^{-p} D(|x_2 - x_2^0|\,|x_1 - x_1^0|^{-q}, \cdot), \tag{10.8.1}$$

where the symbol '$\cdot$' indicates the dependence of D on x^0, θ, and $s_j = (x_j - x_j^0)/|x_j - x_j^0|$, $j = 1, 2$, or on some of these variables.

The condition in (i) that U_j are semi-cones could be extended by allowing U_j to be either a semi-cone or a cone with vertex at the origin. However, this extension turns out to be connected with the only possible point x_0 if U_j is not the whole subspace $\mathbb{R}^{n_j}$. Anyway, the latter case is covered by assumption (i) if $\mathbb{R}^{n_j}$ is regarded as a limit case of an expanded semi-cone; therefore, the indicated extension involves only a very special situation.

Further, we note that in the case where $U_j = \mathbb{R}^{n_j}$, the source function v must be invariant under passage to the variable x_j, i.e., it must depend only on the difference $x_j - x_j^0$. In what follows, $p > 0$ and $q \geq 0$ will be related to the dimensions n_1 and n_2 of the subspaces by means of additional conditions.

We consider (10.7.9) with the condition $D_1 < \infty$, after returning to the variables x and θ:

$$\ln f(k) = \rho \int_{U\times T} \left(e^{i(k,v)} - 1\right) dx P(d\theta), \tag{10.8.2}$$
$$D_1 = \rho \int_{U\times T} \min(1, |v|) dx P(d\theta) < \infty.$$

In the integral (10.8.2), we pass to the polar system of coordinates in the subspaces $\mathbb{R}^n$ by setting ($j = 1, 2$)

$$r_j = |x_j - x_j^0|, \qquad x_j - x_j^0 = r_j s_j,$$

where s_j is a point on the surface of the unit sphere S_j centered at the origin. We obtain

$$\omega_1 = \int_U \left(e^{i(k,v)} - 1\right) dx = \int_{S_1} \int_{S_2} \omega_2 \, ds_2 \, ds_1,$$

where

$$\omega_2 = \int_0^\infty \int_0^\infty [\exp\{i(k, r_1^{-p} D(r_2/r_1^q, \cdot))\} - 1] r_2^{n_2-1} r_1^{n_1-1} dr_2 \, dr_1.$$

We replace r_2 by $r_2 r_1^q$, and, after changing the integration order, obtain

$$\omega_2 = \int_0^\infty r_2^{n_2-1} dr_2 \int_0^\infty [\exp(i\xi r_1^{-p}) - 1] r_1^{n_1+n_2q-1} dr_1,$$

where $\xi = (k, D(r_2, \cdot))$.

Substituting r_1 for r_1^{-p} yields

$$\omega_2 = \frac{1}{p} \int_0^\infty r_2^{n_2-1} dr_2 \int_0^\infty (e^{i\xi r_1} - 1) r_1^{-(n_1+qn_2)/p-1} dr_1.$$

If

$$\alpha = (n_1 + qn_2)/p < 1, \tag{10.8.3}$$

then the inner integral in the expression for ω_2 converges, and

$$\int_0^\infty \left(e^{i\xi r_1} - 1\right) r_1^{-\alpha-1} dr_1 = \Gamma(-\alpha)|\xi|^\alpha \exp\left\{-i(\alpha\pi/2)\operatorname{sign}\xi\right\},$$

which implies

$$\omega_2 = \frac{\Gamma(-\alpha)}{p}\int_0^\infty |(k,s)|^\alpha \exp\left\{-i(\alpha\pi/2)\ \operatorname{sign}(k,s)\right\} |D|^\alpha r_2^{n_2-1} dr_2,$$

where $s = D/|D|$.

Let us introduce the measure χ on the unit sphere S in $\mathbb{R}^n$ by setting

$$\chi(B) = \frac{\rho\Gamma(1-\alpha)}{\alpha p}\int_{s\in B} |D(r_2,\cdot)|^\alpha r_2^{n_2-1}\, dr_2\, ds_1\, ds_2\, P(d\theta) \tag{10.8.4}$$

for any Borel subset B of S (here the integration is also over all r_2 and z).

Considering the expressions obtained for ω_1 and ω_2, we can rewrite the right-hand side of (10.8.2) as

$$\ln f(k) = -\int_S |(k,s)|^\alpha \exp\left\{-i(\alpha\pi/2)\operatorname{sign}(k,s)\right\}\chi(ds), \tag{10.8.5}$$

which corresponds to the canonical form of the characteristic function of an n-dimensional stable law with parameter $0 < \alpha < 1$. The finiteness of the complete measure $\chi(S)$ is an additional condition in the description of n-dimensional stable laws. It is not hard to verify on the basis of (10.8.4) that the condition $\chi(S) < \infty$ is equivalent to the condition $D_1 < \infty$.

The second case concerns transformation of the integral

$$\omega = \int_C \left(e^{i(k,v)} - 1 - i(k,v)\right)\mu(dv)$$

in (10.7.10) and (10.7.13) under the condition $D_2 < \infty$. Repeating above reasoning, we obtain

$$\begin{aligned}\omega &= \rho\int_T P(d\theta)\int_U \left(e^{i(k,v)} - 1 - i(k,v)\right) dx \\ &= \rho\int_T P(d\theta)\int_{S_1}\int_{S_2} \omega_2\, ds_2\, ds_1,\end{aligned}$$

where

$$\omega_2 = \int_0^\infty\int_0^\infty [\exp\{i(k, r_1^{-p}D)\} - 1 - i(k, r_1^{-p}D)] r_2^{n_2-1} r_1^{n_1-1} dr_2\, dr_1.$$

After the substitution of $r_2 r_1^q$ for r_2, the quantity D becomes independent of r_1, and after the substitution of r_1 for r_1^{-p}, the integral ω_2 is transformed to

$$\omega_2 = \frac{1}{p}\int_0^\infty r_2^{n_2-1} dr_2 \int_0^\infty \left(e^{i\xi r_1} - 1 - i\xi r_1\right) r_1^{-\alpha-1} dr_1,$$

where $\xi = (k, D(r_2, \cdot))$ and $\alpha = (n_1 + qn_2)/p$. If $1 < \alpha < 2$, then the inner integral converges; it turns out to be equal to

$$\Gamma(-\alpha)|\xi|^\alpha \exp\{-i(\alpha\pi/2)\ \operatorname{sign}\xi\}.$$

Consequently, introducing the measure χ just as in (10.8.4) but with the opposite sign, we arrive at the following expression for $\omega = \ln f(k) - i(k, M)$ and $\omega = \ln \hat{f}(k)$:

$$\omega = \int_S |(k,s)|^\alpha \exp\{-i(\alpha\pi/2)\operatorname{sign}(k,s)\}\chi(ds). \tag{10.8.6}$$

The finiteness of the complete measure $\chi(S)$ is equivalent to the condition $D_2 < \infty$.

The cases above are among simplest. The stable laws are obtained also on the basis of formulae (10.8.6) and (10.7.12), since in the final analysis the question whether or not the limit distribution is stable is answered only by the form of the spectral measure $\mu(dv)$, which is directly related to the form of the function $v(x^0; x, \theta)$. The laws with $\alpha = (n_1 + qn_2)/p = 1$ also appear to be among these somewhat complicated cases.

REMARK 10.8.1. We put stress on the fact that the field generated by a set of particles is not necessarily homogeneous and isotropic. Its nature depends on the structural characteristics of the source function. Therefore, in the general situation the stable distributions computed for the value of the field $W(x^0)$ at a point x^0 or the value of the previously centered field $\hat{W}(x^0)$ depend on x^0, even though the main parameter α of those distributions does not vary as x^0 changes.

The distributions $W(x^0)$ and $\hat{W}(x^0)$ do not depend on x^0 in the case where the influence function v possesses the property of homogeneity within the set U:

$$v(x^0, x, \theta) = v(0, x - x^0, \theta). \tag{8.7}$$

This should be understood in the sense that $v(x^0, x + x^0, \theta)$ does not depend on x^0 within the bounds of the semi-cone $U_1 \times U_2$, which also does not depend on x^0. In this situation, any interior points of the original region U can be chosen to play the role of the points $x^0 \in \mathbb{R}^n$ satisfying conditions (i) and (ii).

REMARK 10.8.2. Assumption (10.8.1) concerning the structure of the source function can be generalized as follows.

Let $\mathbb{R}^n = \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_k}$, where $0 < n_1 \le n$ and $n_j \ge 0, j = 2, \ldots, k$. In this case, each vector $x \in \mathbb{R}^n$ can be written in the form $x = (x_1, \ldots, x_k)$, $x_j \in \mathbb{R}^{n_j}$.

Assume further that $U_{x^0} = U_1 \times \ldots \times U_k$, where U_j are semi-cones in $\mathbb{R}^{n_j}$, and that the source function v is equal to zero outside U and inside U can be represented in the form

$$v(x^0; x, \theta) = |x_1 - x_1^0|^{-p} D(|x_2 - x_2^0||x_1 - x_1^0|^{-q_2}, \ldots, |x_k - x_k^0||x_1 - x_1^0|^{-q_k}, \cdot). \tag{10.8.7}$$

Here $p > 0$, $q_j \ge 0$, and the symbol '$\cdot$' indicates (as in (10.8.1)) the dependence on the collection of variables x^0, θ, and $s_j = (x_j - x_j^0)/|x_j - x_j^0|$, $j = 1, \ldots, k$ (or some of them).

Just as in the case $k = 2$ considered in (10.8.1), the distributions of the random variables $W(x^0)$, $\tilde{W}(x^0)$, and $\hat{W}(x^0)$ in case (10.8.7) turn out to be stable with parameter $\alpha = (n_1 + q_2 n_2 + \ldots + q_k n_k)/p$. The reasoning is somewhat more complicated than in the case of (10.8.1), but is basically the same.

REMARK 10.8.3. Assume that the function D in (10.8.1) is of the following structure:

$$D(|x_2 - x_2^0|/|x_1 - x_1^0|^q, \cdot) = \varphi(|x_2 - x_2^0|/|x_1 - x_1^0|^q, x^0, \theta)\psi(s_1, s_2, x^0), \tag{10.8.8}$$

where φ is a real-valued function, $s_j = (x_j - y_j)/|x_j - y_j|$, and

$$\psi(-s_1, s_2, x_0) = -\psi(s_1, s_2, x_0) \quad \text{or} \quad \psi(s_1, -s_2, x_0) = -\psi(s_1, s_2, x_0).$$

Under appropriate conditions (i.e., the condition $D_1 < \infty$ for W and the condition $D_2 < \infty$ for $\hat{W}$), the variables $W(x^0)$ and $\hat{W}(x^0)$ in this case have spherically symmetric distributions with characteristic functions of the form

$$\exp(-\lambda |k|^\alpha), \qquad 0 < \alpha < 2,$$

where $\alpha = (n_1 + q n_2)/p$, and λ is a constant depending on v and $P(d\theta)$.

REMARK 10.8.4. The question of the source functions which lead to stable distributions in the model of point sources is apparently not exhausted by functions of types (10.8.1) and (10.8.7). It is hence of interest to look for other types of functions v which generate stable laws. Another interesting direction for investigation has to do with the desire to weaken the constraints (a)–(c) in the general model. The following special case shows that the search in this direction is promising indeed.

Let

$$v(x^0; x, \theta) = |x_1 - x_1^0|^{-p} D(x_2 - x_2^0, \cdot), \qquad p > 0, \tag{10.8.9}$$

where the symbol '$\cdot$' denotes the dependence on s_1, s_2, x^0, and θ, and let conditions (b) and (c) be replaced by the following conditions.

(b*) For any domain U_1 of finite volume, the number of particles lying in U_1 is independent of both the positions $X_1, X_2, \dots$ of these particles and the values $\Theta_1, \Theta_2, \dots$ of the parameter θ characterizing them.

(c*) The pairs (X_1, Θ_1), (X_2, Θ_2), … of random variables are independent and identically distributed.

In the setting of the new conditions, we assume that the joint distribution is of the structure

$$P(dx, d\theta) = dx_1 Q(dx_2, d\theta). \tag{10.8.10}$$

It turns out that in the case where conditions (10.8.9) and (10.8.10) are in effect, the distributions of $W(x^0)$ and $\hat{W}(x^0)$ are stable with parameter $\alpha = n_1/p$. The functions $\ln f(k)$ in (10.8.2) and $\ln \hat{f}(k)$ in (10.7.13) are evaluated by the same scheme as above. The parameter $\alpha = n_1/p$ varies in the interval $(0, 1)$ if the condition $D_1 < \infty$ is used, and in $(1, 2)$ for $D_2 < \infty$.

If, by analogy with (10.8.8),

$$v(x^0; x, \theta) = |x_1 - x_1^0|^{-p} \varphi(|x_2 - x_2^0|, x^0, \theta) \psi(s_1, s_2, x_0), \tag{10.8.11}$$

where φ is real-valued and ψ is chosen so that

$$\psi(-s_1, s_2, x_0) = -\psi(s_1, s_2, x_0) \quad \text{or} \quad \psi(s_1, -s_2, x_0) = -\psi(s_1, s_2, x_0),$$
$$Q(dx_1, d\theta) = Q_1(dr_2, d\theta) ds_2,$$

then the distribution generated by v is spherically symmetric and stable, with parameter varying between 0 and 2.

11

Correlated systems and fractals

11.1. Random point distributions and generating functionals

At the close of the previous chapter, we discussed stochastic properties of fields generated by point sources randomly distributed in space according to the Poisson ensemble model. There exist a large number of problems where the assumption that the positions of sources are independent seems too strong and even unacceptable. Certainly, different models of correlated points in space can be constructed. We consider here one of them, namely the Markov ensemble model. But, before proceeding we concentrate our attention on some basic concepts of the theory of random point distribution following (Harris, 1963).

A finite set of different points $x_1, x_2, \ldots, x_k$ belonging to a Borel subset in $\mathbb{R}^n$ (all subsets or domains used hereafter are Borel) is called point distribution. Each point distribution generates an integer-valued function of domain A

$$N(A; x_1, \ldots, x_k) \equiv N(A) = \sum_{i=1}^{k} \mathbf{1}(x_i; A). \qquad (11.1.1)$$

There exists a one-to-one correspondence between point distributions and functions $N(A)$. Therefore, we let the functions $N(A)$ denote point distributions.

Let us consider now a statistical ensemble of point distributions. It can be specified by setting the set of functions $P(A_1, \ldots, A_k; n_1, \ldots, n_k)$ defined for any integer k, any set of integers $n_1, \ldots, n_k$, and any set of domains $A_1, \ldots, A_k$, so that

$$P(A_1, \ldots, A_k; n_1, \ldots, n_k) = \mathsf{P}\{N(A_1) = n_1, \ldots, N(A_k) = n_k\}.$$

The non-negative functions should satisfy some additional conditions (Harris, 1963; Sevastyanov, 1974).

This way of specifying a random point distribution is not very suitable for computations, and is not unique. A complete description of all stochastic properties of a random point distribution can be obtained by means of a generating functional (GF):

$$\Phi(u(\cdot)) = \mathsf{E}\exp\left\{\sum_i \ln u(X_i)\right\} = \mathsf{E}\exp\left\{\int N(dx)\ln u(x)\right\}, \qquad (11.1.2)$$

where the random function $N(dx)$ is an integer-valued random measure and $u(x)$ is some measurable function, $|u(x)| \le 1$. A detailed description of this approach can be found in (Sevastyanov, 1974). We will give below only some of the properties of GFs which are needed for understanding the following presentation.

(1) A GF $\Phi(u(\cdot))$ exists for all measurable $u(x)$ with $|u(x)| \le 1$. Its values at simple functions

$$0 \le u(x) = \sum_{m=1}^{M} c_m \mathbf{1}(x; A_m) \le 1, \quad A_i \cap A_j = \varnothing, \quad i \ne j,$$

uniquely determine a probability distribution of the random measure $N(\cdot)$.

(2) If $|u(x)| \le 1$, then $|\Phi(u(\cdot))| \le 1$; if $0 \le u(x) \le 1$, then $0 \le \Phi(u(\cdot)) \le 1$. Moreover,

$$\lim_{\theta\downarrow 0}\Phi(\theta^{1(\cdot;A)}) = \mathsf{P}\{N(A) = 0\},$$
$$\lim_{\theta\uparrow 1}\Phi(\theta^{1(\cdot;A)}) = \mathsf{P}\{N(A) < \infty\}.$$

(3) Let random measures $N_1, N_2, \dots, N_k$ be independent and $N = N_1 + \ldots + N_k$. Then the corresponding GFs $\Phi_1, \Phi_2, \dots, \Phi_k$ and Φ are related to each other as follows:

$$\Phi(u(\cdot)) = \Phi_1(u(\cdot))\ldots\Phi_k(u(\cdot)) = \prod_{i=1}^{k}\Phi_i(u(\cdot)). \qquad (11.1.3)$$

Very important characteristics of integer-valued random measures are their factorial moments $\varphi^{[n]}(A_1, \dots, A_n)$. The first factorial moment is defined as the expectation of $N(A)$:

$$\varphi^{[1]}(A) = \mathsf{E}N(A). \qquad (11.1.4)$$

The second factorial moment is

$$\varphi^{[2]}(A_1, A_2) = \mathsf{E}[N(A_1)N(A_2) - N(A_1 \cap A_2)]. \qquad (11.1.5)$$

The third factorial moment is

$$\varphi^{[3]}(A_1, A_2, A_3) = \mathsf{E}[N(A_1)N(A_2)N(A_3) - N(A_1)N(A_2 \cap A_3) - N(A_2)N(A_1 \cap A_3) - N(A_3)N(A_1 \cap A_2) + 2N(A_1 \cap A_2 \cap A_3)].$$

Let $C_1, \dots, C_m$ be disjoint domains. Let each of A_i coincide with some of them, and let C_j occur n_j times among $A_1, \dots, A_n$. Then

$$\varphi^{[n]}(A_1, \dots, A_n) = \mathsf{E}[N^{[n_1]}(C_1) \dots N^{[n_m]}(C_m)], \tag{11.1.6}$$

where

$$N^{[n]}(C) \equiv N(C)[N(C) - 1] \dots [N(C) - n + 1]$$

and $\mathsf{E}N^{[n]}(C)$ is the nth factorial moment of the random integer $N(C)$. This concept was extended to arbitrary sets $A_1, \dots, A_n$ in (Sevastyanov, 1974).

The following property demonstrates the interconnection between factorial moments and GFs.

(4) Let the random integer-valued measure $N(A)$ possess finite factorial moments $\varphi^{[1]}(A), \dots, \varphi^{[m]}(A_1, \dots, A_m)$. Then the GF $\Phi(u(\cdot))$ can be represented in the form

$$\Phi(u(\cdot)) = 1 + \sum_{k=1}^{m-1} \frac{(-1)^k}{k!} \int \dots \int \bar{u}(x_1) \dots \bar{u}(x_k) \varphi^{[k]}(dx_1, \dots, dx_k) + R_m, \tag{11.1.7}$$

where $\bar{u}(x) = 1 - u(x)$ and R_m is the remainder term (Sevastyanov, 1974).

It is easy to see that setting

$$u(x) = \begin{cases} u = \text{const}, & x \in A, \\ 1, & x \notin A, \end{cases}$$

we reduce the GF $\Phi(u(\cdot))$ to the g.f. $\varphi(u)$.

Following the common practice, we pass from the functions of sets to their densities (assuming that they exist, of course) by means of the relation

$$\varphi^{[k]}(dx_1, \dots, dx_k) = f^{[k]}(x_1, \dots, x_k) dx_1 \dots dx_k.$$

As follows from (11.1.7), the densities $f^{[k]}(x_1, \dots, x_k)$ can be expressed in terms of functional derivatives of GFs:

$$f^{[k]}(x_1, \dots, x_k) = \left. \frac{\delta \Phi(u(\cdot))}{\delta u(x_1) \dots \delta u(x_k)} \right|_{u=1}. \tag{11.1.8}$$

Each of them is invariant under permutations of the arguments.

In the case of non-intersecting $dx_1, \dots, dx_k$, from (11.1.6) it follows that $f^{[k]}(x_1, \dots, x_k)dx_1 \dots dx_k$ is the mean number of collections of random points of a given random point distribution such that one of them falls into dx_1, the other falls into dx_2 and so on, and $f^{[1]}(x) \equiv f(x)$ is the average density (concentration) of random points at x.

As an example, we consider a homogeneous Poisson ensemble. Let $u(x)$ be a simple function

$$u(x) = \sum_{m=1}^{M} u_m \mathbf{1}(x; A_m).$$

Inserting this into (11.1.2) and taking into account that

$$\ln \sum_{m=1}^{M} u_m \mathbf{1}(x; A_m) = \sum_{m=1}^{M} \ln u_m \mathbf{1}(x; A_m),$$

$$\int N(dx)\mathbf{1}(x; A_m) = N(A_m),$$

we obtain

$$\begin{aligned} \Phi(u(\cdot)) &= \mathsf{E} \exp\left\{\sum_{m=1}^{M} \ln u_m N(A_m)\right\} = \mathsf{E} \prod_{m=1}^{M} u_m^{N(A_m)} \\ &= \prod_{m=1}^{M} \mathsf{E} u_m^{N(A_m)} = \prod_{m=1}^{M} \exp\{(u_m - 1)\mathsf{E}N(A_m)\} \\ &= \exp\left\{\sum_{m=1}^{M} (u_m - 1)\mathsf{E}N(A_m)\right\} = \exp\left\{\int [u(x) - 1] f(x) dx\right\}. \end{aligned} \tag{11.1.9}$$

Multifold functional differentiation applied to (11.1.9) yields

$$f^{[k]}(x_1, \dots, x_k) = f(x_1) \dots f(x_k).$$

Thus, for the Poisson ensemble the factorial moments of high orders are factorized, which can be used to describe correlations in other ensembles in terms of the differences

$$\theta'_k(x_1, \dots, x_k) = f^{[k]}(x_1, \dots, x_k) - f(x_1) \dots f(x_k).$$

For $k = 2$, the function $\theta'_2(x_1, x_2) \equiv \theta_2(x_1, x_2)$ reflects the simplest type of correlations, namely the pair correlations. For $k = 3$, two types of correlations are possible: where two particles are correlated and the third one is independent, and where all the three particles are correlated. The corresponding terms in $\theta'_3(x_1, x_2, x_3)$ take the form $f(x_i)\theta_2(x_j, x_k)$ and $\theta_3(x_1, x_2, x_3)$. To analyze possible types in the general case, all the ways of partitioning the set $\{x_1, \dots, x_k\}$ into non-intersecting subsets containing at least one particle should be considered. Interpreting each subset or a group (cluster) as a collection of mutually correlated particles being statistically independent of the particles of other subsets, we obtain the following representation for factorial moment densities, which is known as a group expansion:

$$
\begin{aligned}
f^{[2]}(x_1, x_2) &= f(x_1)f(x_2) + \theta_2(x_1, x_2),\\
f^{[3]}(x_1, x_2, x_3) &= f(x_1)f(x_2)f(x_3) + f(x_1)\theta_2(x_2, x_3)\\
&\quad + f(x_2)\theta_2(x_1, x_3) + f(x_3)\theta_2(x_1, x_2) + \theta_3(x_1, x_2, x_3),
\end{aligned}
$$

and so on. The functions $\theta_n(x_1, \dots, x_k)$ are called the irreducible n-particle correlation functions, and each term in the expansions is called the correlation form (Balescu, 1975).

A more detailed description of problems related to the GFs, functional differentiation and multiparticle functions can be found in the monographs (Tatarsky, 1971; Uchaikin & Ryzhov, 1988) and others.

We will present below some results obtained in (Uchaikin, 1977; Uchaikin & Lappa, 1978; Uchaikin & Gusarov, 1997a; Uchaikin & Gusarov, 1997b; Uchaikin & Gusarov, 1997c)

11.2. Markov point distributions

Let a random distribution $N_0(A)$ of particles be defined in a space, and let the GF of this distribution be $\Phi_0(u(\cdot))$:

$$
\Phi_0(u(\cdot)) = \mathsf{E} \exp\left\{\int N_0(dx) \ln u(x)\right\} \qquad (11.2.1)
$$

Let, moreover, each of them generate its own random distribution denoted by $N(X_i \to A)$. Then the cumulative distribution $N(A)$ can be presented as the sum

$$
N(A) = \sum_{i=1}^{\nu} N(X_i \to A) \qquad (11.2.2)
$$

where ν is the progeny, $X_1, \dots, X_\nu$ are the random coordinates of the particles, and the terms with the fixed first arguments $X_i = x_i$ are statistically independent.

Let us calculate the GF of this distribution by the formula of total mathematical expectation. We first calculate the GF for a fixed distribution of initial particles. According to (11.1.3),

$$
\Phi(u(\cdot) \mid x_1, \dots, x_\nu) = \prod_{i=1}^{\nu} G(x_i \to u(\cdot)), \qquad (11.2.3)
$$

where $G(x_i \to u(\cdot))$ is the GF of the distribution $N(x_i \to A)$ including the point x_i. Calculating now the expectation of (11.2.3) in view of (11.1.2) and (11.2.1), we obtain the following expression for the GF of cumulative point distribution:

$$\begin{aligned}
\Phi(u(\cdot)) &= \mathsf{E} \prod_{i=1}^{\nu} G(X_i \to u(\cdot)) \\
&= \mathsf{E} \exp\left\{ \sum_{i=1}^{\nu} \ln G(X_i \to u(\cdot)) \right\} \\
&= \mathsf{E} \exp\left\{ \int N_0(dx) \ln G(x \to u(\cdot)) \right\} \\
&= \Phi_0(G(\cdot \to u(\cdot))).
\end{aligned}$$

With given $\Phi_0(u(\cdot))$, the functional relation

$$\Phi(u(\cdot)) = \Phi_0(G(\cdot \to u(\cdot))) \tag{11.2.4}$$

contains complete information about the random distribution of initial particles in space, and reflects the mutual independence of the families of descendant particles. To find statistical properties of all the points, one should have information concerning the evolution of the families, i.e., a particular expression of the GF $G(x \to u(\cdot))$.

As the basis, we take the model which is convenient from the practical (computational) viewpoint and is well developed in a mathematical sense. It is the Markov ensemble model, yielding, in particular, a functional equation for the GF $G(x \to u(\cdot))$.

According to this model, each particle X_i generates ν_i particles distributed by the GF $K(X_i \to u(\cdot))$ independently of the others, and this process continues with the same GF $K(x \to u(\cdot))$. The random point distribution looked for consists of points of all generations including the initial points.

Thus, the point distribution generated by the particle X_i resembles (11.2.2):

$$N(X_i \to A) = \mathbf{1}(X_i; A) + \sum_{k=1}^{\nu_i} N(X_{ik} \to A), \tag{11.2.5}$$

where $\{X_{i1}, X_{i2}, \dots, X_{ik}, \dots, X_{i\nu_i}\}$ is a point distribution with the GF $K(X_i \to u(\cdot))$. In the same way as we derived (11.2.4), we obtain

$$G(x \to u(\cdot)) = u(x) K(x \to G(\cdot \to u(\cdot))). \tag{11.2.6}$$

With known GFs Φ_0 and K, relations (11.2.4) and (11.2.6) yield a complete description of statistical properties of the point distributions under consideration.

We now assume that the parent particles are distributed as a homogeneous Poisson ensemble with the mean density $f(x) = \text{const}$. Then (11.2.4) is reduced to (11.1.9):

$$\Phi(u(\cdot)) = \exp\left\{\rho_0 \int [G(x \to u(\cdot)) - 1]dx\right\}. \tag{11.2.7}$$

If the parent particles produce no descendants, then

$$K(x \to u(\cdot)) = 1$$

and, by (11.2.6),

$$G(x \to u(\cdot)) = u(x).$$

In this case (11.2.7) is reduced to (11.1.9) with $f(x) = \rho_0$, i.e., we have a simple homogeneous Poisson ensemble of independent particles.

Using (11.1.8), we can derive from (11.2.7) a relation between the irreducible correlation functions θ_k and the factorial moment densities:

$$g^{[k]}(x \to x_1, \dots, x_k) = \left.\frac{\delta^k G(x \to u(\cdot))}{\delta u(x_1)\dots\delta u(x_k)}\right|_{u=1},$$

$$\theta_k(x_1, \dots, x_k) = \rho_0 \int g^{[k]}(x \to x_1, \dots, x_k)\, dx. \tag{11.2.8}$$

If the process of evolution of the families is spatially homogeneous, then the functions $g^{[k]}(x \to x_1, \dots, x_k)$ depend only on the differences $x_1 - x, \dots, x_k - x$, therefore

$$g^{[k]}(x \to x_1, \dots, x_k) \equiv g^{[k]}(x_1 - x, \dots, x_k - x),$$

$$f = \rho_0 \int g^{[1]}(x)\, dx = \text{const}, \tag{11.2.9}$$

$$\theta_2(x_{21}) = \rho_0 \int g^{[2]}(x', x_{21} + x')dx', \qquad x_{21} = x_2 - x_1,$$

$$\theta_3(x_{21}, x_{31}) = \rho_0 \int g^{[3]}(x', x_{21} + x', x_{31} + x'),$$

and so on.

11.3. Average density of random distribution

The first density (concentration)

$$f(x) = \mathsf{E}N(dx)/dx$$

of a random point distribution is necessary (and sufficient) for computation of the mathematical expectation of the additive functions $\sum h(X_i)$:

$$J \equiv \mathsf{E} \sum h(X_i) = \int h(x) f(x)\, dx. \tag{11.3.1}$$

To obtain the corresponding relations, it is necessary to take the functional derivative of (11.2.4) and (11.2.6), and then set $u(x) = 1$; we thus obtain

$$\frac{\delta \Phi(u(\cdot))}{\delta u(x)} = \frac{\delta \Phi_0(F(u(\cdot)|\cdot))}{\delta u(x)}, \tag{11.3.2}$$

$$\frac{\delta \Phi_0(G(\cdot \to u(\cdot)))}{\delta u(x)} = \int \frac{\delta \Phi_0(G(\cdot \to u(\cdot)))}{\delta G(x' \to u(\cdot))} \frac{\delta G(x' \to u(\cdot))}{\delta u(x)} dx'. \tag{11.3.3}$$

If $u \to 1$, then $G(\cdot \to u(\cdot)) \to 1$ and by virtue of (11.1.7) we obtain the relation

$$f(x) = \int dx' f_0(x') g(x' \to x) \tag{11.3.4}$$

whose physical sense is obvious. The function

$$g(x' \to x) \equiv g^{[1]}(x' \to x)$$

is a source function, since it describes the concentration of particles at the point x of the family generated by one parent particle appeared at x', and $f_0(x)$ is the concentration of parent particles at x.

Differentiating the second relation of (11.2.6) in a similar way,

$$\frac{\delta G(x \to u(\cdot))}{\delta u(x_1)} = \delta(x - x_1) K(x \to G(\cdot \to u(\cdot))) + u(x) \frac{\delta K(x \to G(\cdot \to u(\cdot)))}{\delta u(x_1)}$$

and following (11.3.3), we obtain

$$\frac{\delta G(x \to u(\cdot))}{\delta u(x_1)} = \delta(x - x_1) K(x \to G(\cdot \to u(\cdot))) + u(x) \int \frac{\delta K(x \to G(\cdot \to u(\cdot)))}{\delta G(x' \to u(\cdot))} \frac{\delta G(x' \to u(\cdot))}{\delta u(x_1)} dx' \tag{11.3.5}$$

Setting here $u(x) = 1$, we arrive at the integral equation

$$g(x \to x_1) = \delta(x - x_1) + \int k(x \to x') g(x' \to x_1) dx' \tag{11.3.6}$$

with

$$k(x \to x') \equiv \left. \frac{\delta K(x \to u(\cdot))}{\delta u(x')} \right|_{u=1}$$

which is the average spatial density of the offspring of the particle x.

Substituting (11.3.4) into (11.3.1), we obtain

$$J = \int dx\, h(x) \int g(x' \to x) f_0(x')\, dx'.$$

Changing the integration order and introducing

$$f^+(x) = \int dx'\, h(x') g(x \to x'), \tag{11.3.7}$$

for (11.3.1) we obtain

$$J = \int f_0(x) f^+(x)\, dx, \tag{11.3.8}$$

where the function $f^+(x)$, in view of (11.3.6), satisfies the equation

$$f^+(x) = h(x) + \int k(x \to x') f^+(x')\, dx'. \tag{11.3.9}$$

Formulae (11.3.7)–(11.3.9) provide us with a possibility to compute the mathematical expectations of any additive function of random point distribution if the characteristics $f_0(x)$ and $k(x \to x')$ are known. However, this representation is not unique.

Assuming that the Neumann series for the integral equation (11.3.6) converges (say, the condition $\int k(x' \to x) dx = c < 1$ holds), we represent its solution as

$$g(x' \to x) = \delta(x - x') + \sum_{i=1}^{\infty} k^{(i)}(x' \to x), \tag{11.3.10}$$

where the function of two variables $k^{(i)}(x' \to x)$ is the kernel of equation (11.3.6) if $i = 1$, and is the $(i-1)$th convolution of such kernels if $i > 1$:

$$k^{(1)}(x' \to x) = k(x' \to x),$$
$$k^{(i)}(x' \to x) = \int k^{(i-1)}(x' \to x'') k(x'' \to x) dx''.$$

Substituting expansion (11.3.10) into (11.3.4) we obtain

$$f(x) = f_0(x) + \sum_{i=1}^{\infty} \int f_0(x') k^{(i)}(x' \to x)\, dx'. \tag{11.3.11}$$

Being rewritten in the form

$$f(x) = f_0(x) + \int \left[f_0(x') + \sum_{i=1}^{\infty} \int f_0(x'') k^{(i)}(x'' \to x') dx'' \right] k(x' \to x)\, dx'$$

this relation leads us to another integral equation for the average density (concentration) of particles:

$$f(x) = f_0(x) + \int f(x')k(x' \to x)\,dx', \tag{11.3.12}$$

which, together with (11.3.1) allows to find the mathematical expectation of any additive function of random point distribution.

The existence of these two forms is well known and widely used in modern transportation theory, where this phenomenon is referred to as the duality principle, (11.3.12) is called the basic transport equation, (11.3.9), the adjoint (in the Lagrange sense) equation, and its solution $f^+(x)$ is called the adjoint function or importance (Case & Zweifel, 1967; Kolchuzhkin & Uchaikin, 1978; Lewins, 1965; Marchuk, 1980).

It follows from (11.3.6) that

$$\int g(x' \to x)\,dx = 1 + \int dx'' k(x \to x'') \int g(x'' \to x)\,dx.$$

If $k(x \to x'')$ is invariant under the translation

$$k(x \to x'') = k(x'' - x),$$

then

$$\int g(x' \to x)dx = \frac{1}{1 - \int k(x)\,dx} \equiv \frac{1}{1-c}. \tag{11.3.13}$$

Thus, the total mean density (11.3.4) with $f_0(x) = \text{const}$ is finite only if

$$c \equiv \int k(x)\,dx < 1.$$

11.4. Correlation functions

We begin our investigation of correlations generated by the considered model with

$$\theta_2(x_1, x_2) = \rho_0 \int g^{[2]}(x \to x_1, x_2)\,dx. \tag{11.4.1}$$

Differentiating (11.3.5) once more, we obtain

$$\frac{\delta^2 G(x \to u(\cdot))}{\delta u(x_2)\delta u(x_1)} = \delta(x_1 - x)\int \frac{\delta K(x \to G(\cdot \to u(\cdot))}{\delta G(x' \to u(\cdot))}\frac{\delta G(x' \to u(\cdot))}{\delta u(x_2)}dx'$$
$$+ \delta(x_2 - x)\int \frac{\delta K(x \to G(\cdot \to u(\cdot))}{\delta G(x' \to u(\cdot))}\frac{\delta G(x' \to u(\cdot))}{\delta u(x_1)}dx'$$
$$+ u(x)\left\{\iint \frac{\delta^2 K(x \to G(\cdot \to u(\cdot))}{\delta G(x' \to u(\cdot))\delta G(x'' \to u(\cdot))}\frac{\delta G(x' \to u(\cdot))}{\delta u(x_1)}\frac{\delta G(x'' \to u(\cdot))}{\delta u(x_2)}dx'dx''\right.$$
$$\left. + \int \frac{\delta K(x \to G(\cdot \to u(\cdot))}{\delta G(x' \to u(\cdot))}\frac{\delta^2 G(x' \to u(\cdot))}{\delta u(x_1)\delta u(x_2)}dx'\right\}.$$

Setting $u(x) = 1$, we arrive at the integral equation for the second factorial moment density for a single cascade:

$$g^{[2]}(x \to x_1, x_2) = \delta(x_1 - x)\int k(x \to x')g(x' \to x_2)\,dx'$$
$$+ \delta(x_2 - x)\int k(x \to x')g(x' \to x_1)\,dx'$$
$$+ \int dx' \int dx'' k^{[2]}(x \to x', x'')g(x' \to x_1)g(x'' \to x_2)$$
$$+ \int dx' k(x \to x')g^{[2]}(x' \to x_1, x_2). \quad (11.4.2)$$

Equation (11.4.2) can be considered as a special case of (11.3.9) with

$$h(x) \equiv h(x; x_1, x_2) = \delta(x_1 - x)\int k(x \to x')g(x' \to x_2)\,dx'$$
$$+ \delta(x_2 - x)\int k(x \to x')g(x' \to x_1)\,dx'$$
$$+ \int dx' \int dx'' k^{[2]}(x \to x', x'')g(x' \to x_1)g(x'' \to x_2),$$

where x_1 and x_2 are fixed parameters. By virtue of (11.3.7), the solution of this equation is of the form

$$g^{[2]}(x \to x_1, x_2) = g(x \to x_1)\int k(x_1 \to x')g(x' \to x_2)\,dx'$$
$$+ g(x \to x_2)\int k(x_2 \to x')g(x' \to x_1)\,dx'$$
$$+ \int dx' g(x \to x') \int dx'' \int dx''' k^{[2]}(x' \to x'', x''')g(x'' \to x_1)g(x''' \to x_2). \quad (11.4.3)$$

Substituting (11.4.3) into (11.4.1) and taking (11.2.9) into account, we arrive at the expression

$$\theta_2(x_1,x_2) = \rho\left\{\int k(x_1 \to x')g(x' \to x_2)\,dx' + \int k(x_2 \to x')g(x' \to x_1)\,dx' + \int dx' \int dx'' \bar{k}^{[2]}(x',x'')g(x' \to x_1)g(x'' \to x_2)\right\}, \quad (11.4.4)$$

where

$$\bar{k}^{[2]}(x',x'') = \int k^{[2]}(x \to x',x'')\,dx.$$

Making use of (11.3.6), (11.3.10) with

$$g'(x' \to x) = \sum_{i=1}^{\infty} k^{(i)}(x' \to x),$$

we can represent the result in the form

$$\theta_2(x_1,x_2) = \rho\left\{g'(x_1 \to x_2) + g'(x_2 \to x_1) + \int dx' \int dx'' \bar{k}^{[2]}(x',x'')g(x' \to x_1)g(x_2 \to x'')\right\}. \quad (11.4.5)$$

Taking multi-fold functional derivatives of both sides of (11.2.6) rewritten in the form

$$G(x \to u(\cdot)) = u(x) + \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_1 \ldots \int dx_n k^{[n]}(x \to x_1,\ldots,x_n) \times \left[u(x)\overline{G}(x_1 \to u(\cdot))\ldots\overline{G}(x_n \to u(\cdot))\right] \quad (11.4.6)$$

using formula (A.12.1) and setting $u(x) = 1$, we obtain the following system of simultaneous equations for factorial moment densities of arbitrary orders:

$$g^{[n]}(x \to 1,\ldots,n) \overset{s}{=} \int d1' k(x \to 1')g^{[n]}(1' \to 1,\ldots,n) + h_n(x \to 1,\ldots,n), \quad (11.4.7)$$

where

$$h_1(x \to 1) = \delta(x,1)$$

and

$$h_n(x \to 1,\dots,n) = n\delta(x,1)\int d1' k(x \to 1')g^{[n-1]}(1' \to 2,\dots,n)$$
$$+\sum_{m=2}^{n}\frac{1}{m!}\int d1'\dots\int dm' k^{[m]}(x \to 1',\dots,m')$$
$$\times \sum_{n_1\dots n_m \ge 1}\left\{\binom{n}{n_1\dots n_m}\delta_{N_m,n}g^{[n_1]}(1' \to 1,\dots,n_1)\dots g^{[n_m]}(m' \to N_{m-1}+1,\dots,n)\right.$$
$$\left.+n\delta(x,1)\binom{n-1}{n_1\dots n_m}\delta_{N_m,n-1}g^{[n_1]}(1' \to 2,\dots,n_1+1)\dots g^{[n_m]}(m' \to N_{m-2}+2,\dots,n)\right\},$$

$N_m = n_1 + \dots + n_m$, $n > 1$.

Here we use the notations $1, 2, \dots$ and $1', 2', \dots$ for $x_1, x_2, \dots$ and $x'_1, x'_2, \dots$, respectively, and $\delta(x,1)$ for $\delta(x-x_1)$. Recall also that $\stackrel{s}{=}$ means the symmetrization of the right-hand side:

$$f(x_1,\dots,x_n) \stackrel{s}{=} g(x_1,\dots,x_n)$$

means

$$f(x_1,\dots,x_n) = \frac{1}{n!}\sum_{i_1\dots i_n}^{n}{}' g(x_{i_1},\dots,x_{i_n}),$$

where the prime indicates the omission of terms with two or more coinciding indices.

Since the $g^{[1]}(x \to 1) \equiv g(x \to 1)$ obeys the equation

$$g(x \to 1) = \int d1' k(x \to 1')g(1' \to 1) + \delta(x,1), \tag{11.4.8}$$

it can be used as the Green function for all other equations (11.4.7), and we obtain their solutions in the recurring form:

$$g^{[n]}(x \to 1,\dots,n) \stackrel{s}{=} \int dx' g(x \to x')h_n(x' \to 1,\dots,n).$$

If no branching occurs, all $k^{[m]}$, $m \ge 2$, vanish, and we obtain a simple result for functions θ_n. Inserting the function

$$h_n(x \to 1,\dots,n) = n\delta(x,1)\int d1' k(x \to 1')g^{[n-1]}(1' \to 2,\dots,n)$$

into (11.4.7), we obtain

$$g^{[n]}(x \to 1,\dots,n) \stackrel{s}{=} ng(x \to 1)\int d1' k(1 \to 1')g^{[n-1]}(1' \to 2,\dots,n), \qquad n \ge 2.$$

Direct substitution shows that the system of equations is satisfied by the solutions

$$g^{[n]}(x \to 1, \dots, n) \stackrel{s}{=} n!\, g(x \to 1) g'(1 \to 2) \dots g'(n-1 \to n), \qquad (11.4.9)$$

where

$$g'(x \to x') = \int dx'' k(x \to x'') g(x'' \to x')$$

satisfies the equation

$$g'(x \to x') = \int dx'' k(x \to x'') g'(x'' \to x') + k(x \to x') \qquad (11.4.10)$$

that follows from (11.4.8). Thus, by virtue of (11.2.8) and (11.3.13), we obtain

$$\theta_k(x_1, \dots, x_k) \stackrel{s}{=} \rho k! g'(1 \to 2) \dots g'(k-1 \to n), \qquad (11.4.11)$$

where

$$\rho = \frac{\rho_0}{1-c} = \text{const.} \qquad (11.4.12)$$

Two important properties of the models considered follow herefrom.

First, the correlation functions of all orders $k > 2$ are expressed in terms of θ_2:

$$\theta_k(x_1, \dots, x_k) \stackrel{s}{=} \frac{\rho k!}{(2\rho)^{k-1}} \theta_2(x_1, x_2) \dots \theta_2(x_{k-1}, x_k), \qquad (11.4.13)$$

$$\theta_2(x_1, x_2) = 2\rho g'(x_1 \to x_2). \qquad (11.4.14)$$

Second, multiplying (11.4.10) by 2ρ, we see that correlation function (11.4.14) satisfies the integral equation

$$\theta_2(x_1, x_2) = \int dx' k(x_1 \to x') \theta_2(x', x_2) + 2\rho k(x \to x'), \qquad (11.4.15)$$

which is nothing but the Ornstein–Zernike equation (see, e.g. (Stell, 1991)).

11.5. Inverse power type correlations and stable distributions

To demonstrate how this model generates long-range correlations of the inverse power type, we consider a three-dimensional homogeneous case with spherically symmetric

$$k(\mathbf{r}_1 \to \mathbf{r}_2) = cp(\mathbf{r}_2 - \mathbf{r}_1), \qquad c \le 1,$$

where $p(\mathbf{r})$, which satisfies the condition

$$\int p(\mathbf{r})d\mathbf{r} = 1$$

and depends only on $r = |\mathbf{r}|$, can be interpreted as the transition probability density, whereas the constant c can be thought of as the survival probability.

With these assumptions in mind, we write the following equation for the function $f_c(\mathbf{r}) = cg'(\mathbf{r})$, which is also spherically symmetric:

$$f_c(\mathbf{r}) = p(\mathbf{r}) + c \int p(\mathbf{r}')f_c(\mathbf{r} - \mathbf{r}')d\mathbf{r}'. \tag{11.5.1}$$

The most investigated case of this equation is related to neutron transportation problem (Case & Zweifel, 1967) where

$$p(\mathbf{r}) = \frac{e^{-r}}{4\pi r^2}. \tag{11.5.2}$$

The corresponding solution is expressed in the form

$$f_c(\mathbf{r}) = \frac{1}{4\pi r}\left\{\frac{e^{-r/\nu_0}}{\nu_0 N_0} + \int_0^1 \frac{e^{-r/\nu}}{\nu N(\nu)}d\nu\right\},$$

where

$$N(\nu) = \nu[(1 - c\nu \operatorname{Arth} \nu)^2 + c^2\pi^2\nu^2/4],$$

$$N_0 = (c/2)\nu_0^3 \left\{[c/(\nu_0^2 - 1)] - [1/\nu_0^2]\right\}; \tag{11.5.3}$$

$$\nu_0 \approx \left\{\sqrt{3(1 - c)}[1 - (2/5)(1 - c)]\right\}^{-1} \tag{11.5.4}$$

for c close to 1.

It is easy to see that $f_c(\mathbf{r}) \sim f_c^{\rm as}(\mathbf{r})$ for $r \to \infty$, where

$$f_c^{\rm as}(\mathbf{r}) = \frac{1}{\nu_0(c)N_0(c)}\frac{e^{-r/\nu_0(c)}}{4\pi r}, \qquad c < 1,$$

$$f_1^{\rm as}(r) = \frac{3}{4\pi r}, \qquad c = 1.$$

Thus, the solution has two quite different asymptotic expressions for $c < 1$ and $c = 1$. But the results of numerical calculations presented in Fig. 11.1 allow us to re-unite these two cases. One can see that in the case where c is very close to 1 the true function $f(\mathbf{r})$ follows first $f_1^{\rm as}(\mathbf{r})$ and then changes into $f_c^{\rm as}(\mathbf{r})$ beyond some distance $r^{as}(c)$, so the closer c is to 1, the greater $r^{\rm as}(c)$ and the more domain of validity of expression $f_1^{\rm as}(\mathbf{r})$ for the function $f(\mathbf{r})$ with $c \neq 1$.

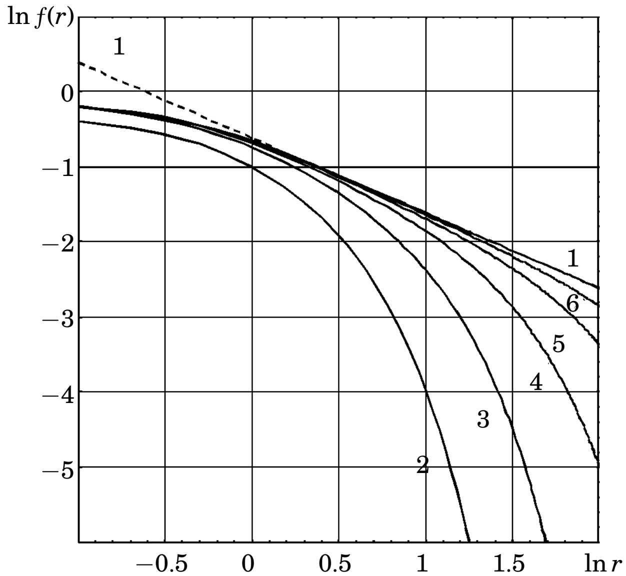


Figure 11.1. The function $f_1^{\mathrm{as}}(r)$ (dashed line) plays the role of intermediate asymptotics for $f_c(r)$ with c close to 1 (solid lines 2 (c = 0.9), 3 (c = 0.99), 4 (c = 0.999), 5 (c = 0.9999), and 6 (c = 0.99999)).

This means that the subcritical cascade with c close to 1 looks like critical one on a large but limited scale.

Mandelbrot investigated a similar problem in connection with the description of random distribution of galaxies in space (Mandelbrot, 1975; Mandelbrot, 1977). As known from observations, their two-point correlations have a long asymptotic tail

$$\theta_2(\mathbf{r}) \propto r^{-3+\gamma}$$

with $1 < \gamma < 2$. Using the walk process he considered the solution of the equation under conditions $c = 1$ and

$$p(r) = \begin{cases} 0, & r < \varepsilon, \\ (\alpha/4\pi)\varepsilon^{\alpha} r^{-3-\alpha}, & r > \varepsilon. \end{cases} \tag{11.5.5}$$

Using the Fourier transformation, Mandelbrot concluded that

$$f_1(r) \sim Cr^{-3+\alpha}, \quad r \to \infty, \tag{11.5.6}$$

where C is a constant (see also (Peebles, 1980)).

To look at this problem more closely we consider the solution of (11.5.1) with the transition probability density $p(\mathbf{r})$ which is a very stable density itself,

$$p(\mathbf{r}) = q_3(\mathbf{r}; \alpha).$$

Representing its solution as Neumann's series

$$f_c(\mathbf{r}) = \sum_{n=1}^{\infty} c^{n-1} p^{(n)}(\mathbf{r})$$

and keeping in mind that for stable distribution

$$q_3^{(n)}(\mathbf{r}) = n^{-3/\alpha} q_3(n^{-1/\alpha}\mathbf{r}; \alpha)$$

we arrive at

$$\begin{aligned} f_c(\mathbf{r}) &= \sum_{n=1}^{\infty} c^{n-1} n^{-3/\alpha} q_3(n^{1/\alpha}\mathbf{r}; \alpha) \\ &= \sum_{n=1}^{\infty} c^{n-1} n^{-3/\alpha} \rho_3(n^{-1/\alpha} r; \alpha), \qquad r = |\mathbf{r}|. \end{aligned} \tag{11.5.7}$$

Using (7.2.16) and changing the order of summation in the case $c < 1$ we obtain

$$f_c(\mathbf{r}) = \frac{1}{2\pi^2 r^3} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \Gamma(n\alpha + 2) \sin(n\alpha\pi/2) \Phi(c, -n, 1) r^{-n\alpha}, \tag{11.5.8}$$

where

$$\Phi(c, -n, 1) = \sum_{k=1}^{\infty} c^{k-1} k^n.$$

In particular,

$$\begin{aligned} \Phi(c, -1, 1) &= (1 - c)^{-2}, \\ \Phi(c, -2, 1) &= (c + 1)(1 - c)^{-3}, \\ \Phi(c, -3, 1) &= (c^2 + 4c + 1)(1 - c)^{-4}, \\ \Phi(c, -4, 1) &= (c^3 + 11c^2 + 11c + 1)(1 - c)^{-5}, \\ \Phi(c, -5, 1) &= (c^4 + 26c^3 + 66c^2 + 26c + 1)(1 - c)^{-6}. \end{aligned}$$

The asymptotic behavior of (11.5.8) far away from the origin is governed by the first term of the series:

$$f_c^{as}(\mathbf{r}) = (2\pi^2)^{-1} \Gamma(\alpha + 2) \sin(\alpha\pi/2)(1 - c)^{-2} r^{-3-\alpha}, \qquad r \to \infty.$$

As $c \to 1$,

$$\Phi(c, -n, 1) \sim n!\,(1 - c)^{-1-n}$$

and

$$f_c(\mathbf{r}) \sim (2\pi^2)^{-1}\sum_{n=1}^{\infty}(-1)^{n-1}\Gamma(n\alpha+2)\sin(n\alpha\pi/2)(1-c)^{-1-n}r^{-3-n\alpha}, \qquad c \to 1.$$

However this result is not applicable to the case $c = 1$, which should be treated separately. Setting $c = 1$ in (11.5.7) and applying the Euler–Maclaurin summation formula, we obtain

$$\begin{aligned} f_1(\mathbf{r}) &= \sum_{n=1}^{\infty} n^{-3/\alpha}\rho_3(n^{-1/\alpha}r;\alpha) \\ &= \int_1^{\infty}\psi(x)dx + \frac{1}{2}\psi(1) - \sum_{m=0}^{\infty}\frac{B_{2m}}{(2m)!}\psi^{(2m-1)}(1), \end{aligned} \tag{11.5.9}$$

where

$$\begin{aligned} \psi(x) &= x^{-3/\alpha}\rho_3(x^{-1/\alpha}r;\alpha) \\ &= \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n-1}}{n!}\Gamma(n\alpha+2)\sin(n\alpha\pi/2)r^{-n\alpha-3}\right]x^n \end{aligned}$$

and B_{2m} are the Bernoulli numbers.

The leading asymptotic term is due to the integral

$$\begin{aligned} f_1(\mathbf{r}) &\sim \int_1^{\infty} x^{-3/\alpha}\rho_3(x^{-1/\alpha}r;\alpha)\,dx \\ &= \alpha r^{-3+\alpha}\int_0^{r} x^{-\alpha+2}\rho_3(x)\,dx \\ &\sim \alpha r^{-3+\alpha}\overline{\rho}_3(2-\alpha;\alpha), \qquad r \to \infty, \end{aligned}$$

where $\overline{\rho}_3(s;\alpha)$ is the Mellin transform of the radial function $\rho_3(r;\alpha)$. Using (7.5.8), we obtain

$$f_1^{as}(\mathbf{r}) = (4\pi)^{-3/2}(r/2)^{-3+\alpha}\Gamma((3-\alpha)/2)/\Gamma(\alpha/2),$$

which is immediately transformed to

$$f_1^{as}(\mathbf{r}) = (2\pi^2)^{-1}\Gamma(2-\alpha)\sin(\alpha\pi/2)r^{-3+\alpha}. \tag{11.5.10}$$

The asymptotic expansion of (11.5.9) is thus of the following form:

$$f_1(\mathbf{r}) = (2\pi^2)^{-1}\left\{\Gamma(2-\alpha)\sin(\alpha\pi/2)r^{-3+\alpha} + \sum_{n=1\ (\mathrm{odd})}^{\infty}\Gamma(n\alpha+2)\sin(n\alpha\pi/2)A_n r^{-n\alpha-3}\right\}, \tag{11.5.11}$$

where

$$A_n = \frac{1}{n!2} - \frac{1}{(n+1)!} - \sum_{m=1}^{(n+1)/2} \frac{B_{2m}}{(2m)!(n-2m+1)!}.$$

In particular, $A_1 = -1/12$, $A_3 = 1/720$, $A_5 = -1/30240$, $A_7 = 1/1209600$, $A_9 = -1/47900160$, and so on. The leading term (11.5.10) and two terms of the sum in (11.5.11) were found in (Slobodenyuk & Uchaikin, 1998).

In view of asymptotics (11.5.8), (11.5.11), one can estimate the distance r_0, where the solution behaves like the leading asymptotics of type $r^{-3\pm\alpha}$. By requiring the contribution of correction terms in (11.5.8), (11.5.11) to be small in comparison with the leading one (to be not greater than $\beta \cdot 100\%$), we obtain

$$r_0 = \left(|\cos(\pi\alpha/2)| \frac{(1+c)\Gamma(2+2\alpha)}{(1-c)\beta\Gamma(2+\alpha)} \right)^{1/\alpha} \qquad c < 1,$$

$$r_0 = \left(\frac{\Gamma(2+\alpha)}{12\beta\Gamma(2-\alpha)} \right)^{1/(2\alpha)}, \qquad c = 1.$$

11.6. Mandelbrot's stochastic fractals

Following (Uchaikin *et al.*, 1998a), we continue the study of infinite trajectories ($c = 1$) generating infinite sets of points $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \dots$ randomly distributed in $\mathbb{R}^3$. The function $g'(r)$ can be considered as a conditional mean density of neighbours of the point $\mathbf{X}_0$ placed at the origin $\mathbf{X}_0$. The corresponding GF of the set with $\mathbf{X}_0 = \mathbf{r}$ satisfies the equation

$$G'(\mathbf{r} \to u(\cdot)) = \int d\mathbf{r}' p(\mathbf{r} \to \mathbf{r}') u(\mathbf{r}') G'(\mathbf{r}' \to u(\cdot)). \tag{11.6.1}$$

To make all points of the set statistically equivalent, we add the symmetric (in the statistical sense) part $\mathbf{X}_{-1}, \mathbf{X}_{-2}, \mathbf{X}_{-3}, \dots$ Owing to the spherically symmetric character of the distribution $p(\mathbf{r})$ of increments $\mathbf{X}_k - \mathbf{X}_{k-1}$ and their independence, the left-hand part of the trajectory can be constructed in the same way as the right-hand part $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \dots$ Namely, $\mathbf{X}_{-1}$ is chosen from the distribution $p(\mathbf{r})$, then $\mathbf{X}_{-2} = \mathbf{X}_{-1} + \Delta\mathbf{X}$, where $\Delta\mathbf{X}$ is chosen from the same distribution $p(\mathbf{r})$ independently of $\mathbf{X}_{-1}$ and so on. As a result we have an infinite to both sides trajectory $\{\dots, \mathbf{X}_{-3}, \mathbf{X}_{-2}, \mathbf{X}_{-1}, 0, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \dots\}$. All the points are statistically equal now, and the conditional characteristics of the random point distribution relative to each of the points are the same. The characteristics can be found from the GF

$$F(u(\cdot)) = G'^2(0 \to u(\cdot)). \tag{11.6.2}$$

Thus, the mean density of neighbours

$$f(\mathbf{r}) = \left. \frac{\delta F(u(\cdot))}{\delta u(\mathbf{r})} \right|_{u=1} = 2g'(\mathbf{r}), \tag{11.6.3}$$

and the expected number of neighbours in the sphere of radius R centered at any point is

$$\langle N(R)\rangle = 8\pi \int_0^R g'(\mathbf{r}) r^2 dr.$$

Substituting here $g'(\mathbf{r})$ from (11.5.6), we arrive at the asymptotic expression

$$\langle N(R)\rangle \sim AR^{\alpha}, \quad R \to \infty, \quad 0 < \alpha < 2. \tag{11.6.4}$$

It is obvious that the random point distribution possesses some peculiarities. Indeed, it may be thought of as a homogeneous set because all its points are stochastically equivalent but the conditional density (11.6.3)

$$f(\mathbf{r}) = g'(\mathbf{r}) \sim 2Cr^{-3+\alpha}, \quad r \to \infty, \quad 0 < \alpha < 2, \tag{11.6.5}$$

is not homogeneous.

Scale invariance plays the fundamental role in many natural phenomena and is often related to the appearance of irregular forms which cannot be described by means of usual differential geometry. A classic example is given by the Brownian motion which led Jean Perrin (Perrin, 1909) to understanding the physical relevance of non-differentiable curves and surfaces.

The necessity of introducing a new class of geometrical objects, the fractals, has subsequently arisen in various problems. Indeed, some aspects of 'fractality' were already present in the ideas of some scientists at the beginning of this century like Perrin himself, Hausdorff, Wiener, Richardson, but the concept of 'fractal object' was explicitly formulated and made popular in the scientific community in the recent decade or so by Mandelbrot.

A rough definition of a fractal object can be given by referring to the scale invariance displayed by these structures. In this sense, we say that a fractal is a geometric structure which looks always the same (at least in the statistical sense) irrespective of the resolution at which it is observed (Borgani, 1995, p. 49).

A more formal and correct definition of a fractal set, as given by Mandelbrot (Mandelbrot, 1983), consists in that it is a mathematical object whose fractal (Hausdorff) dimension D_H is strictly greater than its topological dimension D_T. Thus, for a fractal point distribution in a d-dimensional ambient space, $D_T = 0$ and $0 < D_H \le d$. The fractal dimension $D_H = d$ characterizes a space-filling and homogeneous distribution. As an example of the latter, a homogeneous Poisson ensemble can be given, for which

$$\langle N(R)\rangle = (4/3)\pi\rho R^3. \tag{11.6.6}$$

Note that in this case there is no difference whether or not the counting sphere is centered at any point of the set or at any other point of space.

Formulae (11.6.4) and (11.6.6) led Mandelbrot to another definition of a fractal dimension. Concerning the distribution of galaxies in the Universe, he write in *Is there a global density of matter?* , that to define and measure density, one starts with the mass $M(R)$ in a sphere of radius R centered on Earth. The approximate density, defined as

$$M(R)/[(4/3)\pi R^3],$$

is evaluated. After that, the value of R is made to tend toward infinity, and the global density is defined as the limit toward which the approximate density converges.

But need the global density to converge to a positive and finite limit? If so, the rate of convergence leaves a great deal to be desired. Furthermore, the estimates of the limit density had behaved very oddly. As the depth of the world perceived by telescopes increased, the approximate density diminished in a surprisingly systematic manner. According to de Vaucouleurs (1970), it has remained $\propto R^{D-3}$. The observed exponent D is much smaller than 3, the best estimate on the basis of indirect evidence being $D = 1.23$.

The thesis of de Vacouleurs is that the behavior of the approximate density reflects reality, meaning that $M(R) \propto R^D$. This formula recalls the classical result that a ball of radius R in a Euclidean space of dimension E has the volume $\propto R^E$. We encounter the same formula for the Koch curve, with the major difference that the exponent is not the Euclidean dimension $E = 2$ but a fraction-valued fractal dimension D. For the Cantor dust on the time axis (for which $E = 1$), $M(R) \propto R^D$.

All these precedents suggest very strongly that the de Vacouleurs' exponent D is a fractal dimension (Mandelbrot, 1983, p.85).

Mandelbrot used the walk model described above (with one-sided trajectories) to simulate such a distribution. The random walk model of the distribution of galaxies implements any desired fractal dimension $D < 2$ using a dust, i.e., a set of correct topological dimension $D_T = 0$. To achieve this goal, a random walk is used wherein the mathematical expectation $\langle U^2 \rangle$ is infinite, because $U(= |\Delta \mathbf{X}|)$ is a hyperbolic random variable, with an inner cutoff at $u = 1$. Thus, for $u \leq 1$, $\mathsf{P}\{U > u\} = 1$, while for $u > 1$ $\mathsf{P}\{U > u\} \propto u^{-D}$, with $0 < D < 2$.

A major consequence is that $\langle M(R) \rangle \propto R^D$ when $R \gg 1$. It allows any dimension likely to be suggested by fact or theory (Mandelbrot, 1983, p.289).

Using this approach as a basis for fractal interpretation of the random set of points $\{\dots, \mathbf{X}_{-3}, \mathbf{X}_{-2}, \mathbf{X}_{-1}, 0, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \dots\}$ and understanding $\langle \dots \rangle$ as averaging over the whole ensemble of random realizations, we consider some properties of the random fractal.

As follows from (11.6.1) and (11.6.2), the nth factorial moment density is

$$f_n(\mathbf{r} \to 1, \dots, n) \stackrel{s}{=} \sum_{k=0}^{n} \binom{n}{k} g'^{[k]}(\mathbf{r} \to 1, \dots, k) g'^{[n-k]}(\mathbf{r} \to k+1, \dots, n)$$

and the nth factorial moment of the random number M of points occurring in the sphere U_R of radius R is

$$\langle M^{[n]}\rangle = \int_{U_R}\dots\int_{U_R} f^{[n]}(0\to z_1,\dots,z_n)\,d\mathbf{r}_1\dots d\mathbf{r}_n = \sum_{k=0}^{n}\binom{n}{k}\langle N^{[k]}\rangle\langle N^{[n-k]}\rangle \qquad (11.6.7)$$

where N is the random number of points generated in the sphere by the 'one-sided' trajectory.

In view of (11.6.5),

$$\langle N^{[k]}\rangle = k!\int_{U_R}\dots\int_{U_R} g'(0\to\mathbf{r}_1)\dots g'(\mathbf{r}_{k-1}-\mathbf{r}_n)\,d\mathbf{r}_1\dots d\mathbf{r}_k \sim k!\,(4\pi A/3)^k R^{\alpha k}K_k(\alpha), \qquad R\to\infty,$$

where

$$K_0(\alpha) = 1, \qquad K_1(\alpha) = (3/4\pi)\int_{U_1} r^{-3+\alpha}d\mathbf{r},$$

$$K_k(\alpha) = (3/4\pi)^k\int\cdots\int_{U_1^k}(r_1r_{1,2}\dots r_{k-1,k})^{-3+\alpha}d\mathbf{r}_1d\mathbf{r}_2\dots d\mathbf{r}_k, \qquad k>1, \qquad (11.6.8)$$

$$r_{ij} = |\mathbf{r}_i-\mathbf{r}_j|.$$

Before discussing the algorithm to calculate multiple integrals (11.6.8), we note that (11.6.7) has the asymptotics

$$\langle M^{[n]}\rangle \sim (4\pi A/3)^n R^{\alpha n}Q_n(\alpha), \qquad R\to\infty,$$

where

$$Q_n(\alpha) = \sum_{k=0}^{n} K_k(\alpha)K_{n-k}(\alpha).$$

Asymptotically,

$$\langle M^{[n]}(R)\rangle \sim \langle M^n(R)\rangle, \qquad R\to\infty,$$

and the moments of the normalized random variable

$$Z = \frac{M(R)}{\langle M(R)\rangle} \qquad (11.6.9)$$

do not depend on the radius R:

$$\langle Z^n\rangle \sim n!\frac{Q_n(\alpha)}{(Q_1(\alpha))^n}, \qquad R\to\infty. \qquad (11.6.10)$$

As a result, the asymptotic distribution of $M(R)$ can be rewritten in the following scaled form:

$$\mathsf{P}\left\{Z = m/\langle M(R)\rangle\right\} \sim \frac{1}{\langle M(R)\rangle}\Psi_\alpha\left(\frac{m}{\langle M(R)\rangle}\right), \qquad R \to \infty, \quad (11.6.11)$$

where $\Psi_\alpha(z)$ is the density of the distribution of (11.6.9) with the moments (11.6.10):

$$\langle Z^n \rangle = \int_0^\infty z^n \Psi_\alpha(z)\, dz, \qquad \langle Z \rangle = 1.$$

11.7. Numerical results

Now we turn back to the problem of calculation of multiple integrals (11.6.8) following (Uchaikin *et al.*, 1998a). We consider the set of functions

$$v_n^{(\alpha)}(\mathbf{r}) = \int_{U_1} \cdots \int_{U_1} (|\mathbf{r} - \mathbf{r}_1| r_{1,2} \ldots r_{n-1,n})^{-3+\alpha} d\mathbf{r}_1 d\mathbf{r}_2 \ldots d\mathbf{r}_n$$

such that

$$K_n(\alpha) = (3/4\pi)^n v_n^{(\alpha)}(0).$$

The functions are recursively interrelated as follows:

$$v_n^{(\alpha)}(\mathbf{r}) = \int_{U_1} F^{(\alpha)}(\mathbf{r} - \mathbf{r}') v_{n-1}^{(\alpha)}(\mathbf{r}') d\mathbf{r}', \quad F(\mathbf{r}) = r^{-3+\alpha}, \quad v_0^{(\alpha)}(\mathbf{r}) = 1. \quad (11.7.1)$$

Taking the spherical symmetry of functions F and $v_n^{(\alpha)}$ into account, we can represent (11.7.1) as

$$w_n^{(\alpha)}(r) = \int_0^1 F^{(\alpha)}(r, r') w_{n-1}^{(\alpha)}(r')\, dr', \qquad w_n^{(\alpha)}(r) = r v_n^{(\alpha)}(\mathbf{r}), \quad (11.7.2)$$

where

$$F^{(\alpha)}(r, r') = \frac{2\pi}{\alpha - 1}\left[(r + r')^{\alpha-1} - |r - r'|^{\alpha-1}\right], \qquad \alpha \neq 1,$$
$$F^{(1)}(r, r') = 2\pi\left[\ln(r + r') - \ln|r - r'|\right].$$

It is easy to see that $K_n(\alpha)$ is expressed in terms of $w_n^{(\alpha)}$ as follows:

$$K_n(\alpha) = (3/4\pi)^n \left[dw_n^{(\alpha)}(r)/dr\right]_{r=0}. \quad (11.7.3)$$

This representation is convenient for sequential numerical calculation of $K_n(\alpha)$ for $\alpha > 1$ while using the standard technique of numerical integration.

The situation becomes even more simple in the case $\alpha = 1$. Equation (11.7.1) can be transformed into

$$v_n^{(1)}(r) = \frac{2\pi}{r}\int_0^1 \ln\frac{|r'+r|}{|r'-r|} v_{n-1}^{(1)}(r')\,dr',$$
$$v_1^{(1)}(r) = \frac{2\pi}{r}\left\{r + \frac{1}{2}(1-r^2)\ln\frac{|1+r|}{|1-r|}\right\}.$$

Expanding $v_n^{(1)}(r)$ into power series, we arrive at the recurrence relation

$$v_n^{(1)}(r) = 2\pi v_{n-1}^{(1)}(0)\left(2 - \sum_{i=1}^{\infty} r^{2i}\left(\frac{1}{2i-1} - \frac{1}{2i+1}\right)\right) - (4\pi)^2 v_{n-2}^{(1)}(0)\sum_{i=1}^{\infty}\frac{1}{(2i-1)(2i+1)^2};$$

hence

$$v_n^{(1)}(0) = 4\pi v_{n-1}^{(1)}(0) - (4\pi)^2 v_{n-2}^{(1)}(0)\sum_{i=1}^{\infty}\frac{1}{(2i-1)(2i+1)^2},$$
$$v_0^{(1)}(0) = 1, \qquad v_1^{(1)}(0) = 4\pi.$$

In the domain $\alpha \le 1$, the integrand possesses a singularity at the point $r' = r$ causing an increase in error in the course of numerical integration. To avoid the trouble, we transform (11.7.2) into

$$\int_0^r w_n^{(\alpha)}(r')dr' = W_n^{(\alpha)}(r) - \frac{4\pi}{\alpha(\alpha-1)}\int_0^1 r'^{\alpha} w_{n-1}^{(\alpha)}(r')\,dr',$$
$$W_n^{(\alpha)}(r) = \frac{2\pi}{\alpha(\alpha-1)}\int_0^1 \left[(r+r')^{\alpha} - (r-r')|r-r'|^{\alpha-1}\right] w_{n-1}^{(\alpha)}(r')\,dr';$$

hence (11.7.3) can be rewritten as

$$K_n(\alpha) = (3/4\pi)^n \left[d^2 W_n^{(\alpha)}(r)/dr^2\right]_{r=0}.$$

The numerical results for $K_n(\alpha)$ are presented in Table 11.1.

Using formula (11.6.10), we calculate $\langle Z^n\rangle$ (see Table 11.2). As one can see, the relative fluctuations grow as α decreases. Also it can be seen from Table 11.2 that the following relation holds for the moments $\langle Z^n\rangle$:

$$\langle Z^n\rangle = \big(A(\alpha)n + B(\alpha)\big)\langle Z^{n-1}\rangle.$$

This relation is the characteristic property for the gamma distribution

$$\Psi_\alpha(z) = \frac{1}{\Gamma(\lambda)}\lambda^{\lambda} z^{\lambda-1} e^{-\lambda z};$$

Table 11.1. The coefficients $K_n(\alpha)$ for a single trajectory. The values in brackets are taken from (Peebles, 1980). The values with asterisks are the data for $\alpha = 1.23$ from (Peebles, 1980).

K_n	α							
	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
K_1	12.0	6.00	4.00	3.00	2.40	2.00	1.71	1.50
K_2	140	34.0 (34.0)	14.6	7.80 (7.82)	4.88 (5.02)*	3.31 (3.31)	2.42	1.88 (1.88)
K_3	$163 \cdot 10$	185	50.4	19.1	9.38 (9.5)*	5.22	3.30	2.29
K_4	$188 \cdot 10^2$	985	168	45.8	17.7	8.13	4.46	2.78
K_5	$215 \cdot 10^3$	$516 \cdot 10$	554	108	33.2	12.6	6.03	3.38

Table 11.2. The moments $\langle Z^n \rangle$ for the paired trajectory. The lower values are results of the Monte-Carlo simulation.

$\langle Z^n \rangle$	α						
	0.25	0.50	0.75	1.00	1.25	1.50	1.75
Z_2	1.49	1.44 1.51 ± 0.07	1.41	1.37 1.40 ± 0.06	1.35	1.33 1.36 ± 0.06	1.33
Z_3	2.93	2.70 3.04 ± 0.25	2.55	2.36 2.52 ± 0.21	2.29	2.22 2.38 ± 0.21	2.22
Z_4	7.18	6.18 7.5 ± 1.0	5.59	4.95 5.7 ± 0.8	4.71	4.51 5.3 ± 0.9	4.54
Z_5	21.0	16.7 22 ± 4	14.4	12.2 16 ± 3	11.4	10.8 15 ± 5	11.0

hence we obtain

$$\langle Z^n \rangle = \left(\frac{n}{\lambda} + 1 - \frac{1}{\lambda} \right) \langle Z^{n-1} \rangle.$$

The parameter of gamma distribution λ can be expressed in terms of $\langle Z^2 \rangle$:

$$\lambda(\alpha) = \frac{1}{\langle Z^2 \rangle - 1} = \frac{1}{\sigma_Z^2},$$

that is, the parameter of gamma distribution is determined by the variance $\sigma_Z^2 \equiv \sigma_N^2 / \langle N \rangle^2$.

We also perform Monte-Carlo simulation of random trajectories with transition probability (11.5.5) for $\alpha = 0.5, 1.0, 1.5$. The points are counted inside the sphere of radius $R \gg a$. The trajectory is broken when coming out the boundary of the sphere of the radius $R_{\max} = 1 \gg R$. As a result of simulation, the function $\Psi_\alpha(z)$ and moments $\langle Z \rangle, \ldots, \langle Z^5 \rangle$ for $\alpha = 0.5, 1.0, 1.5$ are obtained.

In Fig. 11.2, the results of simulation are shown for $R = 0.05$, 0.1 and 0.2, $\alpha = 1.0$. It is immediately seen from this histogram that the results of simulation are independent of R. All histograms are satisfactorily described by the gamma distribution.

In conclusion, we should emphasize that, because of fractal self-similarity, the fluctuations of point count inside the sphere (centered around any fractal point) do not vanish with increasing the radius but remain essential at all scales.

11.8. Fractal sets with a turnover to homogeneity

Now, two questions arise:

(1) Can one construct a random point distribution which is a fractal on small scales and homogeneous on large scales?

(2) Can one construct such distributions with fractal dimension $2 < D < 3$ which are observed for example in a turbulence problem (Takayasu, 1984)?

We begin with the first question. P. Coleman and L. Pietronero wrote, concerning the simulation of universe, that a sample which is fractal on small scales and homogeneous on large scales is constructed as follows: A number of random locations are chosen in a large volume. Since the locations are Poisson-distributed, they will have an average separation λ_0. Each of these locations is the starting point for constructing a fractal whose sample length scales up to the limiting value λ_0 (Coleman & Pietronero, 1992, p.334).

It is evident that, rigorously speaking, this large volume must be infinite, otherwise boundary effects are observed. But if the volume is infinite, then only two cases are possible: either $c < 1$, and then $g'(\mathbf{r}) \propto r^{-3-\alpha}$, or $c = 1$ and then $g'(\mathbf{r}) \propto r^{-3+\alpha}$, but

$$\int g'(\mathbf{r})\, d\mathbf{r} = \infty.$$

The first case is not a fractal on any scales; in the second case we cannot use the standard correlation analysis based on the functions (11.4.12) because of divergence of the prefactor

$$\rho = \frac{\rho_0}{1-c} \to \infty, \qquad c \to 1.$$

To investigate the problem in more detail, numerical calculations of $g'(\mathbf{r})$ are performed for c close to 1 (Uchaikin & Gusarov, 1997b). Spherically symmetric stable distributions are used as the kernel $p(\mathbf{r})$. These calculations demonstrate that a certain finite domain exists where the function $g'_c(\mathbf{r})$ coincides numerically with $g'_1(\mathbf{r})$ while c is close enough to 1. So, there is a

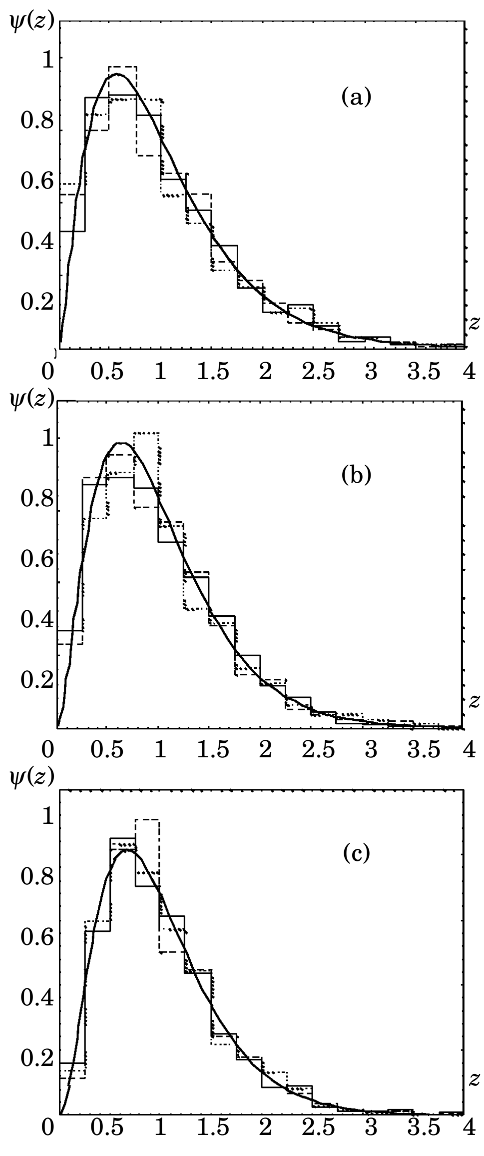


Figure 11.2. The distribution density of the scaled random variable $Z = N(R)/\langle N(R)\rangle$ for $\alpha = 0.5$ (a), $\alpha = 1.0$ (b), $\alpha = 1.5$ (c). Histograms are the results of Monte-Carlo calculations of 1000 realizations ($a = 10^{-3}$): for $R = 0.05$ (—), for $R = 0.1$ (- - -), and for $R = 0.2$ (· · ·). The smooth solid curves show the gamma distribution $\Psi_\alpha(z)$.

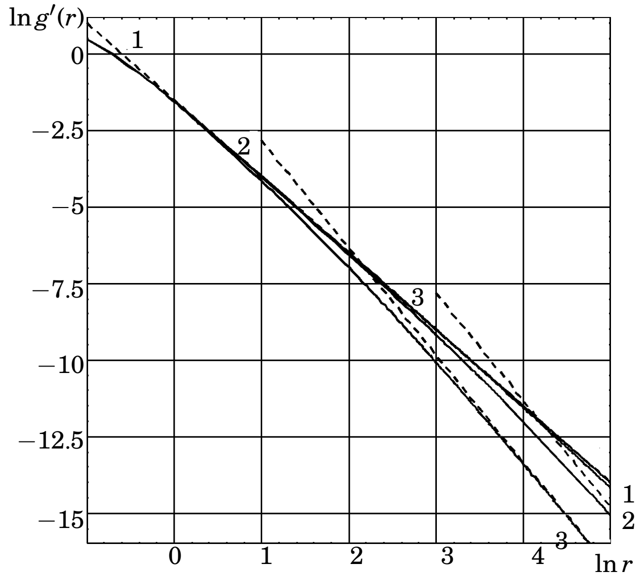


Figure 11.3. The function $g'_c(r)$ (solid lines) and its leading asymptotic term (dashed lines) for $\alpha = 0.5$ and $c = 1$ (1), $c = 0.9$ (2), $c = 0.99$ (3)

remarkable fact that the larger the mean number of steps of the trajectory $(1-c)^{-1}$, the longer the region where the asymptotic formula of $g'_1(r)$ for the infinite trajectory can be approximately used. Stated differently, if the survival probability c is close to 1, formula (11.5.6) plays the role of an intermediate asymptotic expression, whose applicability interval is the larger the closer c is to 1. So, a finite trajectory can be considered as having, in some region, the properties of infinite, and it is possible to construct random point distributions with fractal properties on a large scale and with a finite density

$$\rho = \frac{\rho_0}{1-c}$$

(see Figures 11.3–11.5).

The answer to the second question is also positive. To demonstrate this, one have to return to the branching cascades (Section 11.4). In this case θ_2 is determined by (11.4.5). We choose $k^{[2]}(\mathbf{r} \to \mathbf{r}_1, \mathbf{r}_2)$ in the form

$$k^{[2]}(\mathbf{r} \to \mathbf{r}_1, \mathbf{r}_2) = c_2 k(\mathbf{r}_1 - \mathbf{r}) k(\mathbf{r}_2 - \mathbf{r});$$

then (11.4.5) takes the form

$$\begin{aligned}\theta_2(\mathbf{r}_1, \mathbf{r}_2) &= \rho \left\{ g'(\mathbf{r}_1 \to \mathbf{r}_2) + g'(\mathbf{r}_2 \to \mathbf{r}_1) + c_2 \int g'(\mathbf{r} \to \mathbf{r}_1) g'(\mathbf{r} \to \mathbf{r}_2)\, d\mathbf{r} \right\} \\ &= 2\rho g'(\mathbf{r}_{12}) + c_2 \rho g'^{[2]}(\mathbf{r}_{12}),\end{aligned}$$

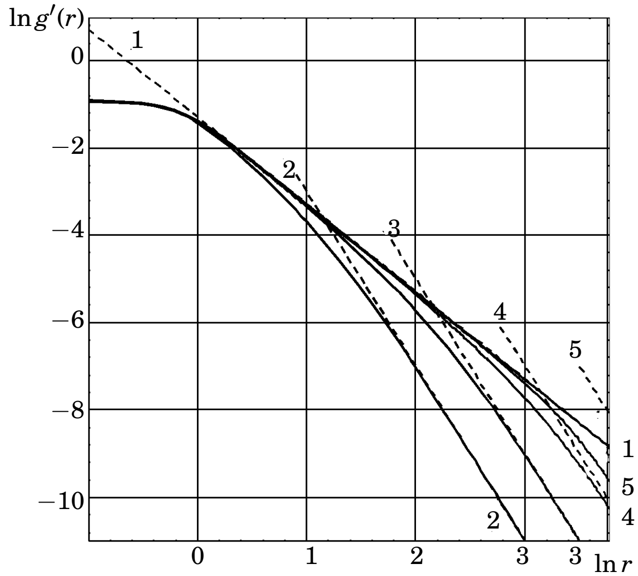


Figure 11.4. The same as in Fig. 11.3, for $\alpha = 1$ and $c = 1$ (1), $c = 1 - 10^{-1}$ (2), $c = 1 - 10^{-2}$ (3), $c = 1 - 10^{-3}$ (4), $c = 1 - 10^{-4}$ (5)

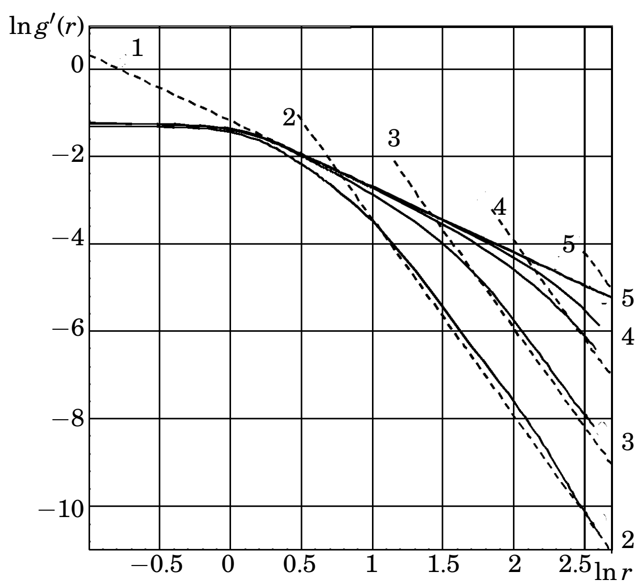


Figure 11.5. The same as in Fig. 11.4, for $\alpha = 1.5$

where the symmetry of the function $g'(\mathbf{r}_1 - \mathbf{r}_2) = g'(\mathbf{r}_{12})$ was used. If $c = 1$ and $p(\mathbf{r})$ is a stable distribution with the characteristic function e^{-k^α}, the Fourier transform of $g'(\mathbf{r})$ is of the form

$$\tilde{g}'(\mathbf{k}) = \frac{e^{-k^\alpha}}{1 - e^{-k^\alpha}}.$$

The convolution

$$g'^{(2)}(\mathbf{r}_{12}) = \int g'(\mathbf{r} \to \mathbf{r}_1) g'(\mathbf{r} \to \mathbf{r}_2)\, d\mathbf{r}$$

has the squared Fourier transform

$$\tilde{g}'^{(2)}(\mathbf{k}) = \frac{e^{-2k^\alpha}}{(1 - e^{-k^\alpha})^2} \sim k^{-2\alpha} e^{-2k^\alpha}, \qquad k \to 0.$$

Thus, for large r we obtain

$$\begin{aligned} g'^{(2)}(\mathbf{r}) &\sim \frac{2}{(2\pi)^2 r^2} \int_0^\infty \sin kr e^{-2k^\alpha} k^{1-2\alpha} dk \\ &= \frac{1}{2\pi^2} \Gamma(2(1-\alpha)) \sin(\alpha\pi) r^{2\alpha-3}, \qquad r \to \infty. \end{aligned}$$

Here α, as before, belongs to the interval $(0, 2)$, and in the case where $\alpha \in (1, 3/2)$ we obtain the fractal dimension $D = 2\alpha \in (2, 3)$.

Therefore, to construct random point distributions with the fractal dimension $D > 2$, one have to use the branching Lévy walk process with $\alpha = D/2$ (Uchaikin *et al.*, 1998b).

In conclusion, we give some results of numerical simulations of two-dimensional point distributions (Figures 11.6–11.9). The reader can see the difference between the Poisson distribution and the stochastic fractal with $\alpha = 1$ as well as the fractal with turnover to homogeneity at large scales.

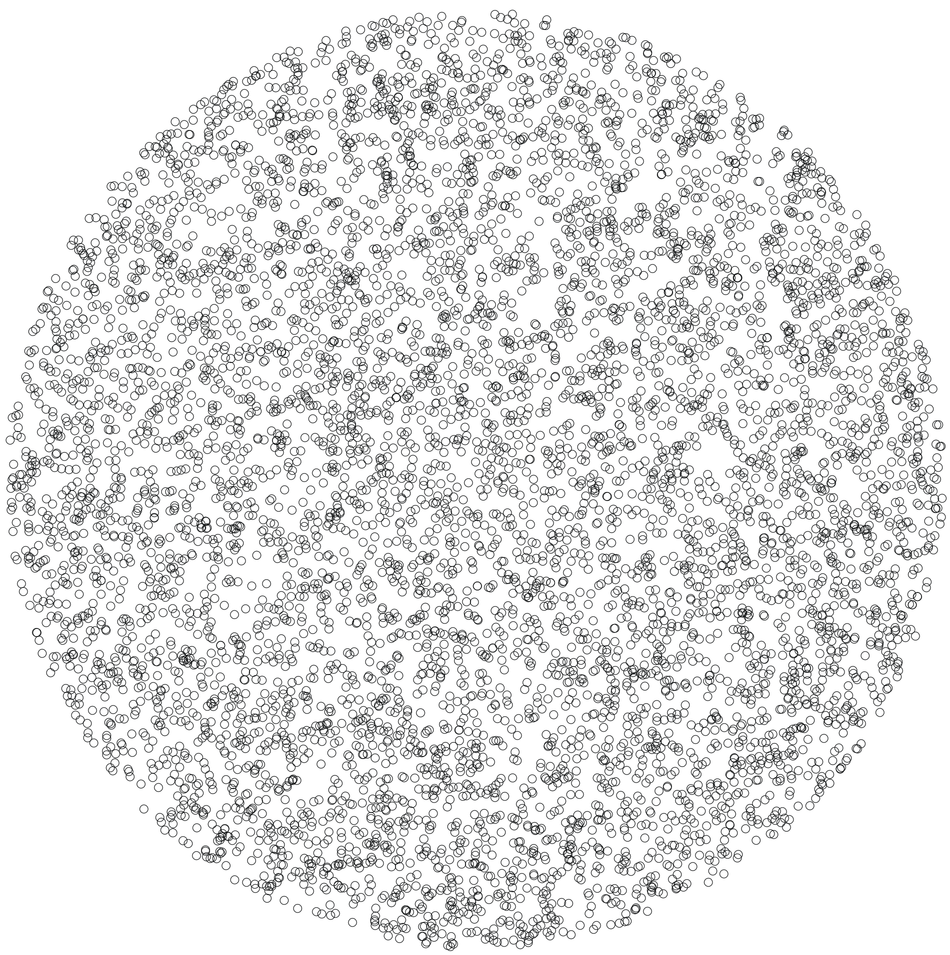


Figure 11.6. Two-dimensional point distribution: Poisson's ensemble

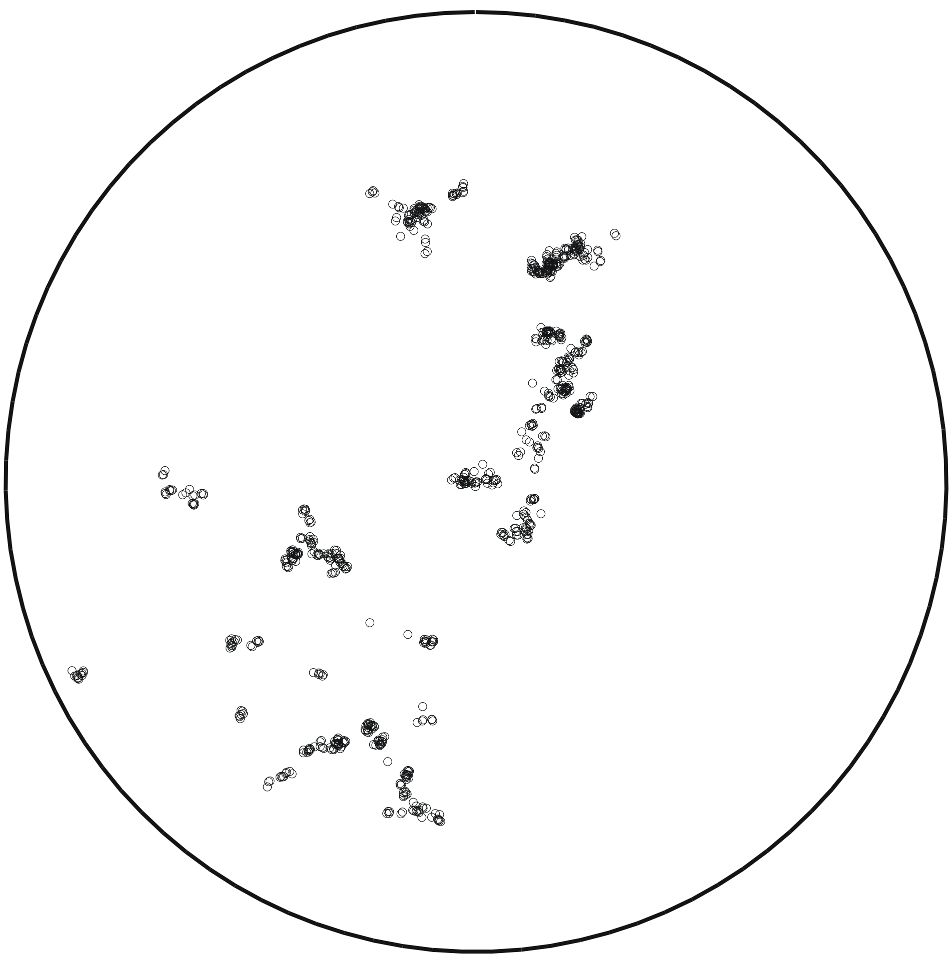


Figure 11.7. Two-dimensional point fractal distribution with $\alpha = 1$

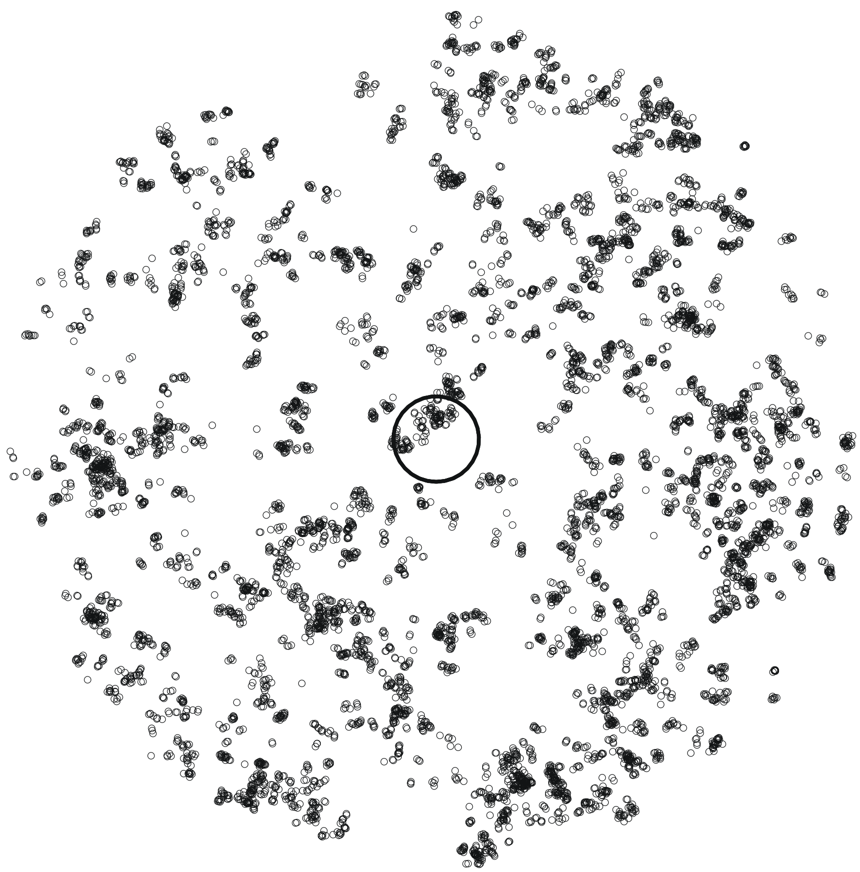


Figure 11.8. Two-dimensional fractal set with a turnover to homogeneity ($\alpha = 1$).

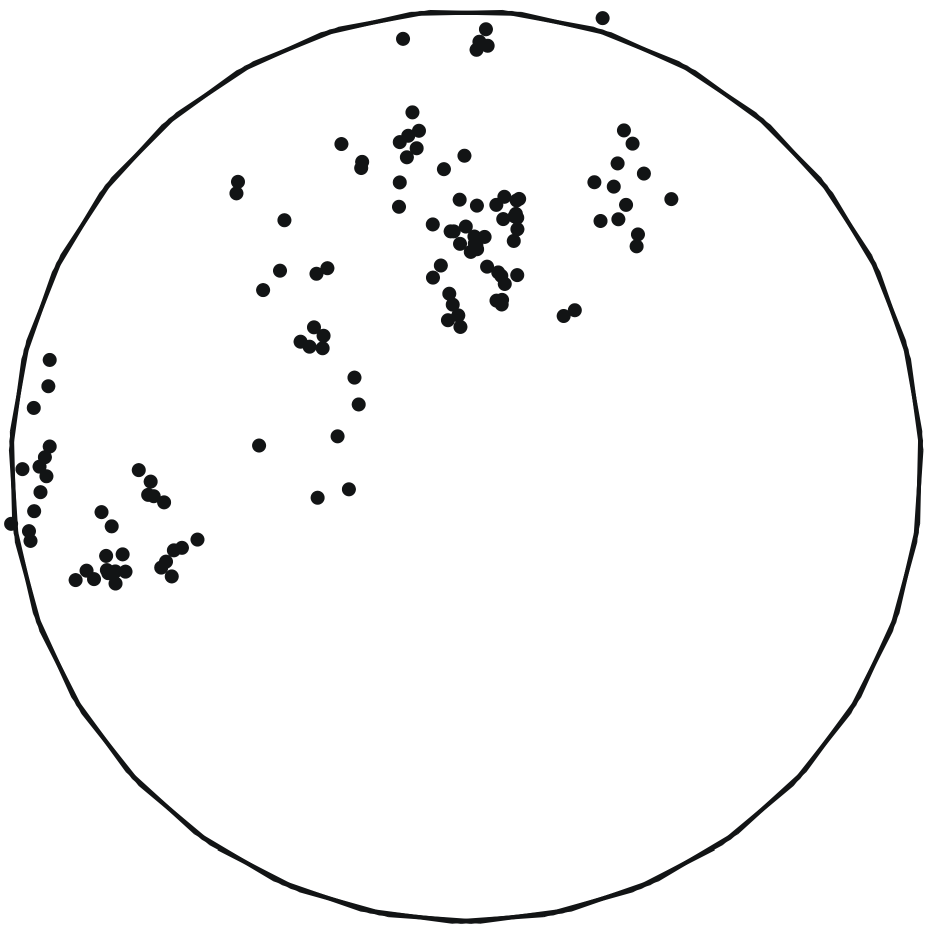


Figure 11.9. The part of the distribution in Fig. 11.8 (encircled) looks like fractal (Fig. 11.7).

12

Anomalous diffusion and chaos

12.1. Introduction

As we have seen above, the classical diffusion in a three-dimensional homogeneous medium is described by the equation

$$\frac{\partial p(\mathbf{r},t)}{\partial t} = D\Delta p(\mathbf{r},t). \qquad (12.1.1)$$

under the initial condition

$$p(\mathbf{r},0) = \delta(\mathbf{r}); \qquad (12.1.2)$$

its solution possesses the two important properties:

(1) automodelling (scaling), i.e.,

$$p(\mathbf{r},t) = (Dt)^{-3/2} q((Dt)^{-1/2}\mathbf{r}); \qquad (12.1.3)$$

(2) the diffusion packet width $\Delta = \sqrt{\langle \mathbf{X}^2 \rangle}$ grows as $t^{1/2}$.

Non-homogeneities of the medium where the process goes can have two types of effects on diffusion properties:

- it may affect only the diffusivity D as compared with the homogeneous medium; or
- it may alter in various ways the laws of diffusion themselves, e.g. the diffusion packet width may no longer grow in time as $t^{1/2}$, and its shape may no longer be described by the Gaussian (8.4.10). In the last case, we deal with anomalous diffusion.

In actual reality, there exists a large number of processes where the width of diffusion packet grows in time faster or slower than $t^{1/2}$. As a rule, this width grows as a power

$$\Delta(t) \propto t^{\nu}, \tag{12.1.4}$$

but

$$\nu \neq 1/2. \tag{12.1.5}$$

Slowed-up diffusion (subdiffusion) with $\nu < 1/2$ occurs in disordered materials and trapping phenomenon in condensed matter physics. Enhanced diffusion (superdiffusion) with $\nu > 1/2$ arises in such cases as phase diffusion in the chaotic regime of a Josephson junction, chaos-induced turbulent diffusion, the relation between the root-mean-square characteristic length of a polymer and the number of monomer units, diffusion of a Brownian particle in a pure shear flow as well as in a turbulent flow field and so on (see reviews (Isichenko, 1992; Bouchaud & Georges, 1990; Klafter *et al.*, 1996; Shlesinger *et al.*, 1993) and references therein).

There exist two approaches to the description of such processes. One of them is based on Gibbs' concept of a statistical ensemble. It assumes that the inhomogeneous medium with diffusivity $D(\mathbf{r})$ varying irregularly in space is one of a large number of copies forming the statistical ensemble. In this case, the solution of the equation

$$\frac{\partial p(\mathbf{r},t)}{\partial t} = \nabla(D(\mathbf{r})\nabla p(\mathbf{r},t))$$

is thought of as a random function of $\mathbf{r}$, t; for comparing with data observed, the ensemble averaging should be performed:

$$\frac{\partial}{\partial t}\langle\langle p(\mathbf{r},t)\rangle\rangle = \nabla\langle\langle D(\mathbf{r})\nabla p(\mathbf{r},t)\rangle\rangle.$$

To obtain $\langle\langle p(\mathbf{r},t)\rangle\rangle$ on the right-hand side of the equation, one usually uses the perturbation theory or other approximation technique.

Thus, the anomalous diffusion is interpreted as a superposition of ordinary diffusion processes in inhomogeneous media (the dispersive theory).

The second approach assumes that anomalous transport properties arise on microscopical scales, and describes them in terms of a walk involving random free paths and random waiting times in traps. On this assumption, the superdiffusion arises as a result of a broad distribution of free paths (Lévy flight), whereas the subdiffusion is caused by a broad distribution of waiting times (Lévy traps).

We consider here the second approach known as the CTRW (continuous time random walk) theory. But before proceeding, we give two illustrative examples of anomalous diffusion.

12.2. Two examples of anomalous diffusion

The first of them is the diffusion of passive particles in a turbulent flow field investigated by Monin (Monin, 1955; Monin, 1956; Monin & Yaglom, 1975). A pair of particles placed at the points $\mathbf{x}_1(0)$ and $\mathbf{x}_2(0)$ at the moment $t = 0$ is considered, and the distribution $p(\mathbf{r}, t)$ of the relative vector $\mathbf{R}(t) = \mathbf{X}_1(t) - \mathbf{X}_2(t)$ is calculated according to the integral equation

$$p(\mathbf{r}, t) = \int A(\mathbf{r} - \mathbf{r}', t) p(\mathbf{r}', 0)\, d\mathbf{r}'. \tag{12.2.1}$$

For the particles which are initially very close to each other, we take

$$p(\mathbf{r}, 0) = \delta(\mathbf{r}),$$

and the Fourier transformation of (12.2.1) yields

$$f(\mathbf{k}, t) = A(\mathbf{k}, t) f(\mathbf{k}, 0) = A(\mathbf{k}, t) = \int e^{i\mathbf{kr}} A(\mathbf{r}, t)\, d\mathbf{r}.$$

Assuming that

(1) $A(\mathbf{k}, t) = a(ck^{2/3}t)$ in a quasi-asymptotic time domain ($t_{\min} < t < t_{\max}$);

(2) the integral operators $\hat{A}_t$ with kernels $A(\mathbf{r}, t)$ generate a semigroup:

$$\hat{A}_{t_1}\hat{A}_{t_2} = \hat{A}_{t_1+t_2},$$

Monin obtained

$$f(\mathbf{k}, t) = e^{-c|\mathbf{k}|^{2/3}t}. \tag{12.2.2}$$

This is nothing but the characteristic function of the spherically symmetric stable distribution with $\alpha = 2/3$. Function (12.2.2) satisfies the equation

$$\frac{\partial f}{\partial t} = -c|\mathbf{k}|^{2/3} f.$$

Rewriting the corresponding equation for $p(\mathbf{r}, t)$ in the form

$$\frac{\partial p}{\partial t} = \hat{L} p \tag{12.2.3}$$

Monin interpreted $\hat{L}$ as a linear operator 'proportional to Laplacian raised to the power 1/3' (Monin & Yaglom, 1975).

As we know, the second moment of the distribution diverges; but if we redefine the width Δ by virtue of a given probability $\mathsf{P}\{|\mathbf{X}| < \Delta\} = \text{const}$, then we arrive at the superdiffusion mode

$$\Delta(t) \propto t^{3/2}.$$

The second example is taken from (Akhiezer *et al.*, 1991) dealing with multiple scattering of charged particles in a crystal. In the case under studying, unlike an amorphous medium, multiple scattering occurs by atomic groups rather than by single atoms, i.e., by crystal atom strings located parallel to the crystallographic axis near which a particle moves. The process is studied using the model of random strings, where it is assumed that the representation of particle scattering in the periodic field of atom strings can be replaced by that of collisions with irregularly located but, nevertheless, parallel atom strings. Besides, an elementary object defining the interaction of a particle with a crystal is its interaction with a single atom string, whereas the interaction with different atom strings can be considered using the methods of statistical physics.

When a fast particle collides with an atom string, its scattering occurs mainly along the azimuthal angle φ in the (x, y) plane orthogonal to the string axis (the z-axis). As a consequence of multiple scattering by different strings, the particles are redistributed over the angle φ (Fig. 12.1). We denote the particle distribution density in a crystal over the azimuthal angle φ at depth z as $p(\varphi, z)$,

$$\int_{-\pi}^{\pi} p(\varphi, z) d\varphi = 1.$$

To derive the distribution, the authors of (Akhiezer *et al.*, 1991) used the kinetic equation

$$\frac{\partial p(\varphi, z)}{\partial z} = na\psi \int_{-\infty}^{\infty} db[p(\varphi + \varphi(b), z) - p(\varphi, z)], \tag{12.2.4}$$

where n is the atomic density, a is the interatomic distance along the z-axis, b denotes the string impact parameter, $\varphi(b)$ is the particle deflection function in the atom string field and ψ is the angle of incidence of a beam onto a crystal with respect to the crystal axis (the z-axis).

Equation (12.2.4) was introduced in (Golovchenko, 1976; Beloshitskii & Kumakhov, 1973) to describe scattering of positively charged particles in a crystal at small angles ψ. The solution of (12.2.4) satisfying the condition $p(\varphi, 0) = \delta(\varphi)$ is of the form

$$p(\varphi, z) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \cos(k\varphi) \exp\left\{-na\psi z \int_{-\infty}^{\infty} db[1 - \cos(k\varphi(b))]\right\}. \tag{12.2.5}$$

An important numerical characteristic of particle scattering in a crystal is the mean square scattering angle

$$\langle \vartheta^2(z) \rangle = 4\psi^2 \int_{-\pi}^{\pi} d\varphi \, p(\varphi, z) \sin^2(\varphi/2).$$

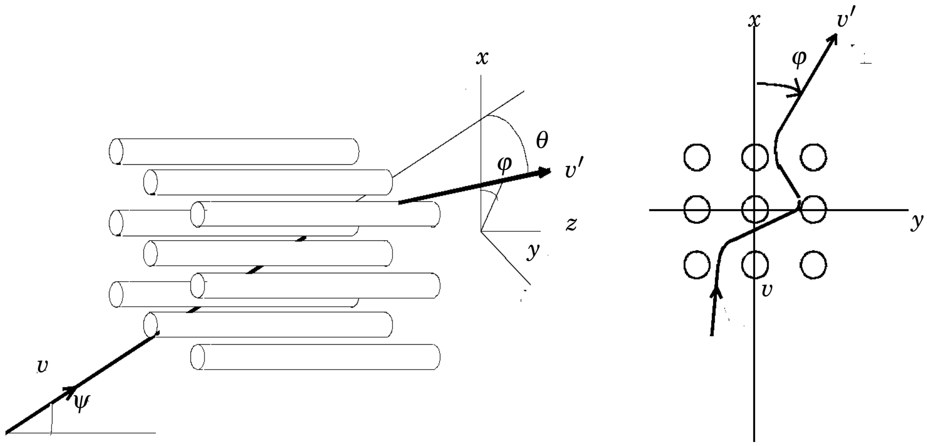


Figure 12.1. The geometry of channeling: multiple scattering of a fast particle by atom strings (taken from (Akhiezer *et al.*, 1991))

Using (12.2.5), we obtain

$$\langle \vartheta^2(z)\rangle = 2\psi \left[1 - \exp\left\{-2naz\psi \int_{-\infty}^{\infty} db\ \sin^2(\varphi(b)/2)\right\}\right]. \qquad (12.2.6)$$

If

$$\langle \varphi^2\rangle = 2naz\psi \int_0^{\infty} db\, \varphi^2(b) \ll 1,$$

then in the range of angles $\varphi \sim \langle\varphi^2\rangle^{1/2}$ or less, the sum in (12.2.5) can be replaced by an integral. In this case, the particle distribution over the angle φ will be of Gaussian form

$$p(\varphi, z) = \frac{1}{\sqrt{2\pi\langle\varphi^2\rangle}} \exp\left\{\frac{\varphi^2}{2\langle\varphi^2\rangle}\right\}. \qquad (12.2.7)$$

Fig. 12.2 taken from (Akhiezer *et al.*, 1991) presents the results of calculations of the function

$$\eta(\psi, z) = \sqrt{\langle\vartheta^2\rangle/\langle\vartheta_a^2\rangle}, \qquad (12.2.8)$$

where $\langle\vartheta_a^2\rangle$ is the mean space angle of particle scattering in an amorphous (homogeneous) medium. Calculations were performed for positrons and electrons with energy $E = 30\,\mathrm{GeV}$ moving in silicon crystals of $z = 50\mu\mathrm{m}$ thickness in the vicinity of the $\langle 111\rangle$ axis.

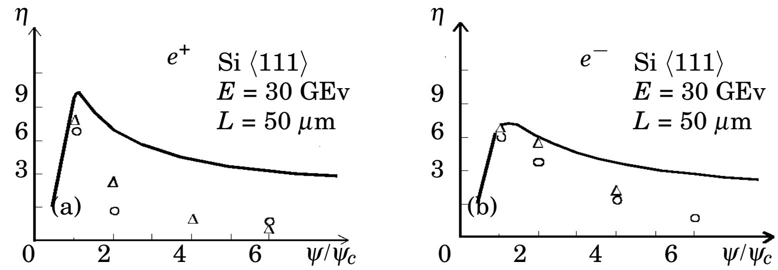


Figure 12.2. Orientation dependence of the mean value of the angle of scattering by silicon crystal atom strings for (a) positrons and (b) electrons; solid lines are the random strings approximation; the symbols △ and ○ denote the result of the numerical simulation of particle scattering in the periodic field of atom strings for different orientations of the crystal x and y axes with respect to the incident beam (taken from (Akhiezer *et al.*, 1991))

We give the details from the cited work to discuss a possible factor responsible for such a difference between the results of the numerical simulation (triangles and circles) and the random string approximation according to (12.2.4) (solid line). Let us look at Fig. 12.1 again. While a particle moves along the z-axis, its angle φ remains almost the same, or stated differently, the coordinate φ is in a trap. When the particle comes into collision with a string, the coordinate φ comes into moving. This is illustrated by (12.2.4). Moreover, its first term $\partial p(\varphi,z)/\partial z$ shows that the random free path of the particle between sequential collisions with strings is distributed exponentially. It is quite possible that the last assumption causes the observed difference, because it is well known that in the case of ordered location of spheres the distribution of free paths can have a long tail of the inverse power kind (Bouchaud & Le Doussal, 1985; Bouchaud & Georges, 1990; Bunimovich & Sinai, 1981). As a free path along z-axis is equivalent to the waiting time for φ, we arrive at the subdiffusion model producing the diffusion packet with width smaller than in the normal case (12.2.7).

12.3. Superdiffusion

The Lévy process considered in Section 10.4 seems to be a very natural generalization of the normal diffusion process to superdiffusion mode. It preserves the first property of the diffusion process noted at the beginning of Section 12.1 and replaces the second one by the relation $\Delta \propto t^{1/\alpha}$ which for $\alpha \leq 2$ characterizes the superdiffusion modes. The shape of the diffusion packet becomes different from the Gaussian one, but remains to be of stable distribution type.

However, there exists some problem which we are going to discuss here.

The problem is that in the case $\alpha < 1$ the width of the diffusion packet grows faster than in the ballistic regime (i.e., in free motion with finite speed). This obviously non-physical result is caused by the self-similarity of the Lévy process in which there is no place for the concept of a free motion velocity. And the matter is not only in the condition $\alpha < 1$ when this effect becomes more bright, it is exhibited up to the limiting value $\alpha = 2$. At any time as much as close to the initial time (when the particle is at the origin of coordinates), the distribution $p(\mathbf{r}, t)$ is different from zero in the whole space (in the normal diffusion theory this 'imperfection' was noticed by Einstein).

To avoid this effect, we pass from the Lévy process to a random walk of a particle with a finite speed of a free motion v.

We consider the following model. At the time $t = 0$, the particle is at the origin $\mathbf{r} = 0$ and stays there during a random time T_0; then it travels along a random vector $\mathbf{R}_1$ at a constant velocity v and stays again in rest during a random time T_1; then the process continues in a similar way. All the random variables $T_0, \mathbf{R}_1, T_1, \mathbf{R}_2, T_2, \dots$ are independent, the times T_i have the same probability density of exponential form

$$q(t) = \mu e^{-\mu t}, \quad \mu > 0,$$

and the three-dimensional vectors $\mathbf{R}$ are also identically distributed with the density $p(\mathbf{r})$.

It is more convenient sometimes to talk about a set of independent trajectories instead of one particle, treating $p(\mathbf{r}, t)$ as a density of the number of particles.

The density $p(\mathbf{r}, t)$ consists of two components $p_0(\mathbf{r}, t)$ and $p_v(\mathbf{r}, t)$ relating the particles being in rest (in traps) or in motion respectively:

$$p(\mathbf{r}, t) = p_0(\mathbf{r}, t) + p_v(\mathbf{r}, t). \tag{12.3.1}$$

An increment of the density of $p_0(\mathbf{r}, t)$ in time dt

$$dp_0(\mathbf{r}, t) = p_0(\mathbf{r}, t + dt) - p_0(\mathbf{r}, t),$$

consists of two parts. The first part is negative, it is caused by the particles leaving traps,

$$[dp_0(\mathbf{r}, t)]_- = -\mu p_0(\mathbf{r}, t)dt,$$

The second one is positive, it is brought by particles falling down into traps:

$$[dp_0(\mathbf{r}, t)]_+ = \int d\mathbf{r}' p(\mathbf{r}')\mu p_0(\mathbf{r} - \mathbf{r}', t - r'/v)dt.$$

As a result, we come to the following equation for $p_0(\mathbf{r}, t)$:

$$\frac{\partial p_0}{\partial t} = -\mu p_0 + \mu \int d\mathbf{r}' p(\mathbf{r}') p_0(\mathbf{r} - \mathbf{r}', t - r'/v). \tag{12.3.2}$$

We recall that $p(\mathbf{r})d\mathbf{r}$ is the probability for the particle leaving trap at the origin of coordinates to undergo the first collision in the volume $d\mathbf{r} = dS \cdot dr$. Let $P(\mathbf{r})dS$ be the probability that the particle crosses the element dS of the sphere of radius r without interaction on a path r. The contribution to the density p_v of the particle is equal to $(1/v)P(\mathbf{r})\delta(t - r/v)$. Replacing here $\mathbf{r}$ by $\mathbf{r}'$ and applying this result to all the particles leaving traps and occurring at time t at the point $\mathbf{r}$, we obtain

$$p_v(\mathbf{r}, t) = \frac{1}{v}\int d\mathbf{r}' \int dt' P(\mathbf{r}')\delta(t' - r'/v)\mu p_0(\mathbf{r} - \mathbf{r}', t - t')$$
$$= \frac{1}{v}\int d\mathbf{r}' P(\mathbf{r}')\mu p_0(\mathbf{r} - \mathbf{r}', t - r/v'). \qquad (12.3.3)$$

In the three-dimensional space with exponential distribution of r, equations (12.3.2)–(12.3.3) describe the non-stationary transport of neutrons with delay and, to within some details (absence of absorption and scattering, constancy of a velocity), are equivalent to equations (1.13)–(1.14) of (Shykhov, 1973). If $\mu \to 0$, then $\rho_0 \to 0$, and (12.3.3) turns into time-dependent one-velocity kinetic equation with isotropic scattering widely used in neutron physics (Beckurts & Wirtz, 1964; Davison, 1957; Case & Zweifel, 1967). So, we call (12.3.2)–(12.3.3) the kinetic equations.

In the limit $v = \infty$, another term of sum (12.3.1) remains different from zero:

$$p(\mathbf{r}, t) = p_0(\mathbf{r}, t).$$

It satisfies the Kolmogorov equation

$$\frac{\partial p}{\partial t} = -\mu p + \mu \int d\mathbf{r}' p(\mathbf{r}')p_0(\mathbf{r} - \mathbf{r}', t), \qquad (12.3.4)$$

describing the generalized Poisson process (Feller, 1966).

Returning to the probabilistic interpretation of the equations, we rewrite the general initial condition as

$$p(\mathbf{r}, 0) = p_0(\mathbf{r}, 0) = \delta(\mathbf{r}).$$

Applying the Fourier transformation with respect to coordinates, we transform (12.3.4) to the equation for the characteristic function $f(\mathbf{k}, t)$ of the distribution $p(\mathbf{r}, t)$:

$$\frac{\partial f}{\partial t} = -\mu\left[1 - \varphi(\mathbf{k})\right] f(\mathbf{k}, t), \qquad f(\mathbf{k}, 0) = 1. \qquad (12.3.5)$$

The solution of equation (12.3.5) is of the form

$$f(\mathbf{k}, t) = \exp\left\{-\left[1 - \varphi(\mathbf{k})\right]\mu t\right\}, \qquad (12.3.6)$$

and its asymptotic behavior as $t \to \infty$ is determined by the behavior of $f(\mathbf{k})$ for small k.

If the second moment of the distribution $p(\mathbf{r})$

$$\int p(\mathbf{r}) r^2 d\mathbf{r} = \sigma^2 \tag{12.3.7}$$

is finite, then

$$1 - \varphi(\mathbf{k}) \sim (\sigma^2/2)k^2, \qquad k \to 0,$$

and the characteristic function (12.3.6) in the domain of large t behaves as

$$f(\mathbf{k}, t) \sim f^{\mathrm{as}}(\mathbf{k}, t) = \exp\left\{-\left(\mu t \sigma^2/2\right) k^2\right\}. \tag{12.3.8}$$

Since

$$\frac{\partial f^{\mathrm{as}}(\mathbf{k}, t)}{\partial t} = -\left(\mu \sigma^2/2\right) k^2 f^{\mathrm{as}}(\mathbf{k}, t),$$

the asymptotic density $p^{\mathrm{as}}(\mathbf{r}, t)$ satisfies the ordinary diffusion equation

$$\frac{\partial p^{\mathrm{as}}}{\partial t} = D \Delta p^{\mathrm{as}}(\mathbf{r}, t)$$

with the diffusivity

$$D = \mu \sigma^2/2,$$

and the initial condition

$$p^{\mathrm{as}}(\mathbf{r}, 0) = \delta(\mathbf{r}).$$

We cite this, generally speaking, trivial fact to emphasize that the asymptotic behavior of the solution of (12.3.4) under condition (12.3.7) is an exact solution of the diffusion equation.

If the second moment (12.3.7) is infinite, but the condition

$$\int_{r>R} p(\mathbf{r}) d\mathbf{r} \sim A R^{-\alpha}, \quad R \to \infty, \quad \alpha < 2, \tag{12.3.9}$$

holds true, then

$$1 - \varphi(\mathbf{k}) \sim A' k^{\alpha}, \qquad k \to 0,$$

and we obtain

$$f(\mathbf{k}, t) \sim f^{\mathrm{as}}(\mathbf{k}, t) = \exp\left\{-\mu t A' k^{\alpha}\right\}, \qquad t \to \infty,$$

instead of (12.3.8). This expression immediately leads us to the three-dimensional spherically symmetric stable distribution

$$p^{\mathrm{as}}(\mathbf{r}, t) = (Dt)^{-3/\alpha} q_3(\mathbf{r}(Dt)^{-1/\alpha}; \alpha), \tag{12.3.10}$$

where $D = \mu A'$.

Solution (12.3.10) can be obtained also in a simpler way with the use of properties of stable laws: under condition (12.3.9) the normalized sum of a large number n of independent random vectors $\mathbf{R}_i$

$$\mathbf{S}_n = (b_1 n^{1/\alpha})^{-1} \sum_{i=1}^{n} R_i, \tag{12.3.11}$$

is distributed by the stable law with the characteristic α. The distribution of random number N of terms at time t is given by the Poisson law with the average value $\langle N \rangle = \mu t$ and the relative fluctuations $(\mu t)^{-1/2}$. So, as $\mu t \to \infty$, it is possible to replace n in (12.3.11) by μt. Because after such a replacement sum (12.3.11) turns into the random vector $\mathbf{X}(t)$ pointing to a particle position at time t, one can write

$$\mathbf{X}(t) \stackrel{d}{=} (Dt)^{1/\alpha}\mathbf{Y}(\alpha), \qquad D = b_1^{\alpha}\mu,$$

where $\mathbf{Y}(\alpha)$ is a random vector with the symmetric stable density $q_3(\mathbf{r};\alpha)$. Thus, we arrive at (12.3.10) again.

To correct for an influence of a finite velocity of free motion, it is necessary to perform the simultaneous asymptotic analysis of (12.3.2) and (12.3.3) as $t \to \infty$. Since the rigorous calculations are somewhat cumbersome, we confine the presentation to an elementary derivation leading to the same result.

In the case of a finite velocity v, the sum

$$\mathbf{S}_n = (b_1 n^{1/\alpha})^{-1} \sum_{i=1}^{n} R_i$$

takes the random time

$$\Theta = \sum_{i=1}^{n} (T_i + R_i/v).$$

In the case $\alpha > 1$, the expectation $a = \mathsf{E}R$ is finite, and for large n it is possible to set

$$\Theta \approx t = n(1/\mu + a/v) \tag{12.3.12}$$

due to the law of large numbers. Obtaining herefrom

$$n = (1 + \mu a/v)^{-1}\mu t$$

and introducing

$$t_v = (1 + \mu a/v)^{-1} t, \tag{12.3.13}$$

we arrive at (12.3.10) with t replaced by t_v:

$$p^{\text{as}}(\mathbf{r},t) = (Dt_v)^{-3/\alpha} q_3\left(\mathbf{r}(Dt_v)^{-1/\alpha};\alpha\right), \qquad \alpha > 1. \tag{12.3.14}$$

This result is physically obvious: the presence of a finite velocity of free motion decelerates the expansion of a diffusion packet as compared with the case $v = \infty$. The replacement of the time t by a smaller t_v is due to this deceleration (in an asymptotic sense).

As the diffusivity and the time enter the asymptotic density as a product, (12.3.14) can be rewritten as

$$p^{\mathrm{as}}(\mathbf{r}, t) = (D_v t)^{-3/\alpha} q_3 \left(\mathbf{r}(D_v t)^{-1/\alpha}; \alpha\right), \qquad \alpha > 1,$$

where

$$D_v = (1 + \mu a/v)^{-1} D.$$

As shown in Section 8.4, function (12.3.10) satisfies fractional differential equation (8.4.8), which allows us to write

$$\frac{\partial p^{\mathrm{as}}}{\partial t} = -D_v(-\Delta)^{\alpha/2} p^{\mathrm{as}}(\mathbf{r}, t) \tag{12.3.15}$$

provided that $v < \infty$.

Thus, the finiteness of velocity influences only the diffusivity in the equation; its solution remains a spherically symmetric stable law. But this conclusion is fair only under the condition $\alpha > 1$, which has been used to replace (12.3.12). If $\alpha < 1$, the situation is essentially different: there is no linear transformation which could transform the solution with $v < \infty$ into a solution with $v = \infty$.

The last conclusion can be easily understood. The width of a diffusion packet of a particle with $v = \infty$ grows as $t^{1/\alpha}$. A finite velocity makes the density vanish outside the sphere of radius vt. Therefore, if $\alpha > 1$, then the influence of the last (ballistic) constraint becomes weak as time grows, since the diffusion radius grows slower than the ballistic one. However, by passing to the domain $\alpha < 1$ the situation becomes inverse and the diffusion radius predominates. Being bounded by the radius vt, a solution with $v < \infty$ has a completely different form as compared with a stable distribution. Obviously, this means that equation (12.3.15) with Laplacian raised to the power $\alpha/2$ for $\alpha < 1$ is generally inapplicable to the description of real diffusion processes.

The Monte-Carlo simulation of the one-dimensional particle walk confirms the conclusion above: with $\alpha = 3/2$, the replacement of D with D_v guarantees the asymptotic goodness-of-fit of solutions of (12.3.4) and (12.3.15) whereas with $\alpha = 1/2$ the solutions are essentially different (see Figures 12.3–12.4).

Let us sum up the abovesaid.

- The superdiffusion equation describes the asymptotic behavior of a generalized Poisson process with instantaneous (jump-like) independent increments, whose absolute value is distributed with density $p(r) \propto r^{-\alpha-1}$, $0 < \alpha < 2$.

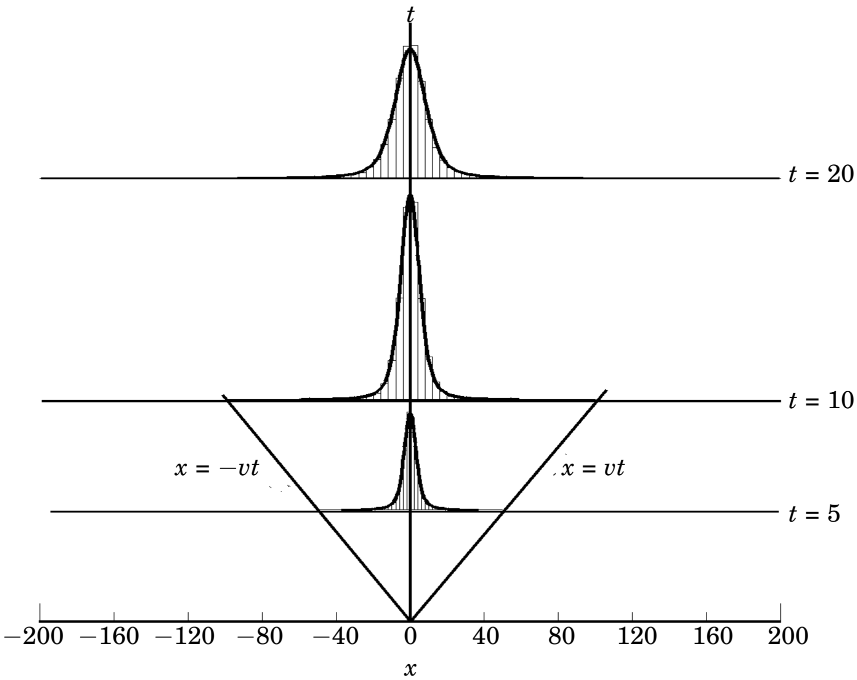


Figure 12.3. The superdiffusion packet in the case $\alpha = 3/2$. The histograms are Monte-Carlo results, the curves are the stable distributions, the straight lines show ballistic constraints

- The solutions of the equation belong to the class of spherically symmetric stable distributions $q_3(\mathbf{r}; \alpha)$.

- For $\alpha \in [1, 2]$, this equation describes also the asymptotic behavior of a walking particle with a finite free motion velocity v (provided that D is replaced by $D_v = (1 + \mu a/v)^{-1}D$, where a is the mean free path, and $1/\mu$ is the mean waiting time).

- For $\alpha < 1$, the superdiffusion packet spreads in space faster than the packet of freely moving particles, and solutions of the diffusion equation (12.3.15) and kinetic equation (12.3.4) have completely different asymptotic expressions.

The last serves as a basis for the conclusion about inapplicability of superdiffusion equations to the description of real physical processes in the domain $\alpha < 1$.

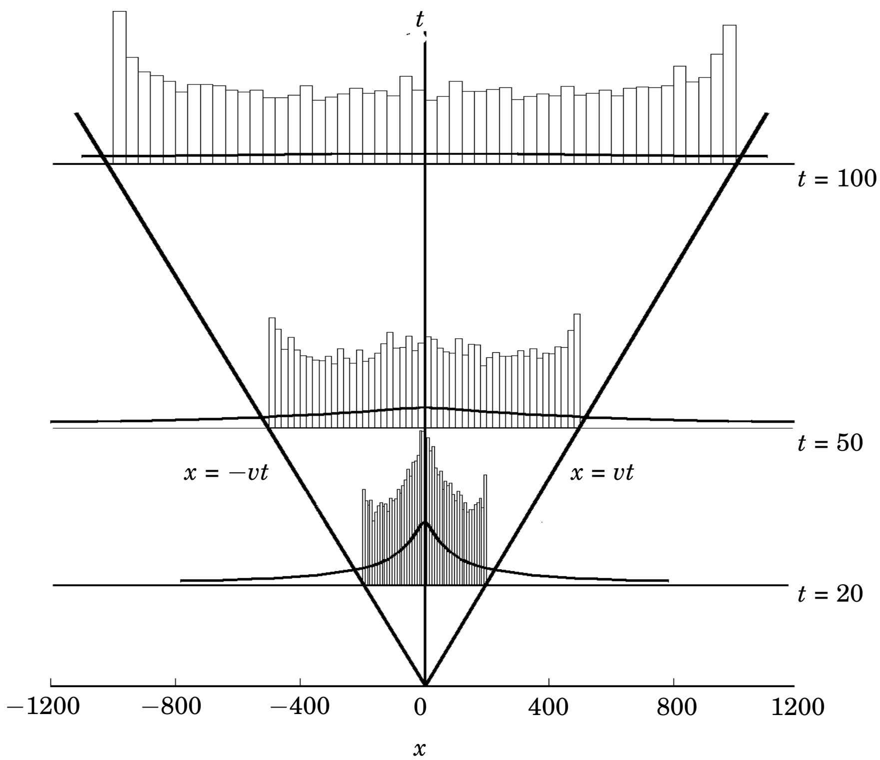


Figure 12.4. The same as in Fig. 12.3 but for $\alpha = 1/2$

12.4. Subdiffusion

Now we consider another model. As before, let there exist two possible states and let one of them be the state of rest (trapping). But in contrast to the preceding case, the other state is ordinary diffusion with diffusivity D:

$$\rho(\mathbf{r}, t) = (4\pi Dt)^{-3/2} e^{-r^2/(4Dt)}.$$

Let $q_1(t)$ be the distribution density of random diffusion time, $q_0(t)$ be the distribution of waiting time, and let both these times be independent. This process is described by the following simultaneous equations:

$$p_0(\mathbf{r}, t) = Q_0(t)\delta(\mathbf{r}) + \int_0^t d\tau\, q_0(\tau) p_1(\mathbf{r}, t - \tau), \tag{12.4.1}$$

$$p_1(\mathbf{r}, t) = Q_1(t)\rho(\mathbf{r}, t) + \int_0^t d\tau\, q_1(\tau)\rho(\mathbf{r}, \tau) * p_0(\mathbf{r}, t - \tau), \tag{12.4.2}$$

where

$$Q_i(t) = \int_t^\infty q_i(\tau) d\tau,$$

and $*$ denotes the convolution operation with respect to spatial variables:

$$\rho(\mathbf{r},\tau) * p_0(\mathbf{r},t-\tau) = \int \rho(\mathbf{r}',\tau)p_0(\mathbf{r}-\mathbf{r}',t-\tau)\,d\mathbf{r}'. \tag{12.4.3}$$

Assuming that the spatial distribution of traps is a homogeneous Poisson ensemble, i.e.,

$$q_1(\tau) = \mu e^{-\mu\tau}, \tag{12.4.4}$$

we can find necessary and sufficient conditions for the subdiffusion mode to occur. Introducing, for the sake of brevity, the notation

$$s_i(t) = \int r^2 p_i(\mathbf{r},t)\,d\mathbf{r},$$

we write the equations that follow from (12.4.1)–(12.4.2)

$$s_0(t) = \int_0^t d\tau q_0(\tau)s_1(t-\tau), \tag{12.4.5}$$

$$s_1(t) = Q_1(t)at + \int_0^t d\tau q_1(\tau)[a\tau + s_0(t-\tau)], \tag{12.4.6}$$

where

$$a = 6D. \tag{12.4.7}$$

Substituting (12.4.4) into (12.4.6) and using the Laplace transformation

$$s_i(\lambda) \equiv \tilde{s}_i(\lambda) = \int_0^\infty e^{-\lambda t}s_i(t)dt,$$

we arrive at the algebraic equations for the transforms:

$$s_0(\lambda) = q_0(\lambda)s_1(\lambda),$$
$$s_1(\lambda) = \frac{a}{\lambda(\mu+\lambda)} + \frac{\mu}{\mu+\lambda}s_0(\lambda)$$

with the solution

$$s_0(\lambda) = \frac{aq_0(\lambda)}{\lambda\left\{\lambda + \mu[1-q_0(\lambda)]\right\}}, \tag{12.4.8}$$

$$s_1(\lambda) = \frac{a}{\lambda\left\{\lambda + \mu[1-q_0(\lambda)]\right\}}. \tag{12.4.9}$$

It is worthwhile to notice that the passage to continuous ordinary diffusion is carried out by setting $\mu = 0$ in the last equation, which yields

$$s_1(\lambda) \sim \frac{a}{\lambda^2}, \quad \lambda \to 0, \quad s_1(t) \sim at, \quad t \to \infty, \tag{12.4.10}$$

which correlates well with the normal diffusion.

Due to the Tauberian theorems, the relation

$$s_i(t) \sim A_i t^{\omega}, \qquad t \to \infty, \tag{12.4.11}$$

yields

$$s_i(\lambda) \sim \Gamma(\alpha+1) A_i \lambda^{-\omega-1}, \qquad \lambda \to 0, \tag{12.4.12}$$

and vice versa. Substituting (12.4.12) into (12.4.8) and (12.4.9), and solving the equations obtained for $1 - q_0(\lambda)$, we arrive at the necessary condition for the subdiffusion to occur:

$$1 - q_0(\lambda) \sim b\lambda^{\omega}, \quad \lambda \to 0, \quad b = \frac{a}{\mu\Gamma(\omega+1)A}, \tag{12.4.13}$$

and $A_1 = A_2 = A$ (the asymptotic behavior of a subdiffusion package width does not depend on the initial state of a particle). By virtue of invertibility of the Tauberian theorems, condition (12.4.13) is sufficient as well.

To reformulate the condition for the density $q_0(\tau)$, we again call for the Tauberian theorems and apply them to the functions $Q_0(t)$ and $Q_0(\lambda) = (1 - q_0(\lambda))/\lambda$. We obtain

$$Q_0(t) = \int_t^{\infty} q_0(\tau)d\tau \sim Bt^{-\omega}, \quad t \to \infty, \quad B = \frac{a}{\mu[\Gamma(1-\alpha)]^2 A}$$

or, for the density,

$$q_0(t) \sim \omega B t^{-\omega-1}, \qquad t \to \infty. \tag{12.4.14}$$

Thus, in the model with exponential distribution of diffusion times (12.4.4) subdiffusion arises in the case where the waiting time distribution has inverse power tail (12.4.14) with $\omega < 1$. In particular, this means that the average waiting time should be infinite:

$$\int_0^{\infty} \tau q_0(\tau)d\tau = \infty, \qquad \omega < 1. \tag{12.4.15}$$

If it is finite,

$$\int_0^{\infty} \tau q_0(\tau)d\tau \equiv \tau_0,$$

then $q_0(\lambda)$ behaves as follows:

$$q_0(\lambda) \sim 1 - \tau_0\lambda, \qquad \lambda \to 0. \tag{12.4.16}$$

Substituting (12.4.16) into (12.4.9), we see that in this case

$$s_1(\lambda) \sim \frac{a}{\lambda^2[1+\mu\tau_0]}, \qquad \lambda \to 0,$$

and the trapping effect reduces only the diffusivity factor

$$D \to \frac{D}{1 + \mu\tau_0},$$

keeping the packet form unchanged.

To investigate the spatial probability distribution in a subdiffusion packet we apply the Fourier–Laplace transformation

$$p_i(\mathbf{k}, \lambda) = \int_0^\infty dt \int d\mathbf{r}\ \exp\{-\lambda t + i\mathbf{kr}\} p_i(\mathbf{r}, t) \qquad (12.4.17)$$

to (12.4.1) and (12.4.2), which yields

$$p_0(\mathbf{k}, \lambda) = [1 - q_0(\lambda)]/\lambda + q_0(\lambda) p_1(\mathbf{k}, \lambda),$$
$$p_1(\mathbf{k}, \lambda) = \rho(\mathbf{k}, \lambda + \mu)[1 + \mu p_0(\mathbf{k}, \lambda)].$$

The solution is

$$p_0(\mathbf{k}, \lambda) = \frac{1 - q_0(\lambda)[1 - \lambda\rho(\mathbf{k}, \lambda + \mu)]}{\lambda[1 - \mu q_0(\lambda)\rho(\mathbf{k}, \lambda + \mu)]}, \qquad (12.4.18)$$

$$p_1(\mathbf{k}, \lambda) = \frac{\rho(\mathbf{k}, \lambda + \mu)\{\lambda + \mu[1 - q_0(\lambda)]\}}{\lambda[1 - \mu q_0(\lambda)\rho(\mathbf{k}, \lambda + \mu)]}. \qquad (12.4.19)$$

We recall that

$$\rho(\mathbf{k}, t) = e^{-k^2 D t};$$

hence

$$\rho(\mathbf{k}, \lambda + \mu) = \frac{1}{\lambda + \mu + Dk^2}.$$

Using condition (12.4.13) in formulae (12.4.18), (12.4.19), for the leading asymptotic terms we obtain the expression

$$p^{\mathrm{as}}(\mathbf{k}, \lambda) = \frac{\lambda^\omega}{\lambda[D' k^2 + \lambda^\omega]}, \qquad D' = D/(\mu b), \qquad (12.4.20)$$

regardless of the initial conditions. Now, inverting transformation (12.4.17) and changing the integration variables, we see that the density $p^{\mathrm{as}}(\mathbf{r}, t)$ is expressed in a scalable way:

$$p^{\mathrm{as}}(\mathbf{r}, t) = (D' t^\omega)^{-3/2} \Psi(\mathbf{r}(D' t^\omega)^{-1/2}), \qquad (12.4.21)$$

where

$$\Psi(\mathbf{r}) = \frac{1}{(2\pi)^4 i} \int d\mathbf{k} \int d\lambda \frac{\lambda^{\omega-1} \exp\{\lambda - i\mathbf{kr}\}}{\lambda^\omega + k^2}. \qquad (12.4.22)$$

We issue the identity

$$\frac{1}{\lambda^{\omega} + k^2} = \int_0^{\infty} e^{-[\lambda^{\omega}+k^2]\xi} d\xi.$$

With this, we bring (12.4.22) to the form

$$\Psi(\mathbf{r}) = \frac{1}{(2\pi)^4 i} \int_0^{\infty} d\xi \int d\mathbf{k}\, e^{-i\mathbf{kr}-k^2\xi} \int e^{\lambda-\xi\lambda^{\omega}} \lambda^{\omega-1} d\lambda$$
$$= \frac{1}{2\pi i} \int_0^{\infty} d\xi \xi^{-3/2} q_3(\mathbf{r}\xi^{-1/2}; 2) \int e^{\lambda-\xi\lambda^{\omega}} \lambda^{\omega-1} d\lambda.$$

Calculating the inner integral by parts

$$J \equiv \int e^{\lambda-\xi\lambda^{\omega}} \lambda^{\omega-1} d\lambda = \frac{1}{\xi\omega} \int e^{\lambda-\xi\lambda^{\omega}} d\lambda$$

we obtain

$$\Psi(\mathbf{r}) = \frac{1}{\omega 2\pi i} \int d\lambda e^{\lambda} \int_0^{\infty} d\xi e^{-\xi\lambda^{\omega}} \xi^{-5/2} q_3(\mathbf{r}\xi^{-1/2}; 2).$$

The subdiffusion distribution (12.4.21) can be derived, as well as the usual diffusion distribution, by simple probabilistic reasoning based on the use of the limit theorem in its generalized form. If the independent random variables T_i are distributed with density $g_0(t)$ satisfying (12.4.14), the normalized sum

$$S_n = \sum_{i=1}^{n} T_i / \left[nB\Gamma(1-\omega)\right]^{1/\omega}$$

for large n is distributed with the density $q_B(t; \omega, 1)$. In other words, the density $q_0^{(n)}(t)$ for the sum $\sum_{i=1}^n T_i$ of a large number of terms n has the asymptotic form

$$q_0^{(n)}(t) \sim [nB^*]^{-1/\omega} q_B\left((nB^*)^{-1/\omega} t; \omega, 1\right),$$

where

$$B^* = B\Gamma(1-\omega).$$

Neglecting the diffusion time of particle while evaluating the distribution of N as the observation time $t \to \infty$, we obtain

$$\mathsf{P}\{N = n\} \approx Q_0^{(n)}(t) - Q_0^{(n+1)}(t)$$
$$= G_B\left((nB^*)^{-1/\omega} t; \omega, 1\right) - G_B\left(\left[(n+1)B^*\right]^{-1/\omega} t; \omega, 1\right).$$

Representing the argument of the subtrahend as

$$\left[(n+1)B^*\right]^{-1/\omega} t = \left[nB^*\right]^{-1/\omega} t - \left[nB^*\right]^{-1/\omega} t(n\omega)^{-1}$$

and expanding it into a series, we arrive at the asymptotical expression

$$\mathsf{P}\{N = n\} \sim \left[nB^*\right]^{-1/\omega} t(n\omega)^{-1} q^A \left(\left[nB^*\right]^{-1/\omega} t; \alpha, 1\right), \qquad t \to \infty.$$

With fixed $N = n$, the conditional distribution of the particle coordinate is expressed in terms of the ordinary diffusion density by the relation

$$p(\mathbf{r}, t \mid n) \sim p(\mathbf{r}, n/\mu)$$

Here the random time of diffusion is replaced by the average value n/μ for reasons well understood. Averaging now over the number of continuous diffusion acts

$$p(\mathbf{r}, t) = \sum_n p(\mathbf{r}, t \mid n)\mathsf{P}\{\nu = n\}$$

and passing from summation over n to integration over $\xi = \left[nB^*\right]^{-1/\omega} t$, we arrive at the same result (12.4.21).

The obtained distribution possesses all spatial moments expressible in an explicit form. Let

$$m_{2n}(t) = \int r^{2n} p^{\mathrm{as}}(\mathbf{r}, t)\, d\mathbf{r},$$
$$M_{2n} = \int r^{2n} \Psi(\mathbf{r})\, d\mathbf{r}.$$

It follows from (12.4.21) that they are inter-related as

$$m_{2n}(t) = (D' t^{\omega})^n M_{2n},$$

where

$$M_{2n} = \int_0^\infty q_B(t; \omega, 1) t^{-n\omega} dt \int r^{2n} q_3(\mathbf{r}; 2)\, d\mathbf{r}$$

is the product of moments of the two stable distributions $q_B(t)$ and $q_3(\mathbf{r}; 2)$. The negative moments of $q_B(t; \omega, 1)$ exist and are

$$\int_0^\infty t^{-n\omega} q_B(t; \omega, 1) dt = \frac{\Gamma(n)}{\omega\Gamma(n\omega)}.$$

For the other moments,

$$\int r^{2n} q_3(\mathbf{r}; 2) d\mathbf{r} = \frac{2^{2n+1}}{\sqrt{\pi}} \Gamma(n + 3/2);$$

hence

$$m_{2n}(t) = \frac{2^{2n+1}\Gamma(n)\Gamma(n + 3/2) D'^n}{\sqrt{\pi}\omega\Gamma(n\omega)} t^{\omega n}.$$

12.5. CTRW equations

We saw above that the superdiffusion and subdiffusion regimes arise as large time asymptotics from some walk processes. Following this way, we consider now a more general case of walk with arbitrary distributions of free path and waiting time of a walking particle. This approach is known as the CTRW (continuous-time random walks) theory (Shlesinger *et al.*, 1982; Montroll & Shlesinger, 1982b; Montroll & West, 1979; Montroll & Scher, 1973).

A walk in three-dimensional space which we are going to consider is defined in the following way.

(1) There exist only two possible states of a particle: the state of rest (trapping) $i = 0$ and the state of moving $i = 1$ with a constant speed v (flight): $v_0 = 0$, $v_1 = v$.

(2) The external source produces only one particle at the origin $\mathbf{r} = 0$ at the time $t = 0$. The particle is born in the state of rest with the probability p_0 and in the state of moving with the probability p_1. In the last case, the angular distribution of the particles satisfies the probability density $W(\mathbf{\Omega})$:

$$\int W(\mathbf{\Omega})\, d\mathbf{\Omega} = 1. \qquad (12.5.1)$$

(3) The free path distribution for the state of moving and the waiting time distribution for the state of rest are given by the densities $p(\xi)$ and $q(\tau)$ respectively.

(4) After each collision, the particle is trapped with probability $c \le 1$, or it is immediately scattered the probability $1 - c$:

$$p_{01} = c, \quad p_{11} = 1 - c, \quad p_{10} = 1, \quad p_{00} = 0.$$

(5) The random direction $\mathbf{\Omega}$ of the scattered or trap-leaving particle is distributed as the primary direction

$$W_{11}(\mathbf{\Omega}' \to \mathbf{\Omega}) = W_{10}(\mathbf{\Omega}' \to \mathbf{\Omega}) = W(\mathbf{\Omega})$$

no matter what the preceding direction $\mathbf{\Omega}'$ was.

We are interested now in the function $p(\mathbf{r}, t)$ giving the probability distribution of the particle position $\mathbf{r}$ at time t. In order to obtain the desired equations, we consider first the two-group model with slow (v_0) and fast ($v_1 = v$) states of the particle and then let $v_0 \to 0$. Taking

$$W_{01}(\mathbf{\Omega}' \to \mathbf{\Omega}) = W(\mathbf{\Omega})$$

for slow particles and using assumptions (2)–(5) for them, we obtain

$$p(\mathbf{r},t) = \int d\mathbf{\Omega} \left\{ \int_0^t d\tau Q(\tau) f_0(\mathbf{r} - v_0\mathbf{\Omega}\tau, \mathbf{\Omega}, t-\tau) + (1/v) \int_0^{vt} d\xi\, P(\xi) f_1(\mathbf{r} - \mathbf{\Omega}\xi, \mathbf{\Omega}, t - \xi/v) \right\}, \tag{12.5.2}$$

$$f_0(\mathbf{r},\mathbf{\Omega},t) = \left\{ c \int_0^{vt} d\xi p(\xi) \int d\mathbf{\Omega}' f_1(\mathbf{r} - \mathbf{\Omega}'\xi, \mathbf{\Omega}', t - \xi/v) + p_0 \delta(\mathbf{r})\delta(t) \right\} W(\mathbf{\Omega}), \tag{12.5.3}$$

$$f_1(\mathbf{r},\mathbf{\Omega},t) = \left\{ (1-c) \int_0^{vt} d\xi p(\xi) \int d\mathbf{\Omega}' f_1(\mathbf{r} - \mathbf{\Omega}'\xi, \mathbf{\Omega}', t - \xi/v) + \int_0^t d\tau q(\tau) \int d\mathbf{\Omega}' f_0(\mathbf{r} - v_0\mathbf{\Omega}'\tau, \mathbf{\Omega}', t-\tau) + p_1 \delta(\mathbf{r})\delta(t) \right\} W(\mathbf{\Omega}). \tag{12.5.4}$$

Denoting the braced functions by $F_0(\mathbf{r},t)$ and $F_1(\mathbf{r},t)$ respectively, we write

$$f_i(\mathbf{r},\mathbf{\Omega},t) = F_i(\mathbf{r},t) W(\mathbf{\Omega}).$$

Substitution of this expression into (12.5.2) yields

$$p(\mathbf{r},t) = \int_0^t d\tau \int d\mathbf{\Omega} Q(\tau) W(\mathbf{\Omega}) F_0(\mathbf{r} - v_0\mathbf{\Omega}\tau, t-\tau) + (1/v) \int_0^{vt} d\xi \int d\mathbf{\Omega} P(\xi) W(\mathbf{\Omega}) F_1(\mathbf{r} - \mathbf{\Omega}\xi, t - \xi/v).$$

On the other hand, the functions $F_i(\mathbf{r},t)$ satisfy the equations following from (12.5.3)–(12.5.4):

$$F_0(\mathbf{r},t) = c \int_0^{vt} d\xi\, p(\xi) \int d\mathbf{\Omega} W(\mathbf{\Omega}) F_1(\mathbf{r} - \mathbf{\Omega}\xi, t - \xi/v) + p_0\delta(\mathbf{r})\delta(t),$$

$$F_1(\mathbf{r},t) = (1-c) \int_0^{vt} d\xi\, p(\xi) \int d\mathbf{\Omega} W(\mathbf{\Omega}) F_1(\mathbf{r} - \mathbf{\Omega}\xi, t - \xi/v) + \int_0^t d\tau\, q(\tau) \int d\mathbf{\Omega} W(\mathbf{\Omega}) F_0(\mathbf{r} - v_0\mathbf{\Omega}\tau, t - \tau) + p_1\delta(\mathbf{r})\delta(t).$$

Letting $v_0 \to 0$ and taking normalization (12.5.1) into account, we arrive at the final result

$$p(\mathbf{r},t) = \int_0^t d\tau Q(\tau) F_0(\mathbf{r}, t-\tau) + (1/v) \int d\mathbf{r}' P(\mathbf{r}') F_1(\mathbf{r} - \mathbf{r}', t - r'/v), \tag{12.5.5}$$

$$F_0(\mathbf{r},t) = c \int d\mathbf{r}' p(\mathbf{r}') F_1(\mathbf{r} - \mathbf{r}', t - r'/v) + p_0\delta(\mathbf{r})\delta(t), \tag{12.5.6}$$

$$F_1(\mathbf{r},t) = (1-c) \int d\mathbf{r}' p(\mathbf{r}') F_1(\mathbf{r} - \mathbf{r}', t - r'/v) + \int_0^t d\tau q(\tau) F_0(\mathbf{r}, t-\tau) + p_1\delta(\mathbf{r})\delta(t), \tag{12.5.7}$$

where

$$P(\mathbf{r}') = P(\xi)W(\mathbf{r}'/\xi)/\xi^2, \quad p(\mathbf{r}') = p(\xi)W(\mathbf{r}'/\xi)/\xi^2, \quad \xi = |\mathbf{r}'|,$$

and $F_i(\mathbf{r}, t)$ vanish for $t < 0$.

In order to reduce the equations to the one-dimensional case, we have to take

$$W(\mathbf{\Omega}) = c_1\delta(\mathbf{\Omega} - \mathbf{e}) + c_2\delta(\mathbf{\Omega} + \mathbf{e})$$

where $c_1,\ c_2 > 0$, $c_1 + c_2 = 1$ and $\mathbf{e}$ is the unit vector of the x-axis along which the particle is walking. In the symmetric case $c_1 = c_2 = 1/2$, and we obtain

$$p(\mathbf{r}') = (1/2)p(\xi)\delta(y')\delta(z'), \qquad \xi = |x'|.$$

Introducing the one-dimensional density

$$p(x,t) = \iint_{-\infty}^{\infty} p(x,y,z,t)\,dy\,dz,$$

from (12.5.5)–(12.5.7) we obtain

$$p(x,t) = \int_0^t d\tau\, Q(\tau)F_0(x, t-\tau) + (2v)^{-1}\int_0^{vt} d\xi P(\xi)[F_1(x-\xi, t-\xi/v) + F_1(x+\xi, t-\xi/v)], \tag{12.5.8}$$

$$F_0(x,t) = (c/2)\int_0^{vt} d\xi p(\xi)[F_1(x-\xi, t-\xi/v) + F_1(x+\xi, t-\xi/v)] + p_0\delta(x)\delta(t), \tag{12.5.9}$$

$$F_1(x,t) = [(1-c)/2]\int_0^{vt} d\xi p(\xi)[F_1(x-\xi, t-\xi/v) + F_1(x+\xi, t-\xi/v)] + \int_0^t d\tau q(\tau)F_0(x, t-\tau) + p_1\delta(x)\delta(t). \tag{12.5.10}$$

For better understanding of the equations, we refer to the space-time diagram given in Fig. 12.5. Every random trajectory is made up of a set of segments, each parallel to one of the straight lines $x = 0$, $x = vt$, $x = -vt$. Hence the density $p(x,t)$ is split into the terms

$$p(x,t) = p^{(0)}(x,t) + p^{(+)}(x,t) + p^{(-)}(x,t).$$

Since $F_0(x,t)\,dx\,dt$ is the probability for the particle to fall into trap in the space-time domain $dx\,dt$ and $Q(\tau)$ is the probability to wait here longer than τ, we obtain

$$p^{(0)}(x,t) = \int_0^t d\tau Q(\tau)F_0(x, t-\tau).$$

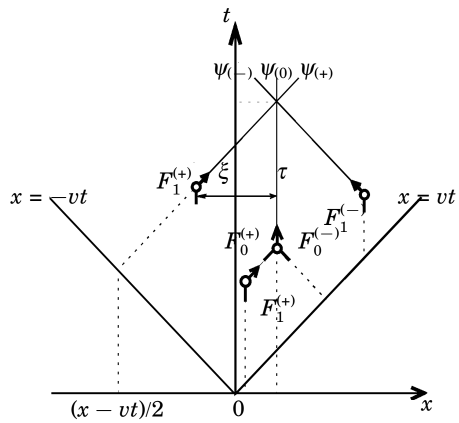


Figure 12.5.

This is the first term of sum (12.5.8). Reasoning along similar lines for moving particle, we arrive at the others.

To explain (12.5.9), we refer to the diagram again. As one can see, the density $F_0(x,t)$ for $t > 0$ in turn is split into two parts: $F_0^{(+)}(x,t)$ and $F_0^{(-)}(x,t)$ relating to the particles moving before collision to the right and to the left respectively. With this in mind, we immediately obtain

$$F_0^{(+)}(x,t) = (c/2)\int_0^{vt} d\xi p(\xi) F_1(x-\xi, t-\xi/v),$$
$$F_0^{(-)}(x,t) = (c/2)\int_0^{vt} d\xi p(\xi) F_1(x+\xi, t-\xi/v),$$

where $p(\xi)d\xi$ is the probability that the random free path falls into $(\xi, \xi + d\xi)$. Similar reasoning clears up the sense of (12.5.10), too.

12.6. Some special cases

Let $v = \infty$, $c = 1$ and $p_0 = 1$. The walking particle hence begins its history from the quiescent state and falls in traps after each jump. If we set $q(\tau) = \delta(\tau - 1)$, (12.5.5)–(12.5.7) take the form

$$p(\mathbf{r},t) = \int_0^1 d\tau F_0(\mathbf{r}, t-\tau), \qquad (12.6.1)$$
$$F_0(\mathbf{r},t) = \int d\mathbf{r}' p(\mathbf{r}') F_0(\mathbf{r}-\mathbf{r}', t-1) + \delta(\mathbf{r})\delta(t).$$

The latter equation has the solution

$$F_0(\mathbf{r},t) = \sum_{k=0}^{\infty} p_k(\mathbf{r})\delta(t-k), \tag{12.6.2}$$

where

$$p^{(k)}(\mathbf{r}) = \int d\mathbf{r}' p(\mathbf{r}') p^{(k-1)}(\mathbf{r}-\mathbf{r}'),$$
$$p^{(1)}(\mathbf{r}) \equiv p(\mathbf{r}), \qquad p^{(0)}(\mathbf{r}) = \delta(\mathbf{r}).$$

Substitution of (12.6.2) into (12.6.1) yields

$$p(\mathbf{r},t) = \int_{t-1}^{t} dt' F_0(\mathbf{r},t') = p^{([t])}(\mathbf{r}), \tag{12.6.3}$$

where $[t]$ stands for the integer part of t; (12.6.3) is merely the distribution density of the sum $\mathbf{S}_{[t]}$ of a fixed number $[t]$ of independent random vectors $\mathbf{r}_i$:

$$\mathbf{S}_{[t]} = \sum_{i=1}^{[t]} \mathbf{R}_i.$$

We immediately conclude that

$$p(\mathbf{r},t+1) = \int d\mathbf{r}' p(\mathbf{r}') p(\mathbf{r}-\mathbf{r}',t), \qquad t > 0,$$

is merely the Chapman–Kolmogorov equation, which can be rewritten as

$$p(\mathbf{r},t+1) - p(\mathbf{r},t) = \int d\mathbf{r}' p(\mathbf{r}')[p(\mathbf{r}-\mathbf{r}',t) - p(\mathbf{r},t)],$$

and conforms with equation (5) from (Chukbar, 1995), where only the one-dimensional case was considered.

We take now $q(t) = \mu e^{-\mu\tau}$ under the same rest conditions:

$$p(\mathbf{r},t) = \int_0^t d\tau \tilde{\mu} e^{-\mu\tau} F_0(\mathbf{r},t-\tau), \tag{12.6.4}$$

$$F_0(\mathbf{r},t) = \delta(\mathbf{r})\delta(t) + \mu \int d\mathbf{r}' p(\mathbf{r}') p(\mathbf{r}-\mathbf{r}',t), \tag{12.6.5}$$

which yield the integral equation

$$p(\mathbf{r},t) = p(\mathbf{r})e^{-\mu t} + \mu \int_0^t d\tau e^{-\mu\tau} \int d\mathbf{r}' p(\mathbf{r}') p(\mathbf{r}-\mathbf{r}',t-\tau).$$

By differentiation with respect to t, we bring it to the form of the forward Kolmogorov equation (Feller, 1966):

$$\frac{\partial p}{\partial t} = -\mu p(\mathbf{r}, t) + \mu \int d\mathbf{r}' p(\mathbf{r}') p(\mathbf{r} - \mathbf{r}', t), \qquad p(\mathbf{r}, 0) = \delta(\mathbf{r}).$$

Its solution is

$$p(\mathbf{r}, t) = \sum_{k=0}^{\infty} w_k(t) p^{(k)}(\mathbf{r}), \tag{12.6.6}$$

where

$$w_k(t) = \frac{(\mu t)^k}{k!} e^{-\mu t}$$

is the Poisson distribution. This means that density (12.6.6) determines the distribution of the sum $\mathbf{S}_N$ of a random Poisson's number N of independent random vectors $\mathbf{R}_i$ distributed with density $p(\mathbf{r})$:

$$\mathbf{S}_N = \sum_{i=1}^{N} \mathbf{R}_i.$$

Let us consider the one-dimensional symmetric walk with deterministic unit step and an arbitrary distribution of waiting time under the conditions $v = \infty$, $p_0 = 1$, and $c = 1$. From (12.5.8)–(12.5.10) it follows that in this case

$$p(x, t) = \int_0^t d\tau Q(\tau) F_0(x, t - \tau), \tag{12.6.7}$$

$$F_0(x, t) = \delta(x)\delta(t) + (1/2) \int_0^t d\tau q(\tau)[F_0(x - 1, t - \tau) + F_0(x + 1, t - \tau)]. \tag{12.6.8}$$

Substituting (12.6.8) into (12.6.7) and changing the integration order, we obtain

$$p(x, t) = Q(t)\delta(x) + (1/2) \int_0^t d\tau' q(\tau') \times \int_0^{t-\tau'} d\tau Q(\tau) \left[F(x - 1, t - \tau - \tau') + F(x + 1, t - \tau - \tau')\right].$$

By (12.6.7),

$$\int_0^{t-\tau'} d\tau Q(\tau) \left[F(x - 1, t - \tau - \tau') + F(x + 1, t - \tau - \tau')\right] = p(x - 1, t - \tau') + p(x + 1, t - \tau'),$$

hence

$$p(x,t) = Q(t)\delta(x) + (1/2)\int_0^t d\tau\, q(\tau)[p(x-1,t-\tau) + p(x+1,t-\tau)]. \tag{12.6.9}$$

If many particles are born at the initial time with the distribution density $n_0(x) = n(x,0)$, then the density

$$n(x,t) = \int_{-\infty}^{\infty} p(x-x',t)n_0(x')\,dx'$$

satisfies the equation

$$n(x,t) = n_0(x)Q(t) + \tfrac{1}{2}\int_0^t d\tau q(\tau)[n(x-1,t-\tau') + n(x+1,t-\tau')],$$

which follows from (12.6.9).

Let us turn to the three-dimensional walk with $c = 1$, and $p_0 = 1$. In this case, (12.5.6)–(12.5.7) are reduced to

$$F_0(\mathbf{r},t) = \int d\mathbf{r}' \int dt' w(\mathbf{r}',t')F_0(\mathbf{r}-\mathbf{r}',t-t') + \delta(\mathbf{r})\delta(t), \tag{12.6.10}$$

$$F_1(\mathbf{r},t) = \int d\tau q(\tau)F_0(\mathbf{r},t-\tau) + \delta(\mathbf{r})\delta(t),$$

$$w(\mathbf{r},t) = p(\mathbf{r})q(t - r/v). \tag{12.6.11}$$

Let us split the density $p(\mathbf{r},t)$ into two components: $p_0(\mathbf{r},t)$ related to the quiescence state and $p_*(\mathbf{r},t)$ related to the motion state. It is easily seen that

$$p_0(\mathbf{r},t) = \int d\tau\, Q(\tau)F_0(\mathbf{r},t-\tau), \tag{12.6.12}$$

$$p_0(\mathbf{r},t) = \int d\mathbf{r}' \int dt' w(\mathbf{r}',t')p(\mathbf{r}-\mathbf{r}',t-t') + Q(t)\delta(\mathbf{r}). \tag{12.6.13}$$

Relations (12.6.10), (12.6.12), and (12.6.13) are identical to (17), (18), and (19), respectively, from (Klafter *et al.*, 1987) devoted to continuous-time random walk models with coupled memories. As shown in (Compte, 1996; Compte *et al.*, 1997), the long-time (or long-distance) limit behavior of the distribution is described in terms of fractional equations (see also (Nigmatullin, 1986)).

Reverting to the Fourier–Laplace space, $(\mathbf{r},t) \to (\mathbf{k},\lambda)$, (12.6.13) becomes

$$p_0(\mathbf{k},\lambda) = w(\mathbf{k},\lambda)p_0(\mathbf{k},\lambda) + Q(\lambda)$$

with the solution

$$p_0(\mathbf{k},\lambda) = \frac{Q(\lambda)}{1 - w(\mathbf{k},\lambda)} = \frac{1 - q(\lambda)}{\lambda(1 - w(\mathbf{k},\lambda))},$$

where

$$w(\mathbf{k}, \lambda) = \int d\mathbf{r} \int dt\, e^{i\mathbf{kr}-\lambda t} p(\mathbf{r}) q(t - r/v).$$

In the limit as $v \to \infty$, we obtain decoupled memory,

$$w(\mathbf{k}, \lambda) = p(\mathbf{k}) q(\lambda),$$

and arrive at the Montroll–Weiss result (Montroll & Weiss, 1965)

$$p(\mathbf{k}, \lambda) = \frac{1 - q(\lambda)}{\lambda[1 - p(\mathbf{k})q(\lambda)]}. \tag{12.6.14}$$

12.7. Asymptotic solution of the Montroll–Weiss problem

The above presentation allows us to express the asymptotic solution of the Montroll–Weiss problem in terms of stable distributions.

Under conditions (12.3.9) and (12.4.13), the asymptotic expression for the Montroll–Weiss transform $p(\mathbf{k}, \lambda)$ in the domain of small values of its arguments takes the form

$$p^{\mathrm{as}}(\mathbf{k}, \lambda) = \frac{\lambda^{\omega-1}}{\lambda^{\omega} + Dk^{\alpha}}, \qquad D = b/A', \tag{12.7.1}$$

which yields the corresponding density in the scalable (automodel) form:

$$\begin{aligned} p^{\mathrm{as}}(\mathbf{r}, t) &= \frac{1}{(2\pi)^4 i} \int d\mathbf{k} \int d\lambda\, e^{-i\mathbf{kr}+\lambda t} p^{\mathrm{as}}(\mathbf{k}, \lambda) \\ &= (Dt^{\omega})^{-3/2} \Psi^{(\alpha,\omega)} \left\{ (Dt^{\omega})^{-1/2} \mathbf{r} \right\}, \end{aligned} \tag{12.7.2}$$

where

$$\Psi^{(\alpha,\omega)}(\mathbf{r}) = \frac{1}{(2\pi)^4 i} \int d\mathbf{k} \int d\lambda e^{-i\mathbf{kr}+\lambda} \frac{\lambda^{\omega-1}}{\lambda^{\omega} + k^{\alpha}}. \tag{12.7.3}$$

Four cases arise here.

CASE 1 ($\alpha = 2$, $\omega = 1$). This case concerns the normal diffusion. Recall that it covers all CTRW processes with arbitrary distributions $p(\mathbf{r})$ and $p(t)$ possessing the finite moments

$$\int p(\mathbf{r}) r^2 d\mathbf{r} = \sigma^2, \tag{12.7.4}$$

$$\int_0^{\infty} p(t) t\, dt = \tau. \tag{12.7.5}$$

The integrand in (12.7.3) has a simple pole at the point $\lambda = -k^2$; hence

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{1}{\lambda+k^2}e^{\lambda}\,d\lambda = e^{-k^2},$$

and

$$\Psi^{(2,1)}(\mathbf{r}) = (4\pi)^{-3/2}e^{-r^2/4} \equiv q_3(\mathbf{r};2) \tag{12.7.6}$$

is the three-dimensional stable distribution with $\alpha = 2$ (Gaussian).

CASE 2 ($\alpha < 2$, $\omega = 1$). This case is referred to as the Lévy flight with inverse power type distribution $p(\mathbf{r}) \propto r^{-\alpha-3}$ and an arbitrary distribution $p(t)$ of waiting times T possessing a finite first moment (12.7.5). It does not differ from the first case with the exception of the pole position, being now $\lambda = -k^{\alpha}$. This leads us to the spherically symmetric stable distribution

$$\Psi^{(\alpha,1)}(\mathbf{r}) = q_3(\mathbf{r};\alpha). \tag{12.7.7}$$

Thus, we have the superdiffusion solution discussed in Section 12.3.

CASE 3 ($\alpha = 2$, $\omega < 1$). This is the case of Lévy trapping with the inverse power type distribution $p(t) \propto t^{-\omega-1}$ and an arbitrary distribution $p(\mathbf{r})$ possessing a finite second moment (12.7.4). Now $\lambda = |-k^{\alpha}|^{1/\omega}$ is a branch point, and the residue method cannot be used. Following the way used in Section 12.4, we obtain the same result:

$$\Psi^{(2,\omega)}(\mathbf{r}) = \int_0^{\infty} q_B(t;\omega,1)q_3(\mathbf{r}t^{\omega/2};2)t^{3\omega/2}\,dt. \tag{12.7.8}$$

CASE 4 ($\alpha < 2$, $\omega < 1$). This is the case of Lévy flight and Lévy trapping. Proceeding as in the third case we obtain

$$\Psi^{(\alpha,\omega)}(\mathbf{r}) = \int_0^{\infty} q_B(t;\omega,1)q_3(\mathbf{r}t^{\omega/\alpha};\alpha)t^{3\omega/\alpha}\,dt. \tag{12.7.9}$$

This is the complete asymptotic solution of the Montroll–Weiss problem.

It is worthwhile to notice that the above-obtained asymptotic expressions (12.7.2) with (12.7.7)–(12.7.9) correlate well with fractional differential equations.

Indeed, as one can see from (12.7.1) the Fourier–Laplace transform of $p^{\mathrm{as}}(\mathbf{r},t)$ satisfies the equation

$$\lambda p^{\mathrm{as}}(\mathbf{k},\lambda) = -D\lambda^{1-\omega}k^{\alpha}p^{\mathrm{as}}(\mathbf{k},\lambda) + 1. \tag{12.7.10}$$

The case $\omega = 1$ has been considered in Section 12.3, where the fractional equation

$$\frac{\partial p^{\mathrm{as}}(\mathbf{r},t)}{\partial t} = -D(-\Delta)^{\alpha/2}p^{\mathrm{as}}(\mathbf{r},t) + \delta(\mathbf{r})$$

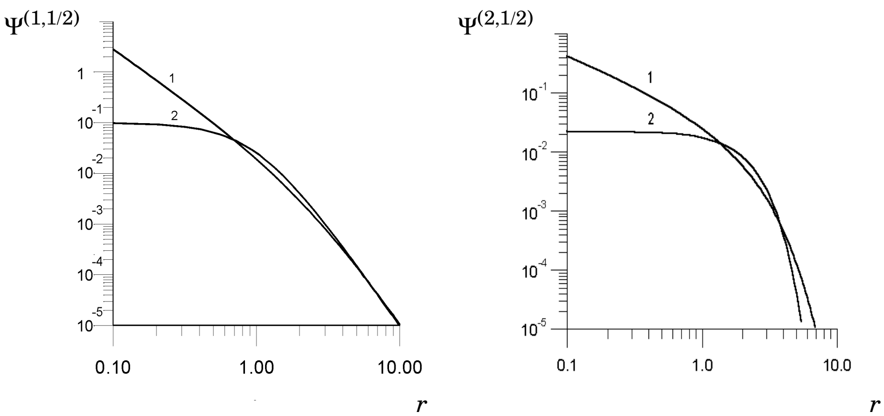


Figure 12.6. The three-dimensional solution $\Psi^{(1,1/2)}(r)$ (1) compared with the corresponding Cauchy density (7.2.7) (2), and the three-dimensional solution $\Psi^{(2,1/2)}(r)$ (1) compared with the corresponding Gauss density (7.2.8) (2)

was established. Now we have to apply the fractional derivative with respect to time. Inverse transformation of (12.7.10) yields

$$\frac{\partial p^{\mathrm{as}}(\mathbf{r},t)}{\partial t} = -D(-\Delta)^{\alpha/2}\frac{\partial^{1-\omega}p^{\mathrm{as}}(\mathbf{r},t)}{\partial t^{1-\omega}} = \delta(\mathbf{r})\delta(t). \quad (12.7.11)$$

Another equivalent form of the anomalous diffusion equation can be obtained with the use of the equation

$$\lambda^{\omega}p^{\mathrm{as}}(\mathbf{k},\lambda) = -Dk^{\alpha}p^{\mathrm{as}}(\mathbf{k},\lambda) + \lambda^{\omega-1} \quad (12.7.12)$$

following from (12.7.10):

$$\frac{\partial^{\omega}p^{\mathrm{as}}(\mathbf{r},t)}{\partial t^{\omega}} = -D(-\Delta)^{\alpha/2}p^{\mathrm{as}}(\mathbf{r},t) + \frac{t^{-\omega}}{\Gamma(1-\omega)}\delta(\mathbf{r}). \quad (12.7.13)$$

The solutions of both of these equations are of the form (12.7.6)–(12.7.9). The functions $\psi^{(1,1/2)}(r)$ and $\psi^{(2,1/2)}$ are plotted in Fig. 12.6 compared with the Cauchy and Gauss densities respectively.

12.8. Two-state model

In the above cases, we thought of particles that move at a constant velocity for a random time (or displacement), then stop and choose a new direction and a new time of sojourn at random according to given probabilities. They refer to this model as the velocity model. If the particle performing one-dimensional walk without trapping proceeds its motion in the same direction, the observer may not distinguish whether the particle has stopped at all or has simply continued its motion until it stops and changes direction. Thus, there exists a one-to-one correspondence between the set of trajectories and the set of sequences of turn points $\{x_i, t_i\}$ on the space–time plane, and it is enough to consider only these points of a trajectory. This case is called the two-state model (Zumofen & Klafter, 1993).

The CTRW approach described in Section 12.5 with regard to the velocity model can be easily adapted to the two-state model. Let us take for the sake of simplicity $v = 1$ and label with R the state with a positive velocity (the particle moves to the right), and by L, the state with a negative velocity (the particle moves to the left). As a consequence, we obtain

$$p(x,t) = p_R(x,t) + p_L(x,t) \tag{12.8.1}$$

where the terms

$$p_R(x,t) = \int_0^{(t+x)/2} d\xi\, P_R(\xi) f_{RL}(x-\xi, t-\xi), \tag{12.8.2}$$

$$p_L(x,t) = \int_0^{(t-x)/2} d\xi\, P_L(\xi) f_{LR}(x+\xi, t-\xi) \tag{12.8.3}$$

are expressed in terms of the distributions of free paths

$$P_A(x) = \int_x^{\infty} p_A(\xi) d\xi, \qquad A = L, R,$$

and the space–time distributions of turn points $f_{AB}(x,t)$. We recall that $f_{AB}(x,t)\,dx\,dt$ is the average number of turn points $B \to A$ in the domain $dx\,dt$ for a single trajectory. These functions satisfy the equations similar to (12.5.9)–(12.5.10):

$$f_{RL}(x,t) = \int_0^{(t-x)/2} d\xi p_L(\xi) f_{LR}(x+\xi, t-\xi) + q_R \delta(x)\delta(t), \tag{12.8.4}$$

$$f_{LR}(x,t) = \int_0^{(t+x)/2} d\xi p_R(\xi) f_{RL}(x-\xi, t-\xi) + q_L \delta(x)\delta(t). \tag{12.8.5}$$

Here q_R and q_L are the probabilities of the initial direction, and the limits of integrals are defined more accurately as compared with those in (12.5.9)–(12.5.10).

The Fourier–Laplace transformation of (12.8.1)-(12.8.5) yields

$$p(k,\lambda) \equiv \int_0^\infty dt \int_{-t}^{t} dx\, e^{-\lambda t + ikx} p(x,t) = p_R(k,\lambda) + p_L(k,\lambda),$$

$$p_R(k,\lambda) = P_R(\lambda - ik) f_{RL}(k,\lambda),$$
$$p_L(k,\lambda) = P_L(\lambda + ik) f_{LR}(k,\lambda),$$
$$f_{RL}(k,\lambda) = p_L(\lambda + ik) f_{LR}(k,\lambda) + q_R,$$
$$f_{LR}(k,\lambda) = p_R(\lambda - ik) f_{RL}(k,\lambda) + q_L.$$

Here

$$p_A(\lambda) = \int_0^\infty e^{-\lambda\xi} p_A(\xi)\, d\xi,$$
$$P_A(\lambda) = \int_0^\infty e^{-\lambda\xi} P_A(\xi)\, d\xi = [1 - p_A(\lambda)]/\lambda.$$

After straightforward transformations we obtain

$$p(k,\lambda) = \left\{ \frac{1 - p_R(\lambda - ik)}{\lambda - ik} [p_L(\lambda + ik) q_L + q_R] + \frac{1 - p_L(\lambda + ik)}{\lambda + ik} [p_R(\lambda - ik) q_R + q_L] \right\} [1 - p_L(\lambda + ik) p_R(\lambda - ik)]^{-1}.$$

It is easy to see that the expression satisfies the normalization

$$p(0,\lambda) = \frac{1}{\lambda}.$$

Choosing now

$$q_R = q_L = 1/2,$$
$$p_L(\lambda) \sim p_R(\lambda) \sim 1 - a\lambda - b\lambda^\alpha, \qquad \lambda \to 0,$$

with $1 < \alpha \le 2$, we conclude that

$$p(k,\lambda) \sim \frac{1}{\lambda + D|k|^\alpha}, \quad \lambda \to 0, \quad D = \text{const} > 0.$$

Therefore, we obtain the asymptotic solution

$$p^{\text{as}}(k,t) = e^{-D|k|^\alpha t}, \tag{12.8.6}$$

which is the characteristic function of the symmetric stable distribution with $\alpha \in (1,2]$. The case $\alpha = 2$ corresponds to the normal diffusion, whereas other values of α are associated with superdiffusion.

12.9. Stable laws in chaos

One of the promising applications of the anomalous diffusion theory leading to stable laws may be the chaos problem.

The word 'chaos' comes from the Greek '$\chi\alpha o\zeta$'. Originally it had the meaning of infinite space which existed before all other appeared. In modern natural sciences, this word means a state of disorder and irregularity (Schuster, 1984). Semantically, this concept can be considered as an opposite one to the word 'order', but the nature is more complicated. Poincaré (1892) (Poincaré, 1892) discovered that chaotic mechanical motion can arise from a regular one described by Hamiltonian equation. 70 years later, meteorologist E.N. Lorenz (1963) (Lorenz, 1963) demonstrated that even a simple system of three non-linear equations of the first order can lead to completely chaotic trajectories. These are examples of a deterministic or Hamiltonian chaos, when chaotic motion is generated by non-linear systems uniquely determining it by its known prehistory. The first cause of the chaotic behavior is the property of non-linear system trajectories initially close to each other to move away of each other exponentially fast (Schuster, 1984). Thus it becomes impossible to predict long-term behavior of such systems.

During last decades, it becomes clear that this phenomenon is often to be found in nature and plays an important role in many processes. Here is the list of them being far from completion (taken from (Schuster, 1984)):

- the forced pendulum ((Humiéres *et al.*, 1982));
- fluids near the turbulence threshold (Swinney & Gollub, 1981);
- non-linear optics and lasers (Haken, 1975; Hopf *et al.*, 1982);
- Josephson junction (Cirillo & Pedersen, 1982);
- chemical reactions (Simoyi *et al.*, 1982);
- three (or more) body problem (Hellemann, 1980);
- charged particle accelerators (Hellemann, 1980);
- channeling of particles in crystals (Kimball *et al.*, 1988; Akhiezer *et al.*, 1991);
- interacting non-linear waves in plasma (Wersinger *et al.*, 1980);
- biological models of population dynamics (May, 1976);
- stimulated cardiac oscillator (Glass *et al.*, 1983).

The impossibility to predict long-time chaotic motion stimulated the development of the stochastic approach to its description (Zaslavsky & Chirikov, 1971). The reason for this can be quantitatively explained by considering the 'standard map' due to Chirikov–Taylor (Chirikov, 1979). This map is obtained while considering the periodically kicked rotor:

$$y_{n+1} = y_n + K \sin x_n, \qquad x_{n+1} = x_n + y_{n+1}, \tag{12.9.1}$$

where y and x are the rotational momentum and the phase of the rotor, respectively, and n, corresponding to the nth kick instance, plays the role of discrete time. If $K \gg 1$, then the phase x, taken always in $(0, 2\pi)$, changes randomly. Averaging over the phase, from equation (12.9.1) one can easily get the moments $\langle \Delta y \rangle = 0$, $\langle (\Delta y)^2 \rangle = K^2/2$, where $\langle \cdot \rangle$ means averaging over x in the interval $(0, 2\pi)$. These simple expressions lead eventually to the diffusion (Fokker–Planck–Kolmogorov) equation

$$\frac{\partial p(y,t)}{\partial t} = D\frac{\partial^2 p(y,t)}{\partial y^2}, \qquad D = K^2/4,$$

which describes the slow evolution of the momentum distribution function $p(y,t)$. This is the simplest manner in which a kinetic description arises in a dynamical system with chaotic behavior. It is due to the randomness of the fast variable phase, generated by non-random equations such as (12.9.1) above (Shlesinger *et al.*, 1993, p. 32). Considering a similar example and arriving at the normal diffusion too, H. Schuster (Schuster, 1984, §2.3) concludes that the diffusion arises not due to the action of a random force as in Brownian motion but because the system 'forgets' its prehistory in the course of chaotic motion.

In reality, however, the situation proves to be more complicated. The models considered in (Afanasiev *et al.*, 1991; Chaikovsky & Zaslavsky, 1991; Zaslavsky & Tippett, 1991; Shlesinger *et al.*, 1993; Zaslavsky, 1994a; Zaslavsky, 1994b; Zaslavsky & Abdullaev, 1995) reveal that the phase space pattern can be described as a connected domain with chaotic dynamics inside (stochastic sea or stochastic webs) and islands immersed into the sea or webs. The islands are filled of nested invariant Kolmogorov–Arnold–Moser curves and isolated stochastic layers, and form a fractal set of non-zero measure. Particle motion inside the stochastic sea or along the stochastic webs can be described as a random process being the result of a complicated competition between traps and flights which depends on the number of degrees of freedom, phase space topology, closeness to the singularity, and other anomalous properties of the dynamics.

This process is characterized by the law

$$\Delta(t) \propto t^{\mu}$$

with transportation exponent μ that can be different from the normal case $\mu = 1/2$ and consequently exhibits the anomalous diffusion behavior.

In the review (Shlesinger *et al.*, 1993), the connection between the theory of Lévy processes (Lévy flight) and dynamical chaos was described, and the phenomenon of the anomalous transportation was considered to be the result of a Lévy-like process rather than that of a Gaussian-like process. For the one-dimensional case, Zaslavsky (Zaslavsky, 1992) derived a generalized Fokker–Planck–Kolmogorov equation which was fractional in time and space, and covered both superdiffusion and subdiffusion types of the process:

$$\frac{\partial^\omega p(x,t)}{\partial t^\omega} = \frac{\partial^\alpha}{\partial(-x)^\alpha}[A(x)p(x,t)] + \frac{1}{2}\frac{\partial^{2\alpha}[B(x)p(x,t)]}{\partial(-x)^{2\alpha}},$$

where

$$\frac{\partial^\alpha g(x,t)}{\partial x^\alpha} \equiv \frac{1}{\Gamma(-\alpha)}\int_{-\infty}^{x}(x-\xi)^{-\alpha-1}g(\xi,t)\,d\xi$$

is one of the definitions of the fractional derivative (Oldham & Spanier, 1974).

Multidimensional spherically symmetric anomalous diffusion is described by equations derived in previous sections and leading to stable distribution as their solutions.

To conclude this section, we would like to notice that long before the beginning of systematical numerical investigations of diverse maps in connection with the chaos problem, the algorithms called the random number generators were invented and widely used in Monte-Carlo calculations. As an example, we give the residual (or congruential) method of D.H. Lehmer (Lehmer, 1949). It is based on the map

$$x_i = m_i/M, \quad m_i = am_{i-1} \pmod{M}, \tag{12.9.2}$$

where m_0, a, and M are some given integers. Sequence (12.9.2) possesses a period not exceeding M. The most important property of such sequences is that for very large M and appropriate values of m_0 and a, the sequence $x_0, x_1, x_2, \ldots$ looks like a sequence of independent uniformly distributed on $(0, 1]$ random variables $U_0, U_1, U_2, \ldots$. Therefore, statistical tests do not distinguish them from true random numbers U_i, although their dynamical origin is evident for us. Consequently, we can say that the dynamical chaos has been widely used in numerous Monte-Carlo calculations for about a half of century, and moreover, it forms the heart of the method!

Supplementing (12.9.2) by the relation

$$S_n = \sum_{i=1}^{n} y_i, \qquad y_i = f(x_i), \tag{12.9.3}$$

where $f(x)$ is a monotonically increasing function on $(0, 1)$ such that

$$\begin{aligned} f(x) &\sim -(x/d)^{-1/\mu}, && x \to 0, \\ f(x) &\sim [(1-x)/c]^{-1/\mu}, && x \to 1, \end{aligned}$$

from dynamical map (12.9.2)–(12.9.3) we immediately obtain a diffusion process (as $n \to \infty$) which is normal for $\mu \geq 2$, or an anomalous one (Lévy process) for $\mu < 2$.

13

Physics

13.1. Lorentz dispersion profile

The most familiar to physicists stable law (excepting the Gaussian law) is the Cauchy law which describes, in particular, the natural widening of a spectral line of a charge under the action of quasi-elastic force.

According to classical electrodynamics (Larmor, 1897), the charge e in one-dimensional motion $x(t)$ emanates energy per unit time as

$$I = \frac{2}{3}\frac{e^2}{c^3}\left[\ddot{x}(t)\right]^2, \tag{13.1.1}$$

where c is the light speed, and the dots above $x(t)$ stand for differentiation with respect to time. The effective brake force created by this radiation is equal to

$$F_{\text{rad}} = \frac{2}{3}\frac{e^2}{c^3}\dddot{x}(t).$$

If the applied force is quasi-elastic, that is,

$$F = -m\omega_0^2 x,$$

then the equation of motion is of the form

$$m\ddot{x}(t) + m\omega_0^2 x(t) = (2/3)(e^2/c^3)\dddot{x}(t).$$

If the brake force described by the right-hand side of this equation is small as compared with the quasi-elastic one, the solution is

$$x(t) = x_0 \exp\left\{i\omega_0 t - \gamma t/2\right\}, \tag{13.1.2}$$

where γ is the classical damping factor

$$\gamma = \frac{2e^2}{3mc}\omega_0^2.$$

According to the Larmor formula (13.1.1) and expression (13.1.2), the total average energy radiated in all directions per unit time is

$$\langle I \rangle = \frac{2}{3c^3}\langle(\ddot{x}(t))^2\rangle \propto \langle[x(t)]^2\rangle,$$

By virtue of the Rayleigh theorem (Rayleigh, 1889),

$$\langle I \rangle = \int_0^\infty I(\omega)\,d\omega,$$

where

$$I(\omega) = |\tilde{x}(\omega)|^2, \tag{13.1.3}$$

and

$$\tilde{x}(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt \tag{13.1.4}$$

is the Fourier transform of the function $x(t)$. Substituting (13.1.2) into (13.1.4), we obtain

$$\tilde{x}(\omega) = \frac{x_0}{i(\omega - \omega_0) + \gamma/2},$$

and then, by (13.1.3), we obtain the energy distribution in linear frequency

$$I(\omega) \propto f(\omega) \equiv \frac{\Delta}{\pi[\Delta^2 + (\omega - \omega_0)^2]}, \tag{13.1.5}$$

where $\Delta = \gamma/2$, and the integral of the function $f(\omega)$ over $-\infty < \omega < \infty$ is equal to one (Lorentz, 1906; Lorentz, 1909).

Distribution (13.1.5), known as the Lorentz dispersion profile, is nothing but the Cauchy distribution with center at ω_0. From the quantum viewpoint, $f(\omega)\,d\omega$ is the probability that a radiated photon has a frequency belonging to the interval $(\omega, \omega + d\omega)$. While an isolated atom is concerned, the only reason of spectral line widening is the radiation loss given by formula (13.1.1).

In actual reality, an atom can be considered as an unperturbed harmonic oscillator until it collides with a perturbing particle. Such collisions lead to the change in phase and, probably, in amplitude of oscillation. If the collisions are distributed in time according to Poisson law with average interval τ, then the probability for random interval to be between t and $t + dt$ is defined by

$$p(t)\,dt = \frac{1}{\tau}e^{-t/\tau}\,dt.$$

For a large collision frequency, the damping factor γ can be neglected, and the radiation spectrum can be written as

$$I(\omega) \propto \int_0^\infty dt\,p(t)|\tilde{x}_t(\omega)|^2,$$

where

$$\tilde{x}_t(\omega) = \int_0^t x(t)e^{-i\omega t}dt = x_0 \int_0^t e^{i(\omega_0-\omega)t}dt.$$

Carrying out the integration, we obtain

$$|\tilde{x}_t(\omega)|^2 \propto (\omega_0 - \omega)^{-2} \sin^2((\omega_0 - \omega)t/2)$$

and therefore,

$$I(\omega) \propto \frac{1}{(\omega_0 - \omega)^2 \tau} \int_0^\infty e^{-t/\tau} \sin^2((\omega_0 - \omega)t/2)\, dt.$$

The evaluation of the last integral yields a formula which coincides with (13.1.5), where the parameter Δ is now defined not by the damping but by the collision frequency:

$$\Delta = 1/\tau.$$

There are also other reasons causing the spectral line widening: heat motion of radiating atoms which changes the observable frequency due to the Doppler shifts and leads to the Gaussian profile; Stark widening caused by an electrical field of ambient atoms. We consider the last effect in more detail.

13.2. Stark effect in an electrical field of randomly distributed ions

For hydrogen-like atoms, the frequency shift $\Delta\omega$ induced by the electrical field with intensity $\mathbf{E}$ is a linear function of E (linear Stark effect) (Stark, 1913). For other atoms, the linear dependence is observed only in strong fields; in the weak ones, $\Delta\omega$ is proportional to E^2 (square-law Stark effect). The radiating atoms, located in different places, are exposed to an action of fields created by various microscopic environments. As a simple model of this situation, they usually consider a single atom placed in a random electrical field $\mathbf{E}$ with distribution density $p(\mathbf{E})$. We denote the three-dimensional distribution density of $\mathbf{E}$ by $p(\mathbf{E})$. In the absence of reasons generating an anisotropy, the distribution $\mathbf{E}$ is considered as a spherically symmetric, with density

$$w(E) = 4\pi p(\mathbf{E})E^2 \tag{13.2.1}$$

Let the shift of the linear frequency $\nu = \omega/(2\pi)$ be a monotonically growing function of E,

$$\nu = \nu_0 + g(E);$$

then distribution (13.2.1) induces the profile of the spectral line

$$f(\nu) = w(g^{-1}(\nu - \nu_0))dg^{-1}(\nu - \nu_0)/d\nu, \tag{13.2.2}$$

where $g^{-1}(\nu)$ is the inverse to $g(E)$ function. In the linear case,

$$g(E) = g_1 \cdot E,$$

and (13.2.2) takes the form

$$f(\nu) = w((\nu - \nu_0)/g_1)/g_1, \tag{13.2.3}$$

and in the case of the square-law,

$$g(E) = g_2 E^2;$$

hence

$$f(\nu) = \tfrac{1}{2} w(\sqrt{(\nu - \nu_0)/g_2})/\sqrt{(\nu - \nu_0)g_2}. \tag{13.2.4}$$

Thus, the problem is reduced to the evaluation of function (13.2.1).

It was solved in 1919 by Holtsmark (Holtsmark, 1919) under the assumption that the given number of point sources of the field (point charges, dipoles or quadrupoles) is distributed within a spherical volume centered at the atom, uniformly and independently of each other.

In the case of point charges (ions) e_j located at $\mathbf{r}_j$, the electric field intensity at the origin

$$\mathbf{E} = -\sum_{j=1}^{N} e_j \mathbf{r}_j / r_j^3, \qquad r_j = |\mathbf{r}_j| \tag{13.2.5}$$

where N is the number of charges given in the spherical volume $V_R = (4/3)\pi R^3$. Holtsmark assumed that $e_j = e$ are deterministic, but we think of e_j as independent random variables. The characteristic function of random vector (13.2.5) is

$$f(\mathbf{k}; R) = \left[\varphi(\mathbf{k}, R)\right]^N, \tag{13.2.6}$$

$$\varphi(\mathbf{k}; R) = \frac{1}{V_R} \left\langle \int_{V_R} e^{-ie\mathbf{kr}/r^3} d\mathbf{r} \right\rangle, \tag{13.2.7}$$

where $\langle \dots \rangle$ means averaging over the random charge e.

Substituting (13.2.7) into (13.2.6) and introducing $\rho = N/V_R$ standing for the density of ions, we obtain

$$f(\mathbf{k}; R) = \left\langle V_R^{-1} \int_{V_R} e^{-ie\mathbf{kr}/r^3} d\mathbf{r} \right\rangle^{\rho V_R}.$$

Since

$$\left\langle \int_{V_R} d\mathbf{r} \right\rangle = V_R,$$

we are able to rewrite $f(\mathbf{k};R)$ as

$$f(\mathbf{k};R) = \left\{1 - V_R^{-1}\left\langle\int_{V_R}\left[1 - e^{-ie\mathbf{kr}/r^3}\right]d\mathbf{r}\right\rangle\right\}^{\rho V_R}. \tag{13.2.8}$$

Letting $R \to \infty$ under the condition ρ = const, we arrive at the formula

$$f(\mathbf{k}) \equiv \lim_{R\to\infty} f(\mathbf{k};R) = \exp\{-\rho\psi(\mathbf{k})\}, \tag{13.2.9}$$

where

$$\psi(\mathbf{k}) = \left\langle\int_{V_R}\left[1 - e^{-iekr/r^3}\right]d\mathbf{r}\right\rangle. \tag{13.2.10}$$

Since the integral over the whole sphere is the integral of an odd function, the integrand is of order $O(r^{-4})$, $\mathbf{r} \to \infty$, and the integral entering into (13.2.10) converges absolutely.

In formula (13.2.10), we pass from integration over the coordinate space to integration over the intensity space in accordance with the equality

$$\mathbf{E} = -e\mathbf{r}/r^3. \tag{13.2.11}$$

The corresponding Jacobian is

$$\begin{aligned}\frac{\partial\mathbf{E}}{\partial\mathbf{r}} &\equiv \frac{\partial(E_x,E_y,E_z)}{\partial(x,y,z)}\\ &= -e^3\begin{vmatrix} r^{-3} - 3x^2r^{-5} & -3xyr^{-5} & -3xzr^{-5}\\ -3xyr^{-5} & r^{-3} - 3y^2r^{-5} & -3yzr^{-5}\\ -3xzr^{-5} & -3yzr^{-5} & r^{-3} - 3z^2r^{-5}\end{vmatrix}\\ &= 2e^3r^{-9}.\end{aligned}$$

Expressing r via E with the use of (13.2.11), we obtain for a volume element

$$d\mathbf{r} = |\partial\mathbf{E}/\partial\mathbf{r}|^{-1}d\mathbf{E} = \tfrac{1}{2}|e|^{3/2}E^{-9/2}d\mathbf{E},$$

so (13.2.10) can be rewritten as

$$\begin{aligned}\psi(\mathbf{k}) &= \tfrac{1}{2}\langle|e|^{3/2}\rangle\int\left[1 - e^{i\mathbf{kE}}\right]E^{-9/2}d\mathbf{E}\\ &= \tfrac{1}{2}\langle|e|^{3/2}\rangle\int\left[1 - \cos\mathbf{kE}\right]E^{-9/2}d\mathbf{E}.\end{aligned}$$

Let us calculate the last integral in the spherical coordinates with z-axis directed along the vector $\mathbf{k}$:

$$\begin{aligned}\psi(\mathbf{k}) &= 4\pi\langle|e|^{3/2}\rangle\int_0^\infty dE\,E^{-5/2}\int_{-1}^{+1}dt\left[1 - \cos(ktE)\right]\\ &= 2\pi\langle|e|^{3/2}\rangle\int_0^\infty dE\,E^{-5/2}\left[1 - (kE)^{-1}\sin(kE)\right]\\ &= 2\pi\langle|e|^{3/2}\rangle k^{3/2}\int_0^\infty(x - \sin x)x^{-7/2}dx.\end{aligned}$$

By integration by parts, the last integral is reduced to

$$\int_0^\infty (x - \sin x)x^{-7/2}dx = \frac{8}{15}\int_0^\infty x^{-1/2}\cos x dx = (4/15)\sqrt{2\pi};$$

hence

$$\psi(\mathbf{k}) = (ak)^{3/2}, \qquad a = 2\pi(4/15)^{2/3}\langle|e|^{3/2}\rangle^{2/3},$$
$$f(\mathbf{k}) = e^{-\rho(ak)^{3/2}}.$$

Thus, we arrive at the three-dimensional symmetric stable law with $\alpha = 3/2$ whose density is

$$\begin{aligned} p(\mathbf{E}) &= \frac{1}{(2\pi)^3}\int e^{-i\mathbf{kE}-\rho(ak)^{3/2}}d\mathbf{k} \\ &= \frac{1}{(2\pi)^2 E}\int_0^\infty e^{-\rho(ak)^{3/2}}\sin(kE)k\,dk. \end{aligned} \tag{13.2.12}$$

Introducing the dimensionless intensity

$$\boldsymbol{\varepsilon} = E/(a\rho^{2/3}),$$

and using formulae (13.2.1) and (13.2.12), we obtain the distribution

$$H(\varepsilon) = \frac{2}{\pi\varepsilon}\int_0^\infty e^{-(x/\varepsilon)^{3/2}}x\sin x\,dx. \tag{13.2.13}$$

Formula (13.2.13) is referred to as the Holtsmark distribution.

In view of (13.2.1), the distribution of the absolute value of the field intensity is of the form

$$w(E) = (a\rho)^{-2/3}H((\rho a)^{-2/3}E); \tag{13.2.14}$$

therefore, profiles (13.2.3) and (13.2.4) induced by the considered mechanism are of the form

$$f(\nu) = (a\rho)^{-2/3}H((\rho a)^{-2/3}(\nu - \nu_0)/g_1)/g_1.$$

in the case of the linear Stark effect, and

$$f(\nu) = \tfrac{1}{2}(a\rho)^{-2/3}H\left((\rho a)^{-2/3}\sqrt{(\nu - \nu_0)/g_2}\right)/\sqrt{(\nu - \nu_0)g_2}$$

in the case of the square-law effect.

Thus, the distribution derived by Holtsmark is related to the observable structure of spectral lines, though it is necessary to say that experiments reveal the combined impact of all reasons on the line width. As calculations show (Lang, 1974), the center of the profile is formed by the Doppler mechanism of widening, giving the normal distribution, and the far wings coincide with the Lorentz distribution. It is understandable that it should be so indeed, because the Holtsmark distribution ($\alpha_H = 3/2$) occupies an intermediate position between the Cauchy distribution ($\alpha = 1$) and the normal law ($\alpha = 2$).

Let us concentrate our attention on some properties and numerical results concerning the Holtsmark distribution.

First, the above distribution of the electric field intensity does not depend on the signs of charges. Indeed, a uniform distribution of one charge e in a sphere of radius R yields, according to (13.2.7),

$$\varphi(\mathbf{k}; R) = 3R^{-3} \int_0^R \frac{\sin(ek/r^2)}{ek} r^4 dr,$$

so the characteristic function of the intensity generated by it does not vary if e is replaced by $-e$.

The following property is typical for the stable laws differing from the normal one: the Holtsmark distribution is basically formed by the nearest ion. To make sure of this, we obtain the distribution function of the intensity created by a single ion uniformly distributed in the volume V_R:

$$\begin{aligned} F(E) &= \mathsf{P}\{|e|r^{-2} < E\} = \mathsf{P}\left\{r > \sqrt{|e|/E}\right\} \\ &= 1 - R^{-3}(|e|/E)^{3/2}, \qquad E > |e|/R^2. \end{aligned}$$

The distribution function of the contribution of the nearest of $N = \rho V_R$ independent ions is equal to the Nth power of the function for a single ion:

$$F(E; N) = \left[1 - R^{-3}(|E|/e)^{3/2}\right]^{\rho V_R} = \left[\left(1 - \frac{1}{y}\right)^y\right]^{(4/3)\pi(|e|/E)^{3/2}\rho}.$$

As $N \to \infty$ and $\rho = \text{const}$, hence we obtain

$$F(\varepsilon; \infty) = e^{-(\varepsilon_0/\varepsilon)^{3/2}},$$

where ε is chosen as above and

$$\varepsilon_0 = (5/2)/\sqrt[3]{5\pi}.$$

The distribution density of ε is

$$p(\varepsilon) = F'(\varepsilon; \infty) = (3/2)e^{-(\varepsilon_0/\varepsilon)^{3/2}}(\varepsilon/\varepsilon_0)^{-5/2}/\varepsilon_0. \tag{13.2.15}$$

It is easy to see that the constant ε_0 is related to the average value $\overline{\varepsilon}$ by the formula

$$\overline{\varepsilon} = \Gamma(1/3)\varepsilon_0 \approx 2.68\varepsilon_0.$$

The comparison of the true Holtsmark density function $H(\varepsilon)$ and the density function of the leading contribution $p(\varepsilon)$ is shown in Fig. 13.1.

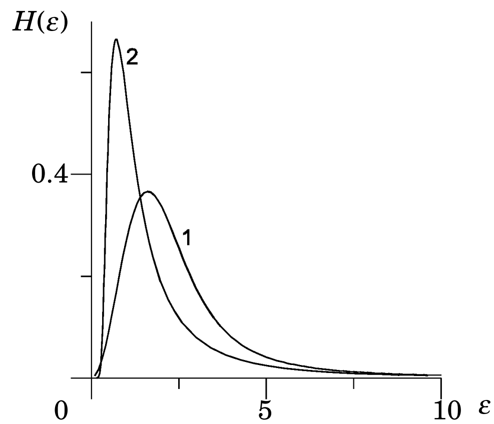


Figure 13.1. Distribution of the field created by the nearest neighbor (2) compared with distribution of the total field (i.e., the Holtsmark distribution) (1)

13.3. Dipoles and quadrupoles

If we deal not with plasma consisting of ions but with gas of neutral atoms, then the electric field is generated by dipole moments $\mathbf{d}_j$ of atoms (or quadrupole moments if the dipole ones are zero). In this case

$$\mathbf{E} = \sum_j \left(\frac{\mathbf{d}_j}{r_j^3} - \frac{3\mathbf{r}_j(\mathbf{r}_j\mathbf{d}_j)}{r_i^5} \right). \tag{13.3.1}$$

Formula (13.2.9) remains true, but the function $\psi(\mathbf{k})$ takes the form

$$\psi(\mathbf{k}) = \left\langle \int \left[1 - \exp\left\{ i\frac{\mathbf{kd}}{r^3} - 3i\frac{(\mathbf{kr})(\mathbf{rd})}{r^5} \right\} \right] d\mathbf{r} \right\rangle.$$

Denoting by $\mathbf{\Omega}$, $\mathbf{\Omega}_k$, and $\mathbf{\Omega}_d$ the unit vectors $\mathbf{r}/r$, $\mathbf{k}/k$, and $\mathbf{d}/d$, respectively, and assuming that the angular distribution of dipoles is isotropic, we can rewrite the last formula as

$$\psi(\mathbf{k}) = \frac{1}{4\pi} \left\langle \int d\mathbf{\Omega}_d \int d\mathbf{\Omega} \int_0^\infty \left[1 - \exp\left\{ ikd\phi(\mathbf{\Omega}, \mathbf{\Omega}_k, \mathbf{\Omega}_d) r^{-3} \right\} \right] r^2 dr \right\rangle, \tag{13.3.2}$$

where

$$\phi(\mathbf{\Omega}, \mathbf{\Omega}_k, \mathbf{\Omega}_d) = \mathbf{\Omega}_k\mathbf{\Omega}_d - 3(\mathbf{\Omega}_k\mathbf{\Omega})(\mathbf{\Omega}, \mathbf{\Omega}_d).$$

The vector $\mathbf{\Omega}_k$ does not vary during integration, and the result of integration, due to the mentioned isotropy, does not depend on it:

$$\psi(\mathbf{k}) = \psi(k).$$

For the same reason $\psi(\mathbf{k})$ is real-valued, so (13.3.2) can be represented as

$$\psi(k) = \frac{1}{4\pi}\int d\mathbf{\Omega}_d \int d\mathbf{\Omega}\int_0^\infty \left[1 - \cos\left\{kd\phi(\mathbf{\Omega},\mathbf{\Omega}_k,\mathbf{\Omega}_d)r^{-3}\right\}\right] r^2 dr.$$

The inner integral is easily calculated:

$$\int_0^\infty \left[1 - \cos\left\{kd\phi(\mathbf{\Omega},\mathbf{\Omega}_k,\mathbf{\Omega}_d)r^{-3}\right\}\right] r^2 dr = (\pi d/6)\,|\phi| k,$$

and we arrive at the formula

$$\psi(k) = ck \tag{13.3.3}$$

where

$$c = \frac{d}{24}\int d\mathbf{\Omega}_d \int d\mathbf{\Omega}|\phi(\mathbf{\Omega},\mathbf{\Omega}_k,\mathbf{\Omega}_d)|.$$

Substituting (13.3.3) into (13.2.9), we obtain

$$\psi(\mathbf{k}) = e^{-\rho ck},$$

and see that the electric field intensity generated by the Poisson ensemble of isotropically distributed dipoles is described by three-dimensional Cauchy distribution:

$$p(\mathbf{E}) = \frac{1}{(2\pi)^3}\int e^{-i\mathbf{k}\mathbf{E}-(\rho c)k} d\mathbf{k} = \frac{\rho c}{\pi^2[(\rho c)^2 + E^2]^2}.$$

The distribution of the magnetic field created by a similar ensemble of point magnets (magnetic dipoles) is also of the same form.

If the dipole moment of neutral atom is zero, then the field generated by it is defined by its quadrupole moment. The absolute value of the quadrupole field decreases as r^{-4}, so the total intensity of the field generated by the Poisson ensemble of isotropically distributed quadrupoles possesses the characteristic function

$$\psi(\mathbf{k}) = e^{-\rho ck^{3/4}}$$

corresponding to the three-dimensional symmetric stable law with $\alpha = 3/4$.

In Fig. 13.2 taken from (Holtsmark, 1919) the distributions of the absolute value of the intensity $w(E)$ for all three cases are shown.

13.4. Landau distribution

When a fast charged particle (electron, positron, proton) passes through some substance layer of thickness z, its final energy (after exiting the layer) E is smaller than the initial E_0 due to the ionization loss:

$$Q = E_0 - E. \tag{13.4.1}$$

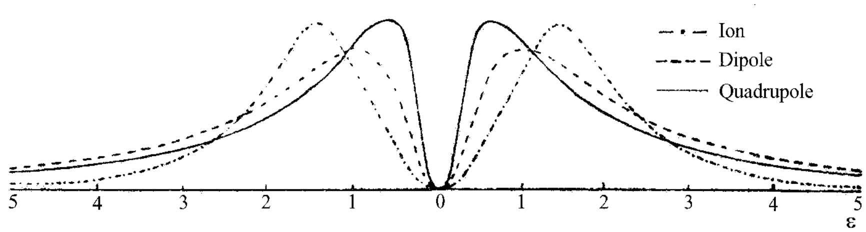


Figure 13.2. The distributions of fields created by ions, dipoles, and quadrupoles (Holtsmark, 1919)

The simplest model of this process can be formulated as follows: a fast particle moves through the layer along a straight line and gives to each electron, being at the distance r away from its trajectory, the energy $q = q(r)$. The r is called the impact parameter, and the function $q(r)$ decreases when it grows. In the domain of values q which are much larger than the ionization energy ε_I, the electrons are supposed to be free, and for them

$$q(r) = Ar^{-2}.$$

On the other hand, the lost energy cannot exceed the initial energy E_0, but for a thin layer the energy loss is small as compared with E_0, and this restriction can be neglected. If $Ar^{-2} \le \varepsilon_I$, ionization is impossible, and the fast particle interacting with an atom as a whole does not lose its energy. One can approximately assume that

$$q(r) = \begin{cases} Ar^{-2}, & r < R_I, \\ 0, & r \ge R_I, \end{cases} \tag{13.4.2}$$

where $R_I = \sqrt{A/\varepsilon_I}$.

Thus, the energy loss in a layer of thickness z looks as the sum

$$Q = \sum_i Ar_i^{-2}, \qquad r_i < R_I. \tag{13.4.3}$$

over all electrons of the layer which are inside the cylinder of radius R_I (Fig. 13.3. The surface electron number density ρ relates to the volume one n by the formula $\rho = nz$. The random distribution of electrons on the plane is supposed to be a homogeneous Poisson ensemble.

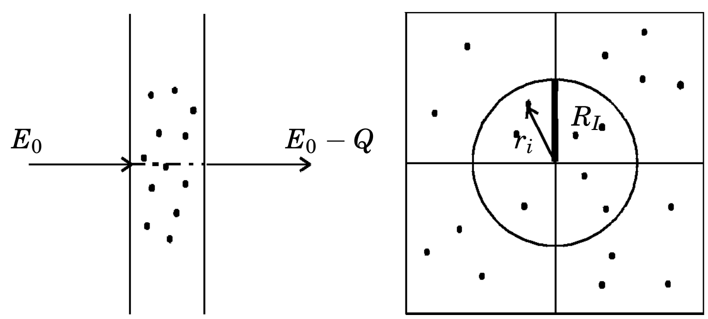


Figure 13.3. Passage of a fast charged particle through a thin layer (dots represent electrons)

Under the indicated conditions, the characteristic function of random variable (13.4.3) takes the form

$$f_Q(k) = \exp\left\{-2\pi\rho\int_0^{R_I}\left[1 - e^{ikAr^{-2}}\right] r\,dr\right\}$$

coinciding with (10.6.13) if we set

$$p(\theta) = \delta(\theta - A)$$

and $\mu = 2$. But the special case with $\mu = 2$ was not considered; we dwell upon this situation here following (Landau, 1944).

Passing to the variable q under the integral sign we obtain, in accordance with (13.4.2),

$$f_Q(k) = \exp\left\{-\rho\int_{\varepsilon_I}^{\infty}\left[1 - e^{ikq}\right] W(q)\,dq\right\}, \qquad (13.4.4)$$

where

$$W(q) = \pi A/q^2.$$

In (Landau, 1944), (13.4.4) is written in terms of the Laplace transform of the kinetic equation for distribution density $p(q,z)$ as a function of layer thickness:

$$\frac{\partial p}{\partial z} = \int_{\varepsilon_I}^{\infty} w(q')\left[p(q-q',z) - p(q',z)\right]\,dq' \qquad (13.4.5)$$

with the boundary condition

$$p(q,0) = \delta(q).$$

Here $w(q) = nW(q)$ is the differential cross-section of the energy loss per unit path length. It follows from (13.4.5) that the Laplace transform

$$\tilde{p}(\lambda, z) = \int_0^\infty e^{-\lambda q} p(q, z)\, dq$$

satisfies the equation

$$\frac{\partial \tilde{p}}{\partial z} = -\left[\int_{\varepsilon_I}^\infty \left(1 - e^{-\lambda q}\right) w(q)\, dq\right] \tilde{p}(\lambda, z), \qquad \tilde{p}(\lambda, 0) = 1.$$

Its solution is

$$\tilde{p}(\lambda, z) = \exp\left[-z \int_{\varepsilon_I}^\infty w(q) \left[1 - e^{-\lambda q}\right] dq\right], \tag{13.4.6}$$

which coincides with (13.4.4).

While evaluating the integral I, Landau splits it into two parts: over $(\varepsilon_I, \varepsilon)$ and over (ε, ∞). The former—I_1—is represented as

$$I_1 = \int_{\varepsilon_I}^\varepsilon w(q) \left[1 - e^{-\lambda q}\right] dq \approx \lambda \int_{\varepsilon_I}^\varepsilon q w(q)\, dq,$$

which corresponds to the average energy loss per unit length

$$\int_{\varepsilon_I}^\varepsilon q w(q)\, dq = \pi n A \ln \varepsilon / \varepsilon_I$$

The latter integral

$$I_2 = n\pi A \int_\varepsilon^\infty \left[1 - e^{-\lambda q}\right] dq/q^2$$

is transformed by integration by parts

$$I_2 = n\pi A \left\{\frac{1}{\varepsilon}\left(1 - e^{-\lambda\varepsilon}\right) + \lambda \int_\varepsilon^\infty \frac{e^{-\lambda q}}{q}\, dq\right\}$$

and for $\lambda\varepsilon \ll 1$ by changing the variable $x = \lambda\varepsilon$ is reduced to

$$I_2 = \pi n A \lambda \left\{1 + \int_{\lambda\varepsilon}^1 \frac{dx}{x} + \int_0^1 \frac{e^{-x} - 1}{x} dx + \int_1^\infty \frac{e^{-x}}{x} dx\right\}.$$

The sum of the two last integrals in braces is equal to $-C$, where $C = 0.577\ldots$ is Euler's constant, so

$$I_2 = \pi n A \lambda \left(1 - C - \ln \lambda\varepsilon\right).$$

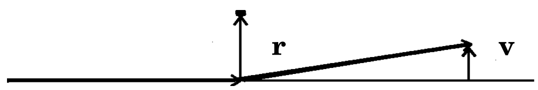


Figure 13.4. The geometry of small-angle scattering

Thus,

$$z \int_{\varepsilon_I}^{\infty} w(q) \left[1 - e^{-\lambda q}\right] dq = \lambda(1 - C - \ln(\lambda \varepsilon_I))\zeta,$$

where $\zeta = \pi n A z$.

The distribution density for the energy loss (Landau distribution) is of the form

$$p(q, z) = \frac{1}{\zeta} \psi(\xi), \tag{13.4.7}$$

where

$$\psi(\xi) = \frac{1}{2\pi i} \int_{-i\infty-\sigma}^{i\infty+\sigma} e^{u \ln u + \xi u} du \tag{13.4.8}$$

and

$$\xi = \frac{q - \zeta(\ln(\zeta/\varepsilon_I) + 1 - C)}{\zeta}.$$

Changing the integration variable in (13.4.8), we conclude that the distribution has the characteristic function of form (3.4.18), and hence the Landau distribution is the stable law with $\alpha = 1$ $\beta = 1$.

For the first time this fact was noted in (Uchaikin & Topchii, 1978) where distribution (13.4.8) was obtained by summing the random (Poisson-distributed) number of random variables. The domain of its validity is investigated in detail in (Asoskov *et al.*, 1982) by experiment (see also (Dudkin *et al.*, 1989)).

13.5. Multiple scattering of charged particles

Multiple elastic scattering of charged particles passing through a layer can be considered in the same way as above. For a small-angle approximation,

$$\mathbf{v} \sim B\mathbf{r}/r^2, \quad B = \text{const},$$

which characterizes the deviation of the particle caused by the nuclei of a single atom being at the point $\mathbf{r}$ (Fig. 13.4).

In this case, the total deviation of the particle $\sum_i \mathbf{v}_i$ leaving the layer is given by the characteristic function

$$f(\mathbf{k}) = \exp\left\{-\rho \int_0^R \int_0^{2\pi} \left[1 - e^{ik(B/r)\cos\varphi}\right] r\,dr\,d\varphi\right\}$$
$$= \exp\left\{-2\pi\rho \int_0^R \left[1 - J_0(kB/r)\right] r\,dr\right\}$$
$$\sim \exp\left\{-2\pi z \int_0^\infty \left[1 - J_0(k\theta)\right] w(\theta)\theta d\theta\right\}, \qquad R \to \infty, \tag{13.5.1}$$

where $\theta = |\mathbf{v}|$,

$$w(\theta) = nB^2\theta^{-4}, \tag{13.5.2}$$

and ρ, n, and z are the same as in Section 13.4. The last formula (13.5.2) is merely the well-known Rutherford cross-section reduced to the small-angle region.

For small scattering angles corresponding to the large impact parameters r, the screen influence of the electron shell of the atom decreases the deviation. In (Molière, 1947; Molière, 1948), this effect is taken into account by introducing into the integrand the factor $q(\theta)$ equal to 1 for the angles θ greater than the screening angle χ and tending to zero as $\theta \to 0$. We represent the characteristic function (13.5.1) as

$$f(\mathbf{k}) = \exp\left\{-2\eta \int_0^\infty \left[1 - J_0(k\theta)\right] q(\theta)\theta^{-3} d\theta\right\}, \tag{13.5.3}$$

where

$$\eta = \pi B^2\rho = \pi B^2 nz.$$

As in the previous case, the integral is separated into two parts corresponding to the intervals $(0, \chi)$ and (χ, ∞). Within the former, the Bessel function is expanded into series

$$I_1 \equiv \int_0^\chi \left[1 - J_0(k\theta)\right] q(\theta)\theta^{-3} d\theta \approx (k^2/4) \int_0^\chi q(\theta)\theta^{-1} d\theta,$$

and after integration by parts we obtain

$$I_1 = (k^2/4)\left[\ln\chi - \int_0^\chi q'(\theta)\ln\theta d\theta\right].$$

We keep in mind that

$$\lim_{\theta\to 0} q(\theta)\ln(\theta) = 0, \qquad q(\chi) = 1.$$

The integral in the square brackets is usually denoted by $\ln \chi_a + 1/2$; thus,

$$I_1 = (k^2/4)\left[\ln(\chi/\chi_a) - 1/2\right]. \tag{13.5.4}$$

The second integral

$$I_2 = \int_\chi^\infty \left[1 - J_0(k\theta)\right] \theta^{-3} d\theta$$

by the change of variable $x = k\theta$ and subsequent integration by parts is reduced to

$$I_2 = (k^2/4)\Big\{2(k\chi)^{-2}\left[1 - J_0(k\chi)\right] + (k\chi)^{-1}J_1(k\chi) - \ln(k\chi)J_0(k\chi) + \int_{k\chi}^\infty \ln x J_1(x)dx\Big\}.$$

Since the lower integration limit is small here, we use the approximate expressions

$$1 - J_0(k\chi) \approx (k\chi)^2/4, \qquad J_1(k\chi) \approx k\chi/2,$$

$$\int_{k\chi}^\infty \ln x J_1(x)dx \approx \int_0^\infty \ln x J_1(x)dx = -C + \ln 2,$$

where C is Euler's constant. Then

$$I_2 \approx (k^2/4)\left[1 - \ln(k\chi) + \ln 2 - C\right]. \tag{13.5.5}$$

Summing (13.5.4) and (13.5.5), and substituting the result into (13.5.3), we obtain

$$\ln f(\mathbf{k}) = -\eta\left(1/2 + \ln 2 - C - \ln(k\chi_a)\right) k^2/2.$$

We set

$$\sqrt{\eta}k = y;$$

then the obtained expression can be rewritten as

$$\ln f(\mathbf{k}) = -\left(b_\theta - \ln(y^2/4)\right) y^2/4,$$

where

$$b_\theta = \ln\left(\eta/\chi_a^2\right) + 1 - 2C.$$

Passing to the constant B_θ related to b_θ by the formula

$$B_\theta = b_\theta + \ln B_\theta \sim b_\theta \propto \ln z, \qquad z \to \infty,$$

we obtain

$$\ln f(\mathbf{k}) = -u^2/4 + \left(u^2/(4B_\theta)\right) \ln\left(u^2/4\right),$$

where

$$u = \sqrt{B_\theta} \cdot y.$$

Performing the inverse Fourier transformation, we arrive at the formula

$$2\pi p(\theta, z)\theta d\theta = \psi(\tilde{\theta}, B_\theta)\tilde{\theta} d\tilde{\theta},$$

where

$$\psi(\tilde{\theta}, B_\theta) = \int_0^\infty J_0(\tilde{\theta}u)e^{-u^2/4} \exp\left\{\left(u^2/(4B_\theta)\right) \ln(u^2/4)\right\} u\, du,$$
$$\tilde{\theta} = \theta/\sqrt{\eta B_\theta}.$$

Expanding the second exponential under the integral sign into the series, we obtain

$$\psi(\tilde{\theta}, B_\theta) = \sum_{n=0}^{\infty} B_\theta^{-n} \psi_n(\tilde{\theta}), \tag{13.5.6}$$

where

$$\psi_\theta(\tilde{\theta}) = (n!)^{-1} \int_0^\infty J_0(\tilde{\theta}u)e^{-u^2/4} \left[(u^2/4)\ln(u^2/4)\right]^n u\, du. \tag{13.5.7}$$

Formulae (13.5.6)–(13.5.7) are called the Molière distribution.

The leading (for large thickness z) term of the expansion is the normal distribution, therefore

$$p^{\mathrm{as}}(\theta, z) = (\pi\eta B_\theta)^{-1} \exp\left\{-\theta^2(B_\theta\eta)^{-1}\right\}, \tag{13.5.8}$$

where the width of the angle distribution

$$\Delta = (B_\theta\eta)^{1/2}$$

increases together with the layer thickness z by the law

$$\Delta \propto \sqrt{z \ln z}. \tag{13.5.9}$$

Since the speed of particle motion along the z-axis, practically, does not change, this coordinate can be interpreted as time, and the process of small angle multiple scattering itself, as diffusion in the two-dimensional space. From this point of view, formula (13.5.8) allows us to talk about anomalous diffusion. The anomaly manifests itself by deviation of law (13.5.8) from the normal case $\Delta \propto \sqrt{z}$, although the form of distribution (13.5.7) still remains normal. Such a type of anomality arises because the variance of single term $\mathbf{u}_i$

$$\mathrm{Var}\,\mathbf{u}_i = 2\pi \int_0^\infty \theta^2 w(\theta)\theta\, d\theta$$

diverges but this divergence, according to (13.5.2), is of logarithmic type. In other words, we deal with the limiting case of summation of the terms with $\alpha = 1$ (see Section 2.5) leading to the normal law.

In the actual reality, the scattering angle θ is restricted by the mass ratio for the incident particle and electron, and besides, the very small-angle approximation holds true only in the domain of small angles $\theta \ll 1$. Therefore, result (13.5.6)–(13.5.8) should not be considered as absolute (in application to a particular problem), and one should rather use it as some intermediate asymptotics valid in a certain thickness interval $(z_{\min}, z_{\max})$. Indeed, experiments show the existence of such an area (Starodubtsev & Romanov, 1962).

13.6. Fractal turbulence

In (Takayasu, 1984) a vortex model of the fully developed turbulence is proposed, which regards the turbulence as the velocity field generated by randomly fractally distributed sources of circulation such as vortex filaments. The idea is inspired by Mandelbrot's comments on the Holtsmark distribution: If we regard the stars as point vortices and the gravitational field as the velocity field, then we can expect that Holtsmark's method is also applicable to the turbulence. In order to consider the effect of the fractal structure of the turbulence, we generalize Holtsmark's method in the case that the point vortices distribute fractally (Takayasu, 1984).

The velocity $\mathbf{u}(\mathbf{r})$ at a point $\mathbf{r}$ is given by the Biot–Savart law

$$\mathbf{u}(\mathbf{r}) = \sum_{j=1}^{N} \mathbf{u}_j,$$

$$\mathbf{u}_j \equiv -\frac{\mathbf{w}_j \times (\mathbf{r} - \mathbf{r}_j)}{4\pi|\mathbf{r} - \mathbf{r}_j|^3},$$

where $\mathbf{r}_j$ and $\mathbf{w}_j$ denote the position and the vorticity of the jth vortex. The velocity distribution is expressed as

$$W_N(\mathbf{u}) = \frac{1}{8\pi^3} \int d\mathbf{q} e^{-i\mathbf{q}\mathbf{u}} \tilde{W}_N(\mathbf{q}),$$

$$\tilde{W}_N(\mathbf{q}) = \prod_{j=1}^{N} \int d\mathbf{w}_j \int d\mathbf{r}_j e^{i\mathbf{q}\mathbf{u}_j} p_j(\mathbf{r}_j, \mathbf{w}_j),$$

where $p_j(\mathbf{r}_j, \mathbf{w}_j)$ governs the probability density of occurrence of the jth vortex at the position $\mathbf{r}_j$ with the vorticity $\mathbf{w}_j$.

In order to take account for the effect of fractal structure of turbulence, they postulate that the vortices are distributed uniformly over a random D-dimensional fractal domain, the number of vortices within a sphere of radius

R, $N(R)$, is statistically proportional to R^D and may be expressed as

$$N(R) = \eta \frac{\pi^{D/2}}{\Gamma(1 + D/2)} R^D.$$

Here η is a constant standing for the density of vortices in the fractal domain, and $\Gamma(x)$ is the gamma function.

The fractal dimension of the real turbulence is estimated both experimentally and theoretically to be 2.6 ± 0.1; hence $2 < D < 3$ in contrast to the distribution of galaxies in the Universe, where $1 < D < 2$. Neglecting the spatial correlations among the vortices, i.e., taking $p_j(\mathbf{r}_j, \mathbf{w}_j)$ in the form

$$p_j(\mathbf{r}_j, \mathbf{w}_j) = \frac{\Gamma(1 + D/2)}{\pi^{D/2} R^D} p(\mathbf{w}_j)$$

on the fractal set, and performing calculation according to Holtsmark's method, one arrives at a spherically symmetric three-dimensional stable distribution with characteristic $D/2$. In the same manner, for a small light particle which drifts with the field element, the anomalous diffusion law takes place: $\langle|\mathbf{X}(t)|\rangle \propto t^{2/D}$. A similar result is obtained for an electron in a uniformly magnetized plasma: the electron motion perpendicular to the magnetic field is the Lévy process with characteristic $D/2$:

$$\langle|\mathbf{R}_\perp(t) - \mathbf{R}_\perp(0)|\rangle \propto t^{2/D}.$$

Even in the case where the particles are distributed uniformly in $\mathbf{R}^3$, that is, in the case $D = 3$, the motion of the electron is still anomalous. This might indicate a connection with the anomalous electron transport phenomena observed experimentally in many fusion devices of magnetic confinement.

13.7. Stresses in crystalline lattices

As is well known, crystalline structures distinguish themselves by a rigid geometric arrangement of their atoms. However, this idealized representation of crystals turns out to be valid only in very small parts of them. Actual crystals always have various disturbances in their structure, either because there are sometimes extraneous atoms at a prescribed location, or because there are no atoms at some locations.

Such anomalies in crystalline lattices are called dislocations. They may be scattered in the body of a crystal, but they may also be concentrated, forming lines and even surfaces of complicated configurations. We consider the case of identical point dislocations uniformly scattered in the body of a crystal with average density ρ (Poisson's ensemble).

The stress tensor σ_{mn} at some point (for example at the origin) is a function of dislocations coordinates $\mathbf{X}_1, \mathbf{X}_2, \ldots$:

$$\sigma_{mn} = \sigma_{mn}(\mathbf{X}_1, \mathbf{X}_2, \ldots).$$

As far as the superpositional principle holds true, the function can be represented in the form of a sum

$$\sigma_{mn} = \sum_i S_{mn}(\mathbf{X}_i)$$

where $S_{mn}(\mathbf{r})$ is the stress tensor generated at the origin by a single dislocation placed at the point $\mathbf{r}$. In the general case of a non-isotropic elastic medium and various models of dislocations

$$S_{mn}(\mathbf{r}) = A_{mn}(\mathbf{\Omega})r^{-3},$$

where $\mathbf{\Omega} = \mathbf{r}/r$. Following the above way, we obtain the characteristic function $f_{mn}(k)$ of the random stress tensor σ_{mn} in the infinite solid body

$$\begin{aligned} f_{mn}(k) &= \exp\left\{\rho \int_0^\infty dr\, r^2 \int d\mathbf{\Omega} \left\{\cos[kA_{mn}(\mathbf{\Omega})r^{-3}] - 1\right\} d\mathbf{\Omega}\right\} \\ &= \exp\left\{-\rho a_{mn} k\right\}, \end{aligned}$$

where

$$a_{mn} = \frac{\pi}{6} \int |A_{mn}(\mathbf{\Omega})| d\mathbf{\Omega}.$$

Therefore the random stress tensor σ_{mn} is distributed by the one-dimensional Cauchy law.

13.8. Scale-invariant patterns in acicular martensites

The name martensite is used to denote metastable configurations in a wide variety of systems ranging from metals to polymeric crystals and crystalline membranes. We give here some results from (Rao & Segupta, 1996) devoted to acicular martensites occurring in Fe–C and Fe–Ni systems. Pure Fe exists in three crystalline phases at atmospheric pressures, denoted by α (ferrite), γ (austenite) and δ. The room temperature structure of Fe is BCC (α), which when heated to a temperature of 910°C, undergoes a structural change to the FCC (γ) phase. Further heating to 1400°C transforms it back to BCC (δ). Fast quenching can preempt the equilibrium solid state transformation $\gamma \to \alpha$ by the formation of a martensite. Alloying elements, e.g. C and Ni, are known to facilitate the formation of this metastable phase. The kinetic and structural features of such martensites are discussed in (Rao & Segupta, 1996).

The model under consideration is restricted to two dimensions. The transformed two-dimensional austenite region is bounded by a one-dimensional grain boundary, taken to be a square of size L. Following a quench below T_{ms}, the austenite region becomes unstable to the production of critical size martensite nuclei. It is reasonable to assume that the spatial distribution of the nuclei is homogeneous (in the absence of correlated defects) and uncorrelated (in the absence of auto catalysis). The detailed substructure of the grain is irrelevant to the study; the high aspect ratio then allows us to represent the lens-shaped grain as a straight line. The model should be viewed as being coarse-grained over length scales corresponding to the width of the martensite grains. This provides a small length scale cut-off, ε.

Thus at time $t = 0$, one starts with p nuclei, uniformly distributed within a square of size L. Once nucleated, the points grow out as lines whose lengths grow with constant velocity v. The tips of these lines grow out with slopes ± 1, with equal probability. As these lines grow, more nuclei are generated at rate I (rate of production of nuclei per unit area). The nucleation events are uniformly distributed in time—thus p nuclei are born at each instant until a total nucleation time t_N; so, finally, a total of N lines are nucleated. A tip of a line, l, is defined as the Euclidean distance between the two tips that have stopped due to a collision with other lines or the boundary. After a time $t > t_N$, when all the lines have stopped growing, one asks for the stationary probability distribution of line lengths, $p(l)$.

There are three time scales which describe the entire physics: $t_v = (N/L^2)^{-1/2} v^{-1}$, $t_I = L^{-2} I^{-1}$ and t_N. Taking t_I as a unit of time, one can construct the dimensionless variables $\tilde{t}_v = t_v / t_I$ and $\tilde{t}_N = t_N / t_I$. As shown in (Rao & Segupta, 1996), there are two extreme geometrical limits. When $\tilde{t}_v^{-1}$, nucleation of N grains occurs in one single burst. The numerically computed $p(l)$ peaks at the characteristic length scale $\propto (N/L^2)^{-1/2}$ and is described by the gamma distribution,

$$p(x) = \frac{a^{\mu} x^{\mu-1} e^{-ax}}{\Gamma(\mu)}.$$

Here $x = l/(N/L^2)^{-1/2}$ and $a = 1.64 \pm 0.02$ and $\mu = 4.59 \pm 0.1$. The lower bound for the exponent μ can be argued to be 3. The simulated microstructure in this limit shows grains of comparable sizes. The other geometric limit is obtained as $\tilde{t}_v^{-1} \to \infty$. In this limit, subsequent nuclei partition space into smaller and smaller fragments, leading to a scale-invariant probability distribution which shows an extremely good fit to a stable distribution $q(x; \alpha, 1)$ with $\alpha \approx 1.51 \pm 0.03$ (Rao & Segupta, 1996).

13.9. Relaxation in glassy materials

Relaxation in amorphous materials, like glassy materials or viscous liquids, is the time-dependent change in any macroscopic material property $R(t)$ (density, entropy, optical properties, or structural factor) following a perturbation (change in temperature, stress, electric, or magnetic field).

It is observed that experimental relaxation behavior in many glassy materials correlates well with the Williams–Watts function

$$R(t) = \exp\{-(t/\tau_e)^\alpha\}, \qquad 0 < \alpha < 1, \tag{13.9.1}$$

where α and τ_e are some constants for a given material: α is the slowness index, and τ_e is the effective relaxation time. This contrasts to the conventional Debye exponential form

$$R(t) = \exp\{-t/\tau_0\}, \tag{13.9.2}$$

where τ_0 is the Debye relaxation time.

The interest in relaxation problem caused by both theoretical and technological reasons provides a number of models explaining the universality of formula (13.9.1). Following (Weron & Weron, 1985; Weron & Weron, 1987; Weron, 1986), we consider one of them leading to stable distributions.

The statistical approach interprets the non-exponential relaxation behavior (13.9.1) of the material in terms of a superposition of exponentially relaxing processes (13.9.2):

$$R(t) = \int_0^\infty e^{-t/\tau} p(\tau)\, d\tau, \tag{13.9.3}$$

where $p(\tau)$ is the density of a distribution of relaxation times τ across different atoms, clusters, or degrees of freedom. If $\mu = \tau_0/\tau$, where τ_0 is a single relevant relaxation time associated with Debye relaxation, then μ is called the relaxation rate and is interpretable as dimensionless time. Substituting $s = t/\tau_0$ into (13.9.3), we obtain

$$R(\tau_0 s) = \int_0^\infty e^{-s\mu} \tau_0 \mu^{-2} p(\tau_0/\mu)\, d\mu = \int_0^\infty e^{-s\mu} w(\mu)\, d\mu, \tag{13.9.4}$$

where

$$w(\mu) = \tau_0 \mu^{-2} p(\tau_0/\mu) \tag{13.9.5}$$

is the density of a distribution of dimensionless rates. Since this approach is microscopically arbitrary, one may consider the random variables $\mu_i = \tau_0^i$, $i = 1, \dots, n$, as the possible relaxation rates of elements in a given complex material. Here n indicates the total number of the elements in the system, and μ_i are independent and identically distributed by (13.9.5) random variables.

Under these hypotheses,

$$\mu = \sum_{i=1}^n \mu_i,$$

and to use the limit theorem in the case of a large number of terms n, we need to introduce the normalization

$$\mu' = \frac{1}{b_n}\left(\sum_{i=1}^n \mu_i - a_n\right), \qquad b_n > 0. \tag{13.9.6}$$

As seen from (13.9.4) and (13.9.1),

$$\int_0^\infty \mu w(\mu)\,d\mu = -\tau_0 \cdot R(0) = \infty.$$

Thus, if the variable μ' has a limit distribution, it should be a stable distribution with $\alpha < 1$ and $\beta = 1$. Then

$$a_n = 0, \qquad b_n = b_1 n^{1/\alpha}$$

and

$$w(\mu) = b_n q(b_n\mu; \alpha, 1), \qquad 0 < \alpha < 1. \tag{13.9.7}$$

Substituting (13.9.7) into (13.9.4) and recalling the generalized limit theorem (Section 2.5), we arrive at formula (13.9.1).

As observed in (Weron, 1986), the statistical approach explains the universal character of formula (13.9.1) as the consequence of the use of universal limit law in macroscopic behavior of the relaxing system.

13.10. Quantum decay theory

The problem considered above is closely related to the general quantum decay theory.

In quantum mechanics, the state of an unstable physical system is described by the so-called state vector $|\psi(t)\rangle$, which is a solution of the time-dependent Cauchy problem for the Schrödinger equation

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle, \tag{13.10.1}$$

where H is the Hamiltonian operator (Hermitian operator) corresponding to the system, and $|\psi(0)\rangle$ is a given initial state vector. The units are chosen so that the Planck constant $\hbar = 1$. Let $\{|\varphi_E\rangle, |\varphi_k\rangle\}$ be the complete system of eigenvectors of the operator H ($|\varphi_E\rangle$ corresponds to the absolutely continuous component of its spectrum, and $|\varphi_k\rangle$ corresponds to the discrete component), i.e.,

$$H|\varphi_E\rangle = E|\varphi_E\rangle, \qquad \langle\varphi_{E'}|\varphi_E\rangle = \delta(E'-E),$$
$$H|\varphi_k\rangle = E_k|\varphi_k\rangle, \qquad \langle\varphi_k|\varphi_l\rangle = \delta_{kl},$$

where $\delta(E'-E)$ is the Dirac delta function, and δ_{kl} is the Kronecker symbol.

We are interested in the probability $P(t)$ that at a time t the system is in the initial state $|\psi_0\rangle$. According to the laws of quantum mechanics,

$$P(t) = |\langle\psi(0)|\psi(t)\rangle|^2.$$

Solving the Cauchy problem (13.10.1) for the Schrödinger equation, we assume that $\langle\psi(0)|\psi(0)\rangle = 1$. In this case, the Fock–Krylov theorem (Krylov & Fock, 1947) yields

$$f(t) = \langle\psi(0)|\psi(t)\rangle = \sum_k |c_k|^2 \exp(-iE_k t) + \int_0^\infty |c(E)|^2 \exp(-iEt)\,dt, \tag{13.10.2}$$

where c_k and $c(E)$ are the Fourier coefficients in the expansion of the vector $|\psi(0)\rangle$ in the complete system $\{|\varphi_E\rangle, |\varphi_k\rangle\}$ of eigenvectors

$$|\psi(0)\rangle = \sum_k c_k|\varphi_k\rangle + \int_0^\infty c(E)|\varphi_E\rangle dE.$$

Thus, $f(t)$ can be interpreted as the characteristic function of some distribution having discrete components (probabilities of isolated values) $|c_k|^2$ and absolutely continuous component (i.e., density) $|c(E)|^2$. Instability of the system means that the probability $P(t) = |f(t)|^2$ of the system returning to the original state at time t tends to zero as $t \to \infty$.

Since $f(t)$ is a characteristic function, $|f(t)| \to 0$ only if the discrete components of the spectrum of H are missing, i.e., $c_k = 0$. In this case

$$f(t) = \int_0^\infty \omega(E) \exp(-iEt)\,dE, \tag{13.10.3}$$

where $\omega(E) = |c(E)|^2$ denotes the density of the energy distribution of the decaying physical system described by equation (13.10.1).

It turns out that for a very broad class of unstable physical systems the densities $\omega(E)$ are meromorphic functions (see (Krylov & Fock, 1947)). For a number of reasons, the case of a function $\omega(E)$ having only two simple poles (they are complex conjugated in view of the condition $\omega(E) \geq 0$) is of great interest. In this case it is obvious that

$$\omega(E) = A[(E - E_0)^2 + \Gamma^2]^{-1}, \qquad E \geq 0,$$

where A is a normalizing constant, and E_0 and Γ are the most probable value and the measure of dispersion (with respect to E_0) of the system's energy. For actual unstable systems[1] the ratio Γ/E_0 is very small, as a rule (10^{-15}, or even smaller). Therefore, to compute $P(t)$ we can, without adverse effects, replace the lower limit 0 in integral (13.10.3) by $-\infty$, after which the density function $\omega(E)$ and the probability $P(t)$ connected with it take the approximate expressions

$$\omega(E) \approx \frac{\Gamma}{\pi}[(E - E_0)^2 + \Gamma^2]^{-1},$$
$$P(t) = |f(t)|^2 \approx \exp(-2\Gamma t).$$

[1] An example of such a system is a neutron with an average lifetime of 18.6 min decaying at the end of its lifetime into a photon, an electron, and a neutrino ($n \to p + e + \nu$).

It is clear from the first relation (the Lorentz distribution of the energy of the unstable system) that we are dealing with the Cauchy distribution, and it is clear from the second relation that the lifetime for unstable systems of the type under consideration behaves according to the exponential law.

Thus, the Cauchy law appears here only as a more or less good approximation of the real energy distribution for unstable systems. And there are situations where the replacement of 0 by $-\infty$ in (13.10.3) is unacceptable, because the corresponding law $P(t)$ of decay of the system differs essentially from the exponential law.

We give here a result of (Hack, 1982). Imposing the constraint normally applied in quantum theory that the self-adjoint Hamiltonian H is lower semi-bounded, i.e., that the energy spectrum is bounded below, Hack established that $P(t)$ cannot decay exponentially fast as $t \to \infty$, i.e.,

$$P(t) > Ce^{-at}$$

for $t > T$, where C, a and T are positive constants.

The following theorem is proved in (Weron & Weron, 1985).

THEOREM 13.10.1. *The non-decay probability function for many-body weakly interacting quantum system is of the form*

$$P(t) = \exp\left\{-at^{\alpha}\right\}, \quad a > 0, \quad 0 < \alpha < 1.$$

Representing the amplitude $f(t)$ as

$$f(t) = \langle\psi|\exp(-Dt)|\psi\rangle = \int_0^{\infty} \exp(-Et)p(E)\,dE$$

where D is the development operator governing the dynamic evolution of the quantum system under investigation and $p(E)$ is the probability density of the state $|\psi\rangle$ associated with the continuous spectrum of the development operator D, the authors conclude that we observe an arbitrariness in the specification of ψ and $p(E)$. In general, one considers ψ to represent a decaying state for a many-body system, and therefore the number of components in the system should not influence the decay. In other words, the same decaying law should be obtained for one portion or several portions of the system. Consequently, in a weakly interacting quantum system, microscopic energies can be considered as independent identically distributed energy random variables. The microscopic energy distribution $p(E)\,dE$ associated with the decaying system is identified to be the limit distribution of normalized sums of the microscopic energy random variables. By the limit theorem (Gnedenko & Kolmogorov, 1954), it is well known that the limit $p(E)\,dE$ has α-stable distribution $0 < \alpha \le 2$. Since $p(E)$ is associated from the above construction with the development operator D, it has to have positive support. This holds only when $p(E)dE$ has a completely asymmetric ($\beta = 1$, $0 < \alpha < 1$) stable distribution (Weron & Weron, 1985).

13.11. Localized vibrational states (fractons)

The inelastic scattering of extended-electronic states off running-wave (phonon) states is well investigated (Ziman, 1962). The analogous quantity, but for localized vibrational states is calculated in (Entin-Wohlman *et al.*, 1985) (see also (Lévy & Souillard, 1987; Terao *et al.*, 1992; Yakubo & Nakayama, 1989)). Localization can occur by virtue of the geometrical connectivity properties of the atomic network, in which case one refers to the vibrational excitations as fractons. Only the vibrations of random fractal networks were considered for which localization of the vibrational states occurred if the fracton dimensionality, d, was less than 2. In this model, the density of states is governed by d, and the vibrational localization length scales with a negative power of frequency. One expects $1 < d < 2$ for a mechanical model where scalar elasticity is dominant. In the case of a purely mechanical rotationally invariant model, d can be less than unity, implying a weak divergence of the vibrational density of states with decreasing frequency. Impurities in an otherwise translationally invariant atomic network can also result in vibrational localization. One refers to these excitations as localized phonons.

Throughout (Entin-Wohlman *et al.*, 1985), the results are expressed in terms which are equally applicable to a vibrational structure with only 'scalar' forces between vibrational atoms and for the structures for which purely mechanical central and 'bending' forces are appropriate. Both assumptions lead to identical scalable forms for the dispersion law and density of vibrational states, with only the constant θ, involved in the range dependence of the force constant, changing its value. In particular, the range dependence of the scalar force constant, K_s, depends on θ as

$$K_s \propto r^{-D-2-\theta},$$

where D is the fractal dimension. The force constant, $K_{c,b}$, for central and bending forces varies as

$$K_{c,b} \propto r^{-\tilde{\zeta}_E}.$$

Hence, if one identifies $\theta_{c,b} = \tilde{\zeta}_E + D - 2$, all the results for scalar forces go over directly to those one would have found for central and bending forces.

In particular, the spatial Fourier transform of the overlap of two wave functions $\phi_a(\mathbf{r})$ of the same mode index a was evaluated:

$$I_a(\mathbf{k}) = \int d\mathbf{r} e^{i\mathbf{kr}} I_a(\mathbf{r}), \tag{13.11.1}$$

$$I_a(\mathbf{r}) = \int d\mathbf{r}' \phi_a(\mathbf{r} + \mathbf{r}') \phi_a^*(\mathbf{r}'). \tag{13.11.2}$$

In order to evaluate $I_a(\mathbf{k})$ within the fracton regime, they write the fracton wave-function assuming simple exponential localization:

$$\phi_a(\mathbf{r}) = A_a \exp\left\{-(1/2)(r/\lambda_a)^{d_\phi}\right\}, \tag{13.11.3}$$

where A_a is the normalization coefficient, and d_ϕ characterizes the localization of the vibrational wave function in a fractal network. From (13.11.1) and (13.11.2), it follows that

$$I_a(\mathbf{k}) = |\phi_a(\mathbf{k})|^2,$$

where $\phi_a(\mathbf{k})$ is the Fourier transform of the wave function in (13.11.3):

$$\phi_a(\mathbf{k}) = 2A_a\pi^{D/2}(2/k)^{D/2-1}\int_0^\infty \exp\left\{-(1/2)(r/\lambda_a)^{d_\phi}\right\} J_{D/2-1}(qr)r^{D/2}dr. \tag{13.11.4}$$

Formulae (13.11.3) and (13.11.4) labelled in (Entin-Wohlman *et al.*, 1985) by (11) and (13), respectively, are nothing but the characteristic function and the distribution density of multivariate spherically symmetric stable law expanded to the fractional dimension D (see (7.5.5)).

13.12. Anomalous transit-time in some solids

Measurements of the transient photocurrent $I(t)$ in organic and inorganic amorphous materials including metallic films and semiconductors display anomalous transport properties. The long tail of $I(t)$ indicates a dispersion of carrier transit times. However, the shape invariance of $I(t)$ to electric field and sample thickness is incompatible with traditional concepts of statistical spreading, i.e., a Gaussian carrier packet. We consider the problem following (Tunaley, 1972; Scher & Montroll, 1975).

If a metallic film is sufficiently thin, the material will be essentially in the form of conducting islands separated by small distances. However if the islands are randomly scattered, some possible paths will be effectively blocked. A similar situation exists in problems of percolation, but in the present treatment the carriers are allowed only to jump in the direction of the applied field, and so the number of available paths is already severely restricted. Furthermore, one does not assume that a path is either open or closed but the probability of a jump is always finite though it may be very small. Thus the model is relevant to the case where the influence of an applied field dominates the situation rather than thermal excitations.

For the sake of simplicity one can consider point sites although in a thin metallic film the sites will of course have a finite area. Following (Tunaley, 1972) we adopt a quasi-classical approach and assume that the carriers have to jump from one site to the next by a tunneling process. The lattice is aperiodic to the extent that no resonant modes for electrons or holes exist except at the sites so that the probability of an electron, for example, to jump from one site to another one 'far away' in one step is extremely small. It is assumed that the sites are distributed at random so that the distance between a site chosen at random and its nearest neighbor has a finite mean and variance.

To illustrate the process, we consider a two-dimensional thin film composed of small metallic islands grown on randomly scattered nuclei. The time T_i for a carrier to jump from one island to another will have a probability distribution which is exponential,

$$F_{T_i}(t) = 1 - \exp(-t/\tau), \tag{13.12.1}$$

where τ is the average time for a jump. However, τ will depend on the random distance X between islands, and so it is a random variable Θ. It is assumed (Harper, 1967) that Θ is related to X, the island spacing, by

$$\Theta = \beta[\exp(\gamma X) - 1], \tag{13.12.2}$$

where β is inversely proportional to the applied potential gradient. Jumps in a direction opposite to the potential gradient are not allowed in this model, and the carriers are supposed to be independent.

Because Θ increases so rapidly with X, jumps to the nearest neighbor island only need be considered, so that X becomes the separation distance of the nearest neighbor in the forward direction. Since the field is greatest in the forward direction provided that the islands are equipotential, the two-dimensional problem can be reduced to the one-dimensional one with some approximation. For one-dimensional islands, the distribution of the gap X between two islands is exponential:

$$F_X(x) = 1 - \exp(-\mu x), \tag{13.12.3}$$

where μ^{-1} is the average island spacing.

Combining (13.12.2) and (13.12.3) for the distribution of the time constants Θ yields

$$\begin{aligned} F_\Theta(t) &= \mathsf{P}\left\{\beta[e^{\gamma X} - 1] < t\right\} \\ &= \mathsf{P}\left\{X < \frac{1}{\gamma}\ln(1 + t/\beta)\right\} \\ &= 1 - \exp\left\{-\frac{\mu}{\gamma}\ln(1 + t/\beta)\right\} \\ &= 1 - (1 + t/\beta)^{-\nu}, \end{aligned} \tag{13.12.4}$$

where $\nu = \mu/\gamma$. The total time for a carrier to reach the point x is the sum of a random number N of independent times T_i, $i = 1, \dots, N$ for jumps. In view of (13.12.3), the sites (points of waiting) constitute the Poisson ensemble, and

$$\mathsf{P}\{N = n\} = e^{-a}a^n/n!$$

with $a = \mu x$. Hence the Laplace transform of the distribution density $p_T(t, x)$ of the total time T is

$$p_T(\lambda, x) = \exp\{-\mu x[1 - q(\lambda)]\} \tag{13.12.5}$$

where

$$q(\lambda) = \int_0^\infty e^{-\lambda t} q(t)\, dt,$$

and

$$q(t) = dF_\Theta(t)/dt = \frac{\nu}{\beta(1 + t/\beta)^{\nu+1}}. \tag{13.12.6}$$

The nth moment of the distribution

$$\int_0^\infty t^n q(t) dt = \nu\beta^n B(n + 1, \nu - n - 1)$$

exists as soon as $\nu > n + 1$. If $\nu > 3$, then

$$q(\lambda) \sim 1 - \bar{\Theta}\lambda + \overline{\Theta^2}\lambda^2/2, \qquad \lambda \to 0. \tag{13.12.7}$$

Inserting (13.12.7) into (13.12.5) as $x \to \infty$, we obtain the asymptotic expression

$$p_T(\lambda, x) \sim \exp\left\{-\mu x(\bar{\Theta}\lambda - \overline{\Theta^2}\lambda^2/2)\right\}, \qquad \lambda \to 0,$$

which yields, after inversion, the normal distribution

$$p_T(t, x) = \frac{1}{\sqrt{2\pi\mu x}\sigma_\Theta} e^{-(t-\mu x\bar{\Theta})^2/2\mu x\sigma_\Theta^2}$$

with $\sigma^2 - \Theta = \overline{\Theta^2} - \bar{\Theta}^2$. This is merely the result of applying the central limiting theorem to the sum $T_1 + \ldots + T_N$. There is no necessity of performing some additional calculations to understand that in the case $1 < \nu < 2$ the random value $(T - \mu x\bar{\Theta})/(b_1(\mu x)^\nu)$ is distributed according to the stable density $q_A(x; \nu, 1)$ and in the case $0 < \nu < 1$ the value $T/(b_1(\mu x)^{1/\nu})$ has a one-sided stable distribution, as follows from the generalized central limit theorem.

Such kind of the transit time distributions are observed in many materials (Scher & Montroll, 1975).

13.13. Lattice percolation

The notion of percolation was introduced in (Broadbent & Hammersley, 1957) as opposing to the diffusion. While diffusion assumes a random walk of a particle in a regular environment, percolation consists in a regular motion (of a liquid or electricity) in a random environment. As a model of such an environment, they usually take a periodic lattice whose each site is 'occupied' with probability p and empty with probability $1-p$. The totality of the occupied sites together with their nearest neighbors forms a cluster. The standard problems of percolation theory are the study of distribution of sizes and other geometric characteristics of the clusters; the determination of the so-called

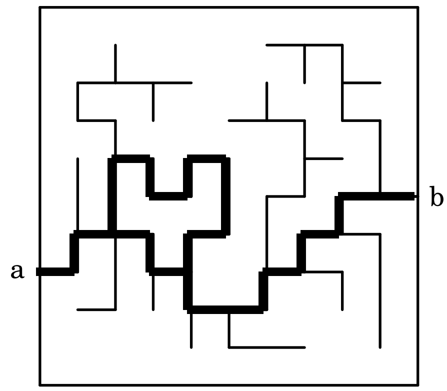


Figure 13.5. Percolation on a two-dimensional lattice (the backbone is shown in heavy lines, the dangling ends in light lines

percolation threshold, $p = p_c$, when the cluster becomes infinite, and so on. Along with the site percolation, the bond percolation is studied, with clusters of connected conducting bonds. The bonds are conducting with the probability p and, correspondingly, blocked with the probability $1 - p$. The model is used for investigation of the problem of the electrical conductivity of a random resistor network. For this problem, another object is relevant—the 'backbone' of an infinite percolation cluster defined as the network of unblocked connected bonds, through which one can go to infinity by at least two non-intersecting paths.

In other words, the backbone is a set of bonds through which the electric current would flow if a voltage is applied to the cluster at infinitely remote electrodes. The rest of the cluster is referred to as a collection of 'dead' or 'dangling ends'. A dangling end can be disconnected from the cluster by cutting a single bond (see Fig. 13.5).

The effect of 'dead ends' on the diffusion and drift of particles along percolation cluster is studied in the frame of a simplified model called the 'comb structure' (Fig. 13.6). The 'teeth' of this comb behave as traps in which x-coordinate of the particle stays for some time while y-coordinate diffuses along the tooth. For infinitely deep teeth, the waiting time distribution $q_0(t)$ is simply the distribution of the first return time at the origin of a one-dimensional Brownian motion (see Section 10.7)

$$q_0(t) \propto t^{-3/2}, \qquad t \to \infty.$$

Thus the diffusion of the particle along x-axes asymptotically does not differ from the subdiffusion process described in Section 12.4.

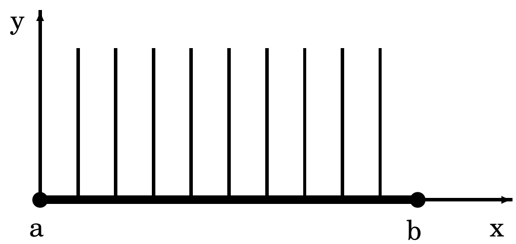


Figure 13.6. A comb-like structure

This problem was introduced in (Nigmatullin, 1986). The equation with the fractional time derivative of the order 1/2 was obtained, and the correct asymptotic behavior of $\langle x^2(t)\rangle \propto t^{1/2}$ was obtained by means of generating function technique, but the Green function was approximated by the Gauss law. The correct Green function obtained in (Arkhincheev & Baskin, 1991) coincides, up to the scale factor, with

$$p^{as}(x,t) = (Dt^{1/2})^{-1/2}\Psi^{(2,1/2)}\left\{(Dt^{1/2})^{-1/2}x\right\}$$
$$= \frac{1}{2\pi\sqrt{Dt}}\int_0^\infty e^{-\frac{\tau^2}{4t}-\frac{x^2}{4D\tau}}\tau^{-1/2}d\tau,$$

which is a one-dimensional analogue of (12.7.8) with $\omega = 1/2$.

In general, the scope of the percolation theory is large enough. As noted in (Isichenko, 1992), a list of problems which percolation theory has been applied to includes hopping conduction in semiconductors (Seager & Pike, 1974; Shklovskii & Efros, 1984), gelation in polymers (de Gennes, 1979; Family, 1984), electron localization in disordered potentials (Ziman, 1969; Ziman, 1979; Thouless, 1974), the quantum Hall effect (Trugman, 1983), flux vortex motion (Trugman & Doniach, 1982), and intergrain Josephson contacts (Gurevich *et al.*, 1988) in superconductors, the gas-liquid transition in colloids (Safran *et al.*, 1985), permeability of porous rocks (Sahimi, 1987; Thompson *et al.*, 1987) and of fractured hard rocks (Balberg *et al.*, 1991), plasma transport in stochastic magnetic fields (Kadomtsev & Pogutse, 1979; Yushmanov, 1990; Isichenko, 1991), turbulent diffusion (Gruzinov *et al.*, 1990), epidemic processes (Grassberger, 1983), and forest fires (MacKay & Jan, 1984).

13.14. Waves in medium with memory

In the statics of a solid body, the strain ε at some point is related to the stress σ at the same point by the formula (for the sake of simplicity, we restrict our consideration to the one-dimensional case)

$$\varepsilon = \sigma/E$$

where $E > 0$ is the elasticity modulus. The same relation is of frequent use in dynamics as well:

$$\varepsilon(t) = \sigma(t)/E.$$

Nevertheless, the domain of applicability of the last relation is limited: in dynamic processes, the medium strain depends not only on the stress $\sigma(t)$ at a given time t, but on the prehistory $\sigma(\tau)$, $\tau < t$, as well. In other words, $\varepsilon(t)$ becomes a functional of $\sigma(\cdot)$. This idea lies in the heart of the theory of residual elasticity, whose most elaborated model is linear:

$$\varepsilon(t) = \left[\sigma(t) + \int_{-\infty}^{t} K(t-\tau)\sigma(\tau)d\tau\right]/E. \tag{13.14.1}$$

Considering relation (13.14.1) as an integral equation in $\sigma(t)$,

$$\sigma(t) = -\int_{-\infty}^{t} K(t-\tau)\sigma(\tau)d\tau + E\varepsilon(t)$$

and assuming that the Neumann series converges, we obtain

$$\sigma(t) = E\left[\varepsilon(t) - \int_{-\infty}^{t} R(t-\tau)\varepsilon(\tau)\,d\tau\right],$$

where

$$R(t) = \sum_{n=1}^{\infty}(-1)^{n-1}K^{(n)}(t).$$

The functions $K(t)$ and $R(t)$ defined for $t > 0$ are referred to as the creepage kernel and the relaxation kernel respectively. They are usually assumed to be non-negative and monotonically decreasing (the last assumption is called the hypothesis about memory decay (Lokshin & Suvorova, 1982). Experiments demonstrate that, as $t \to 0$, these functions rapidly increase, which gives us grounds to make use of singular (i.e., growing without bounds as $t \to 0$) kernels that satisfactorily simulate the experimental data in a certain time interval $0 < a < t < b$.

Let $u(x,t)$ describe the strain of the one-dimensional medium, ρ be its density and $f(x,t)$ be the volume stress density. Then the behavior of a residual elastic medium is determined by the known simultaneous equations

$$\rho\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial\sigma(x,t)}{\partial x} + f(x,t), \tag{13.14.2}$$

$$\partial u(x,t)/\partial x = \varepsilon(x,t), \tag{13.14.3}$$

complemented by the constitutive equation

$$\sigma(x,t) = E\left[\varepsilon(x,t) - \int_{-\infty}^{t} R(t-\tau)\varepsilon(\tau,x)d\tau\right]$$
$$= E\left[\varepsilon(x,t) - R(t) * \varepsilon(x,t)\right], \qquad (13.14.4)$$

where $*$ denotes the convolution in time. Substituting (13.14.4) into (13.14.2) and eliminating $\varepsilon(x,t)$ from the resulting relation with the use of (13.14.3), we arrive at the equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + R * \frac{\partial^2 u}{\partial x^2} = f/\rho, \qquad (13.14.5)$$

where E/ρ is taken equal to one. In (Lokshin & Suvorova, 1982) it was established that, taking the creepage kernel as

$$K(t) = k\frac{t_+^{-\alpha}}{\Gamma(1-\alpha)} + \frac{k^2}{4}\frac{t_+^{-\alpha}}{\Gamma(1-\alpha)} * \frac{t_+^{-\alpha}}{\Gamma(1-\alpha)} \qquad (13.14.6)$$

where $k > 0$, $0 < \alpha < 1$, and

$$t_+^{-\alpha} \equiv \begin{cases} t^{-\alpha}, & t > 0, \\ 0, & t < 0, \end{cases}$$

the fundamental solution $g(x,t)$ of equation (13.14.5) differs from zero in the domain $|x| \le t$ and is expressed in terms of the function

$$u_\alpha(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{-\lambda^\alpha+\lambda t}d\lambda, \qquad \gamma > 0, \qquad (13.14.7)$$

as follows:

$$g(x,t) = \frac{1}{2}\int_0^{h(x,t)} u_\alpha(\xi)\,d\xi + \frac{k}{4}\int_{|x|}^{t} d\tau \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}\int_0^{h(x,\tau)} u_\alpha(\xi)\,d\xi,$$

where

$$h(x,t) = \frac{t-|x|}{(k|x|/2)^{1/\alpha}}.$$

Function (13.14.7) is nothing more nor less than the density of a one-sided stable law $q(x;\alpha,1)$; therefore,

$$g(x,t) = \frac{1}{2}G(h(x,t);\alpha,1) + +\frac{k}{4}\int_{|x|}^{t} d\tau \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}G(h(x,\tau);\alpha,1).$$

A similar result is obtained in (Schneider & Wyss, 1989) in terms of fractional wave equation equivalent to (13.14.5)–(13.14.6).

13.15. The mesoscopic effect

The term 'mesoscopic' is used to speak about a physical system containing a large a large amount of elements, which is too large to trace details of its evolution but, at the same time, is not large enough to apply the methods of statistical mechanics. The distinguishing feature of a mesoscopic system consists of a high level of fluctuations of its characteristics.

The mesoscopic physics is a very young but fast growing direction, which has been noticed by the Nobel Prize Committee: the 1998 Physics Nobel Prize went to Robert B. Laughlin of Stanford, Horst L. Stormer of Columbia, and Daniel C. Tsui of Princeton for their work on the fractional quantum Hall effect—two-dimensional phenomena at the interface between two semiconductor crystals. The reader can gain a rough idea of the problems covered by this line of investigation from the Proceedings of the conferences (Mesoscopic Systems, 1998). We restrict our presentation to a brief formulation of one of such problems considered in (Raikh & Ruzin, 1987).

We consider a flat layer of finite area playing the role of a potential barrier which electrons can tunnel through. In (Raikh & Ruzin, 1987) it was noted that this model is used to describe various phenomena in the p–n barrier, but the base qualitative results hold true in other situations. The problem consists of the fact that the barrier parameters undergone random spatial fluctuations that are due to, say, the roughness of the boundary surface or random fluctuations of the impurity density. Since the transparency of the barrier depends exponentially on its parameters, even small fluctuations can result in a great dispersion of the local transparency as a function of the coordinates of the layer surface. The points with anomal high local transparency are called 'punctures'; their number N and their transparencies V_i are random variables that determine the dispersion of the resulting transparency W of specimen. Because, nevertheless, V_i and W are well lower one, we are able to use the additive approximation, i.e.,

$$W = \sum_{i=1}^{N} V_i,$$

and the conditions $W < 1$ and $V_i < 1$ can be dropped, in the same way as we neglected the constraint $Q < E_0$ in Section 13.4. If we assume that the number of punctures N is distributed by the Poisson law and V_i do not depend on N and are independent, we necessarily arrive at the two-dimensional model of sources considered in Section 10.6. Indeed, formula (9) of (Raikh & Ruzin, 1987) coincides with (10.6.13), whereas the density of transparency distribution (formula (Π.3.1))

$$p_W(w) = \frac{1}{2\pi i} \int d\lambda e^{-\lambda^{\alpha}+\lambda w}$$

is nothing more nor less than the density of a one-sided stable law with $\alpha < 1$;

it relates to the cross-section area A by the formula

$$\alpha = (2\ln A)^2/Q_0^2$$

containing a constant $Q_0 \gg 1$ determined by the band-gap energy ε_g and the average impurity density n:

$$Q_0 \propto \varepsilon_g^{5/4} n^{-1/2}.$$

13.16. Multiparticle production

The problem of multiple hadronic production in heavy ion collisions was approached by means of several kinds of models since the early days of the discovery of strongly interacting particles. A large amount of experimental data were obtained up to now; however, one cannot say that their theoretical interpretation is satisfactory. The problem is caused by somewhat limited understanding of the production mechanism of multiparticle final states, which is related to the involved problems of confinement and non-perturbative quantum chromodynamics

Three basic properties of multiparticle production processes give reason to talk about anomalous fluctuations in them.

(1) The mean number of produced particles $\langle N\rangle$ increases as E^ν with the collision energy E, where α is essentially less than the expected value $\alpha = 1$.

(2) The distribution of the scalable random variable $Z = N/\langle N\rangle$ is not getting narrower while $\langle N\rangle$ grows, but stays the same (the Koba–Nielsen–Olesen scaling).

(3) One of the produced particles takes away about a half of the total energy independently of $\langle N\rangle$ (the leading effect).

It is accepted to consider that all these properties result from dynamic reasons.

Multiple hadronic production processes are usually visualized as proceeding in two main steps. First, a number of more or less well-defined intermediate objects such as strings or fireballs is formed. Their hadronization follows: usually one says that strings 'fragment' and fireballs 'decay' into finally observed hadrons. That terminology reflects the essential dynamic difference between them: strings are supposed to be essentially one-dimensional, whereas fireballs are three-dimensional objects. However, confrontation with experimental data washed out this difference substantially: strings are now allowed to 'bend' in phase space before fragmentation (so they can produce more hadrons than before), whereas fireballs are usually forced to decay anisotropically (therefore reducing their hadronic multiplicities). In both cases, the agreement with data is claimed as satisfactory.

The last fact can be interpreted as an indirect indication that kinematics, i.e., the conservation laws, may play a noticeable role here.

Let us consider the following model discussed in (Uchaikin & Litvinov, 1985). In the spirit of the multiperipherical ideas, we assume that virtual particles arise successively with independent identically distributed energies $E_1, E_2, \dots$ and when the sum $S_n = \sum_{i=1}^n E_i$ falls into the interval $(E, E+\Delta E)$, the virtual particles become real. If the sum S_n jumps over the interval, it starts all over again.

The random multiplicity N has the probability distribution

$$p_n(E) = \mathsf{P}\{N = n\} = \mathsf{P}\{S_n \in \Delta E\} \Big/ \sum_{n=1}^{\infty} \{S_n \in \Delta E\}$$
$$= p^{(n)}(E) \Big/ \sum_{n=1}^{\infty} p^{(n)}(E).$$

Let the distribution density $p(E)$ possess a finite variance σ^2,

$$\int_0^\infty E^2 p(E)dE = \sigma^2 + \varepsilon^2, \qquad \varepsilon = \int_0^\infty Ep(E)dE.$$

It is clear that for large E the expected N is large, too, and one can use the normal approximation for the convolution

$$p^{(n)}(E) \sim \frac{1}{\sqrt{2\pi n}\sigma} e^{-(E-n\varepsilon)^2/(2n\sigma^2)}, \qquad n \to \infty.$$

Therefore,

$$p_n(E) \sim c(E) e^{-(n-E/\varepsilon)^2/(2E\sigma^2\varepsilon^{-3})}, \qquad E \to \infty,$$

where $c(E)$ is the normalizing constant. As one can see, the mean multiplicity grows linearly with energy

$$\langle N \rangle \sim E/\varepsilon,$$

and the relative fluctuations vanish:

$$\sqrt{DN}/\langle N \rangle \sim \frac{\sigma}{\sqrt{\varepsilon E}} \to 0, \qquad E \to \infty.$$

Now we assume that

$$p(E) \sim \alpha a E^{-\alpha-1}, \quad E \to \infty, \quad \alpha < 1. \tag{13.16.1}$$

Using the generalized limit theorem (Section 2.5) and taking the relation between $Y_B(\alpha, \beta)$ and $Y_A(\alpha, \beta)$ (Section 3.7) into account, we obtain

$$p_n(E) \sim \frac{C(E)}{[na\Gamma(1-\alpha)]^{1/\alpha}} q_B\left(\frac{E}{[na\Gamma(1-\alpha)]^{1/\alpha}}; \alpha, 1\right), \qquad E \to \infty. \tag{13.16.2}$$

The moments $\langle N^k \rangle$ can be computed by replacing the summation

$$\langle N^k \rangle = \sum_n n^k p_n(E)$$

by integration

$$\langle N^k \rangle \sim \frac{\alpha E^{\alpha k+\alpha-1} C(E)}{[a\Gamma(1-\alpha)]^{k+1}} \int_0^\infty x^{-k\alpha-\alpha} q_B(x)\, dx = \frac{\alpha E^{\alpha k+\alpha-1} C(E)\Gamma(k+1)}{[a\Gamma(1-\alpha)]^{k+1} \alpha\Gamma((k+1)\alpha)}.$$

The normalization function

$$C(E) = a\Gamma(\alpha)\Gamma(1-\alpha)E^{1-\alpha}$$

is determined from the condition $\langle N^0 \rangle = 1$, the mean multiplicity is of the form

$$\langle N \rangle \sim \frac{\Gamma(\alpha)}{a\Gamma(1-\alpha)\Gamma(2\alpha)} E^\alpha, \qquad (13.16.3)$$

and the higher moments satisfy the relation

$$\frac{\langle N^k \rangle}{\langle N \rangle^k} \sim \frac{\Gamma(\alpha)\Gamma(k+1)}{\Gamma((k+1)\alpha)} \left(\frac{\Gamma(2\alpha)}{\Gamma(\alpha)} \right)^k. \qquad (13.16.4)$$

The obtained relations show that for $\alpha < 1$ all three properties of multiparticle production appear together in this model (the third property is an intrinsic one for a sum of random terms with distribution (13.16.1), see Section 3.6). Moreover, if we take $\alpha = 1/2$, we can find from (13.16.2) that

$$p_z(z) \approx \frac{\pi z}{2} e^{-\pi z^2/4}.$$

This formula is exactly the same formula which was obtained as fitting to some experimental data in the case where $\langle N \rangle \propto E^{1/2}$ (Anoshyn *et al.*, 1979).

Another application of stable distribution to the multiple production problem can be found in (Brax & Peschanski, 1991; Lagutin & Raikin, 1995).

13.17. Tsallis' distributions

At the end of this chapter, we present interesting results obtained by Tsallis (Tsallis *et al.*, 1995). As it is well known, the statistical mechanics can be constructed on the basis of the Boltzmann–Gibbs entropy

$$S(p(\cdot)) = -\sum_{i=1}^{w} p_i \ln p_i \to -\int_{-\infty}^{\infty} dx\, p(x) \ln p(x).$$

We choose $k^B = 1$. The normal distribution

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/(2\sigma^2)}$$

is derived by optimizing the entropy with the constraints

$$\int_{-\infty}^{\infty} dx\, p(x) = 1$$

and

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} dx\, x^2 p(x) = \sigma^2.$$

Tsallis (Tsallis, 1988) generalized the Boltzmann–Gibbs statistics and introduced the following expression for the generalized entropy:

$$S_q^T = \left(1 - \sum_{i=1}^{w} p_i^q\right) /(q-1)$$

which resembles the Renyi entropy

$$S_q^R = \left(\ln \sum_{i=1}^{w} p_i^q\right) /(1-q).$$

One can easily verify that

$$S_q^R = \frac{\ln[1 + (1-q)S_q]}{1-q}, \qquad \lim_{q\to 1} S_q^T = \lim_{q\to 1} S_q^R = -\sum_i p_i \ln p_i.$$

As Tsallis and co-workers assert, the extended formalism based in this definition of entropy has remarkable mathematical properties: among others, it preserves the Legendre transformation structure of thermodynamics, it leaves the Ehrenfest theorem, von Neumann's equation, and Onsager's reciprocity theorem form-invariant for all values of q.

Writing the new entropy for continuous distribution in the form

$$S_q^T(p(\cdot)) = \frac{1 - \int_{-\infty}^{\infty} d(x/\sigma)[\sigma p(x)]^q}{q-1}$$

and using the Lagrange method under the conditions

$$\int_{-\infty}^{\infty} dx\, p(x) = 1,$$

$$\int_{-\infty}^{\infty} dx\, x^2 [p(x)]^q = \sigma^{3-q},$$

Tsallis and collaborators obtained for $1 < q < 3$

$$p(x) = A(q)\frac{1}{[1 + \beta(q-1)x^2]^{1/(q-1)}} \tag{13.17.1}$$

and for $-\infty < q < 1$, the same formula on the segment $|x| \le 1/\sqrt{\beta(1-q)}$ and 0 otherwise.

As one can see, in case $q = 2$ the distribution coincides with a stable distribution (Cauchy). Other distributions given by this formula are not stable, but in the domain $5/3 < q < 3$ they have the asymptotic tails of the same order as symmetric stable laws with $\alpha = (3-q)/(q-1)$.

13.18. Stable distributions and renormalization group

The term 'renormalization group' (RG) is usually employed to denote approaches which apparently do not have much in common. An essential part of the renormalization group is to develop a strategy for dealing with problems that have multiple scales of length. The central concept of such a strategy is to break the main problem down into a sequence of subproblems, with each subproblem involving only a single characteristic length scale (Wilson, 1979).

The fact that stable distributions are fixed points of a RG transformation seems to have been first noted by G. Jona-Lasinio (Jona-Lasinio, 1975). He considered the random variables X_i as the possible values of a collection of continuous spins on a one-dimensional lattice. The index i indicates the position of the spin in the lattice. The problem is then to determine a 'good' variable for describing the collective, i.e., macroscopic behavior of the system.

We divide the lattice in blocks of length n. The total spin of each block is

$$S_n = \sum_{i=1}^{n} X_i.$$

In (Jona-Lasinio, 1975),

$$Z_n = b_n^{-1} S_n$$

was introduced as a 'good' variable with a convenient normalization factor b_n to damp its values at large n. Denoting by R_n the operation of constructing the characteristic function $f(k)$ of the common distribution of the X_i, they obtain

$$R_n(f) = f^n(k/b_n)$$

and then conclude that a 'good' choice of b_n will be one for which

$$\lim_{n\to\infty} R_n(f) = \lim_{n\to\infty} f^n(k/b_n) = \varphi(k),$$

where $\varphi(k)$ is a smooth function but not a constant. The situation may be illustrated with the help of a simple example. Let the characteristic function of the initial distribution be of the form

$$f(k) = \exp\left\{-\sum_{m=1}^{\infty} a_m |k|^{\alpha m}\right\}, \qquad a_m > 0.$$

Then

$$R_n(f) = \exp\left\{-n\sum_{m=1}^{\infty} a_m |k|^{\alpha m} b_n^{-\alpha m}\right\}.$$

If we choose b_n according to

$$b_n \sim n^{1/\alpha}, \tag{13.18.1}$$

all terms, except the first, will disappear from the sum in the exponential, and we obtain

$$\lim_{n\to\infty} R_n(f) = \exp\left\{-a|k|^{\alpha}\right\}.$$

On the other hand, if we take b_n which increases more rapidly than $n^{1/\alpha}$,

$$\lim_{n\to\infty} R_n(f) = 1,$$

which corresponds to a δ-function in the Z variable. Finally, if b_n increases slower than $n^{1/\alpha}$, the limit distribution will be different from zero only at infinity. Therefore, in this example (13.18.1) gives the only 'good' choice for b_n.

Notice that the stable characteristic function $f_\alpha(k) = \exp\left\{-a|k|^{\alpha}\right\}$ satisfies for $b_n = n^{1/\alpha}$ and for arbitrary n

$$R_n(f_\alpha) = f_\alpha,$$

i.e., it is a fixed point of the transformation R_n.

We are able to show that the 'collective' behavior of our one-dimensional lattice of spins is described by a distribution belonging to a rather restricted class, i.e., the class of stable distributions which is parameterized by the characteristics α (Jona-Lasinio, 1975).

This idea is discussed also in (Nonnenmacher, 1989; West, 1994; Zaslavsky, 1994a).

14

Radiophysics

14.1. Transmission line

This section is devoted to a model for a homogeneous electrical line, which can be, for example, an electrical cable or a circuit of cascaded four-pole networks. A number of properties of such a line can be described with the use of the so-called time function $F(t, \lambda)$, $t \geq 0$, which shows the reaction of a line of length $\lambda > 0$ to a perturbation of step function type at the initial time. Its derivative with respect to time is called the pulse reaction, and its Fourier transform, the frequency characteristic of the line.

It is known from electrical circuit theory that, when a homogeneous line of length $\lambda = \lambda_1 + \lambda_2$ is partitioned into sequentially connected sections of lengths λ_1 and λ_2, its time function $F(t, \lambda)$ is formed from the time functions $F(t, \lambda_1)$ and $F(t, \lambda_2)$ of the separate parts via their convolution, i.e.,

$$F(t, \lambda_1 + \lambda_2) = F(t, \lambda_1) * F(t, \lambda_2),$$

which is equivalent to multiplication of the corresponding frequency characteristics:

$$f(\omega, \lambda_1 + \lambda_2) = f(\omega, \lambda_1) f(\omega, \lambda_2).$$

Hence it follows that for any $\lambda > 0$

$$f(\omega, \lambda) = f^{\lambda}(\omega, 1). \tag{14.1.1}$$

The quantities

$$a(\omega, \lambda) = -\Re \ln f(\omega, \lambda) = \lambda a(\omega, 1),$$
$$b(\omega, \lambda) = \Im \ln f(\omega, \lambda) = \lambda b(\omega, 1),$$

called, respectively, the damping and the phase of the frequency characteristic, are closely related to the function $f(\omega, \lambda)$.

It turns out that the cases where the time function of the line does not decrease on the time axis $t > 0$ are not rare. If, in addition, the damping at zero frequency ($\omega = 0$) is equal to zero, then the time function $F(t, \lambda)$ can be regarded as a distribution function concentrated on the half-line $t > 0$. In this case, (14.1.1) implies that $F(t, \lambda)$ is an infinitely divisible distribution with characteristic function $f(\omega, \lambda)$ of the form

$$\ln f(\omega, \lambda) = \lambda \left(i\omega\gamma + \int_0^\infty (e^{i\omega u} - 1)\, dH(u) \right), \qquad \gamma \ge 0.$$

For frequency characteristics of this form, the phase $b(\omega, \lambda)$ can be recovered from the damping $a(\omega, \lambda)$ to within the term $\omega\gamma\lambda$.

Thus, if $a(\omega, \lambda) = \lambda c|\omega|^\alpha$, where $0 < \alpha \le 2$ and c is a positive constant, then the corresponding phase is of the form

$$b(\omega, \lambda) = \lambda c(\omega\gamma + |\omega|^\alpha \tan(\alpha\pi/2) \operatorname{sign} \omega), \qquad \gamma \ge 0,$$

while the time function $F(t, \lambda)$ related to that frequency characteristic is a stable distribution, i.e.,

$$F(t, \lambda) = G^A(t; \alpha, 1, \gamma, c\lambda). \tag{14.1.2}$$

Certain forms of cables which have a power character of damping are known in electrical circuit theory. For example, $a(\omega, \lambda) = \lambda c|\omega|^{1/2}$ for the so-called non-inductive and coaxial cables. Consequently, according to (14.1.2), the time function of such cables is of the form

$$F(t, \lambda) = G^A(t; 1/2, 1, \gamma, c\lambda) = 2[1 - \Phi(c\lambda(t - c\gamma\lambda)^{-1/2})],$$

where Φ is the distribution function of the standard normal law. The pulse reaction $F'_t(t, \lambda)$ in the case possesses the simple explicit expression (Lévy's density)

$$F'_t(t, \lambda) = \frac{c\lambda}{\sqrt{2\pi}}(x - c\gamma\lambda)^{-3/2} \exp\left\{ -\frac{c^2\lambda^2}{2}(x - c\gamma\lambda)^{-1} \right\}.$$

A similar result is obtained in the analysis of noise accumulation in the relay repeater line. It is established that the deviation of the noise value from the average is usually due to 'spike' at some place of the line, whereas the noise accumulated in the remaining part of the line is comparatively small. After partitioning the line into n equal segments and representing the total noise as the sum of independent random contributions on these segments, in (Sindler, 1956) it is concluded that the noise distribution is determined by a stable law, provided that the probability of a spike in a segment behaves as a power function and the number of the constituent segments is large enough.

14.2. Distortion of information phase

The following example concerns the calculation of the performance of systems of radio relay stations (in the engineering practice, they are called radio relay communications lines). In a mathematical setting, the part of the general problem which we give our attention to takes its origin from (Siforov, 1956), and its solution with the use of stable laws makes up the content of several papers, of which (Sindler, 1956) must be regarded as the main one. The transmission of high-quality radio communications over great distances (for example, television transmissions) poses for engineers not only the problem of relaying high-frequency radio signals that can be received only within sight of the transmitter, but also the problem of eliminating noise distortions. The following seems to be one of the simplest models where it is possible to trace both the effects arisen themselves and the ways of their quantitative analysis.

We consider a vector $\mathbf{a} \in \mathbb{R}^2$ rotating with a large angular velocity ω. The projection of $\mathbf{a}$ onto the x-axis at time t, under the condition that its motion began from the position defined by the angle φ, is the periodic function

$$a_x(t) = |\mathbf{a}| \cos(\omega t + \varphi).$$

The oscillatory excitation at the output of the radio transmitter is described by the function $a_x(t)$, where $|\mathbf{a}|$ is the amplitude of the radio signal (its power is proportional to $|\mathbf{a}|^2$), ω is its frequency, and $\omega t + \varphi$ is its phase (at time t). The quantities $|\mathbf{a}|$ and ω stay constant at the transmitter output, while transmission of the useful signal is accomplished by modulating the phase of the signal, i.e., by changing φ.

If the radio signal were received by the receiver without change, then its information content—the phase shift φ—would be recovered without difficulties. However, in actual circumstances the radio signal comes to the receiver in a somewhat distorted and weakened form. This is due to the scattering and absorbing radio waves by the atmosphere. The radio waves hence reach the receiver with modified phases. Taken alone, the changes are small; hence, that in combination with the large frequency ω and the fact that there are many such changes, we arrive at a distribution that is close to uniform with respect to the phases of the vectors associated. As a result, the two-dimensional vector $\mathbf{X}$ describing the received signal has the nature of a random vector with a circular normal distribution. Hence the length $|\mathbf{X}|$ of $\mathbf{X}$ has a Rayleigh distribution with density

$$p(x) = D^{-2} x \exp(-x^2/2D^2), \qquad x \geq 0, \tag{14.2.1}$$

where D^2 is the variance of the components of $\mathbf{X}$.

The influence of the set noise can be represented by adding to $\mathbf{X}$ a two-dimensional random vector $\mathbf{Y}$ having the circular normal distribution with variance of the components equal to σ^2. The vector $\mathbf{X} + \mathbf{Y}$ possesses a phase differing from that of $\mathbf{X}$ by an angle ψ determined by

$$\tan\psi = \frac{|\mathbf{Y}_t|}{|\mathbf{X} + \mathbf{Y}_r|}\xi(\mathbf{Y}_t), \qquad -\pi < \psi < \pi, \tag{14.2.2}$$

where $\mathbf{Y}_r$ and $\mathbf{Y}_t$ are the radial and tangential components of $\mathbf{Y}$ with respect to $\mathbf{X}$, and $\xi(\mathbf{Y}_t)$ is a random variable determined by the direction of $\mathbf{Y}_t$ taking the values +1 and -1 with probabilities 1/2 each. Since the whole phenomenon evolves in time, the vectors $\mathbf{X}$ and $\mathbf{Y}$ and the angle ψ determined by them depend on t, i.e., are random processes. The processes $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ (and, consequently, $\psi(t)$), which are connected with different sections of the relay, can be regarded as independent and stationary.

The total distortion $\bar\psi$ of the information phase φ at time t after passage of N sections of the relay is determined (if the delay in passing from section to section is ignored) by the equality

$$\tan\bar\psi(t) = \tan(\psi_1(t) + \ldots + \psi_N(t)), \qquad -\pi < \bar\psi < \pi. \tag{14.2.3}$$

The distribution of $\psi_j(t)$ is symmetric for each t, as seen from (14.2.2). Therefore,

$$\mathsf{E}\psi_j(t) = 0.$$

The quantity $\bar\psi(t)$, obviously, possesses the same property. With this property in mind, the value of the standard deviation becomes a measure characterizing the level of noise in the transmission. For example, on an interval of 1 sec,

$$\Psi^2 = \int_0^1 \bar\psi^2(t)\,dt.$$

The estimation of Ψ^2 is a complicated problem in the analytic setting. It can be simplified, though. Indeed, in view of (14.2.3),,

$$\begin{aligned}\Psi^2 &\le \int_0^1 (\psi_1(t) + \ldots + \psi_N(t))^2 dt \\ &= \sum_i \int_0^1 \psi_j^2(t)\,dt + \sum_{ij}\int_0^1 \psi_i(t)\psi_j(t)\,dt.\end{aligned} \tag{14.2.4}$$

Then, because the stationary processes $\psi_j(t)$ are independent and $\mathsf{E}\psi_j(t) = 0$, which implies

$$\int_0^1 \psi_i(t)\psi_j(t)\,dt \approx \mathsf{E}\psi_i(t)\psi_j(t) = 0, \qquad i \neq j,$$

we can drop the second term from (14.2.4), and consider the sum

$$\sum_i \int_0^1 \psi_j^2(t)\,dt \tag{14.2.5}$$

as an estimator of Ψ^2. The constituents of this sum, in turn, admit a simplified approximate expression, provided that $\varepsilon_j = \sigma_j/D_j$ are taken to be small, where σ_j^2 and D_j^2 are the variances of the components of $\mathbf{Y}_j$ and $\mathbf{X}_j$ related to the jth part of the relay. Indeed, by (14.2.2),

$$\int_0^1 \psi_j^2(t)dt = \int_0^1 \arctan^2\left(\frac{\varepsilon_j|\mathbf{U}_j|}{|\mathbf{V}_j + \varepsilon_j\mathbf{U}_j'|}\right) dt,$$

where $\mathbf{U}_j = \mathbf{Y}_{tj}/\sigma_j$, $\mathbf{U}_j' = \mathbf{Y}_{rj}/\sigma_j$, and $\mathbf{V}_j = \mathbf{X}_j/D_j$. Therefore,

$$\int_0^1 \psi_j^2(t)\,dt \le \int_0^1 \frac{\varepsilon_j^2|\mathbf{U}_j|^2}{|\mathbf{V}_j + \varepsilon_j\mathbf{U}_j'|^2}\,dt \approx \int_0^1 \varepsilon_j^2\frac{|\mathbf{U}_j|^2}{|\mathbf{V}_j|^2}dt.$$

The next step in simplifying the estimator $\tilde{\Psi}_2$ has to do with the circumstance that the vectors $\mathbf{V}_j(t)$ and $\mathbf{U}_j(t)$ vary with sharply different intensity. For example, $\mathbf{V}_j(t)$ is practically constant on a time interval of the length being considered, while $\mathbf{U}_j(t)$ performs an enormous number of rotations on the same interval (of the order of 10^6). For this reason,

$$\int_0^1 \varepsilon_j^2\frac{|\mathbf{U}_j(t)|^2}{|\mathbf{V}_j(t)|^2}dt \approx \frac{\varepsilon_j^2}{|\mathbf{V}_j|^2}\int_0^1 |\mathbf{U}_j(t)|^2dt$$
$$\approx \frac{\varepsilon_j^2}{|\mathbf{V}_j|^2}\mathsf{E}|\mathbf{U}_j(1)|^2 = \varepsilon_j^2|\mathbf{V}_j|^{-2}.$$

Consequently, a simplified estimator of Ψ^2 can be represented as the sum

$$\tilde{\Psi}^2 = \sum_j \varepsilon_j^2|\mathbf{V}_j|^{-2},$$

where the ε_j are constants small in magnitude, and $\mathbf{V}_j$ are independent random vectors distributed by the normal law with identity covariance matrix. Thus, as an estimator $\tilde{\Psi}^2$ of the random variable Ψ^2 we take a sum of N independent random variables $\varepsilon_j^2|\mathbf{V}_j|^{-2}$ whose distribution functions are of the form (as follows from (14.2.1))

$$F_j(x) = \exp(-\varepsilon_j^2/2x), \qquad x > 0.$$

Since the terms $\varepsilon_j^2|\mathbf{V}_j|^{-2}$ are small, while their number is large enough, the distribution of $\tilde{\Psi}^2$ can be well approximated by an infinitely divisible law whose spectral function is not difficult to compute. Indeed, $H(x) = 0$ if $x < 0$, and

$$H(x) \approx \sum_j (F_j(x) - 1) \approx -\tfrac{1}{2}\sum_j \varepsilon_j^2 x^{-1}$$

if $x > 0$. This spectral function corresponds, according to (3.5.16), to the stable distribution with parameters $\alpha = 1$ and $\beta = 1$.

14.3. Signal and noise in a multichannel system

In radiolocation theory, the following scheme is well known (Dobrushin, 1958).

Let there be n channels fed by a random voltage X_i each, $i = 1, \dots, n$. The random variables X_i are independent and identically distributed by the Rayleigh law

$$p_{X_i}(x) = (2x/\lambda_i)e^{-x^2/\lambda_i}, \qquad x \ge 0,$$

where

$$\lambda_i = \mathsf{E}X_i^2.$$

As concerns the parameters λ_i, there exist two rival hypotheses.

HYPOTHESIS A (signal absence). All λ_i are identical and equal to $d > 0$. This means that the source of voltages X_i consists of mere noise, and d is the average noise intensity in the channel.

HYPOTHESIS B (signal presence). All λ_i, excepting λ_j, are equal to d, whereas

$$\lambda_j = d + \bar{d},$$

where $\bar{d} > 0$. The index j is random, and takes each of n possible values equiprobably. Therefore, the source feeding all channels, excepting the jth one, is a mere noise, whereas the voltage in the jth channel is provided by the sum of a signal and a noise. The quantity $\bar{d}$ is the average power of the signal. We neglect the possibility of simultaneous occurrence of a signal in several channels.

The joint distributiuon density of $X_1, \dots, X_n$ under the hypotheses A and B is of the form

$$p_A(x_1, \dots, x_n) = \prod_{i=1}^{n}(2x_i/d)e^{-x_i^2/d},$$

$$p_B(x_1, \dots, x_n) = \frac{1}{n}\sum_{j=1}^{n}\left[2x_j/(d+\bar{d})\right]e^{-x_j^2/(d+\bar{d})}\prod_{i\ne j}(2x_i/d)e^{-x_i^2/d}.$$

respectively. Considering the likelihood ratio p_B/p_A, we conclude that the optimal test to distinguish the hypotheses A and B is based on the statistic

$$\varphi(X_1, \dots, X_n) = \sum_{j=1}^{n}\exp\left\{\frac{\bar{d}}{d(d+\bar{d})}X_j^2\right\}.$$

We introduce the ratio

$$\beta = \bar{d}/d,$$

where β is a dimensionless characteristic of the ratio of the average signal power to the average noise intensity. It is easily seen that, if X_i is distributed by the Rayleigh law with parameter λ_i, then

$$Y_j = \exp\left\{\frac{\bar{d}}{d(d+\bar{d})}X_i^2\right\}$$

satisfies the relations

$$\mathsf{P}\{Y > x\} = x^{-\gamma}, \qquad 1 \le x < \infty,$$
$$\gamma = \frac{d(d+\bar{d})}{\bar{d}\lambda_i} = \frac{(1+\beta)d}{\beta\lambda_i}.$$

Under hypothesis A, the statistic $\varphi(X_1, \dots, X_n)$ is distributed exactly as the sum

$$S_n = Y_1 + \dots + Y_n \tag{14.3.1}$$

of n independent summands Y_i,

$$\mathsf{P}\{Y_i > x\} = x^{-(1+\beta)/\beta}, \qquad 1 \le x < \infty, \tag{14.3.2}$$

whereas under hypothesis B that statistic is distributed as

$$\tilde{S}_n = Z + Y_2 + \dots + Y_n, \tag{14.3.3}$$

where $Z, Y_2, \dots, Y_n$ are independent, and

$$\mathsf{P}\{Y_i > x\} = x^{-(1+\beta)/\beta}, \qquad 1 \le x < \infty,$$
$$\mathsf{P}\{Z > x\} = x^{-\beta}, \qquad 1 \le x < \infty. \tag{14.3.4}$$

As expected, the distribution of the statistic $\varphi(X_1, \dots, X_n)$ depends on the parameter β only.

It is well known that any test to distinguish hypotheses is based on determination of two complementary sets Ω_A and Ω_B in the n-dimensional space such that if $\{X_1, \dots, X_n\} \in \Omega_A$, hypothesis A is taken to be true, otherwise, if $\{X_1, \dots, X_n\} \in \Omega_B$, then hypothesis B is accepted. The probability

$$\iint_{\Omega_B} p_A(x_1, \dots, x_n)\, dx_1 \dots dx_n$$

of acceptance of hypothesis B in the case where hypothesis A is true is referred to as the false alarm probability. The probability

$$\iint_{\Omega_B} p_B(x_1, \dots, x_n)\, dx_1 \dots dx_n$$

of acceptance of hypothesis B in the case where it is true is referred to as the right signal detection probability.

Let F and D be given, where $0 < F \leq D < 1$. We say that we are able to distinguish the hypotheses with probabilities F and D, if there exists a test to differentiate these hypotheses such that the false alarm probability does not exceed F, whereas the right signal detection probability is no smaller than D. The general theorems of mathematic statistics imply that it is sufficient to consider tests based on the statistic φ, which was done in (Dobrushin, 1958). Following this way, the ability to discriminate the hypotheses with given probabilities depends on the parameters n and β only.

We fix n, F, and D. From continuity considerations, it is obvious that there exists boundary $\beta_n(F,D)$ such that for $\beta \geq \beta_n(F,D)$ the hypotheses distinguishing with probabilities F and D is possible, whereas for $\beta < \beta_n(F,D)$ the hypotheses are indistinguishable. If for $\beta = \beta_n(F,D)$ we make use of an optimal test based on the likelihood ratio to distinguish hypotheses A and B, then for some y the test with hypothesis region

$$\Omega_A = \{\varphi(x_1, \dots, x_n) < y\}$$

yield errors exactly equal to F and D.

The quantity $\beta_n(F,D)$ determining the least excession of the signal power above the noise which allows for signal detection against the background of the noise with given probabilities of errors was studied in (Dobrushin, 1958):

- for any fixed F and D, as $n \to \infty$,

$$\beta_n(F,D) \sim \frac{\ln n}{\ln\left(\frac{1-F}{D-F}\right)} - 1 + \frac{\int_{-\infty}^{K} \ln(K-x) q(x;1,1) dx}{(1-F)\ln\left(\frac{1-F}{D-F}\right)},$$

  where $q(x;1,1)$ is the stable density with characteristic function

$$g(k;1,1) = \exp\left\{\int_0^\infty \left(e^{iku} - 1 - \frac{iku}{1+u^2}\right)\frac{du}{u^2}\right\},$$

  and K is a solution of the equation

$$\int_K^\infty q(x;1,1)\,dx = F;$$

- for any fixed D and n, as $F \to 0$,

$$\beta_n(F,D) \sim \frac{\ln F - \ln n}{\ln D} - 1.$$

The base point is the application of the generalized limit theorem to sums (14.3.1) and (14.3.3) of the addends distributed by the Zipf–Pareto law.

14.4. Wave scattering in turbulent medium

The theory of multiple wave scattering is exposed in monographs (Tatarsky, 1971; Rytov *et al.*, 1978; Ishimaru, 1978), and others. We concern here the problem only in its most simple formulation.

The wave field $u(\mathbf{r}, t)$ is supposed to be scalar and monochromatic:

$$u(\mathbf{r}, t) = u(\mathbf{r})e^{-i\omega t},$$

and the medium inhomogeneities to be time-independent. Under the indicated conditions the wave propagation through the inhomogeneous medium is described by the Helmholtz equation

$$\Delta u(\mathbf{r}) + k_0^2 \varepsilon(\mathbf{r}) u(\mathbf{r}) = 0, \qquad (14.4.1)$$

where k_0 is the wave number in the non-perturbed medium, and $\varepsilon(\mathbf{r})$ is the optical dielectric constant at the point $\mathbf{r}$. For randomly distributed inhomogeneities, $\varepsilon(\mathbf{r})$ can be represented as

$$\varepsilon(\mathbf{r}) = \bar{\varepsilon} + \tilde{\varepsilon}(\mathbf{r}),$$

where $\bar{\varepsilon}$ = const is the ensemble-averaged $\varepsilon(\mathbf{r})$. We thus rewrite the Helmholtz equation as

$$\Delta u(\mathbf{r}) + k_0^2 \bar{\varepsilon} u(\mathbf{r}) = -k_0^2 \tilde{\varepsilon}(\mathbf{r}) u(\mathbf{r}). \qquad (14.4.2)$$

Let $u_0(\mathbf{r})$ describe the primary wave that satisfies the non-perturbed Helmholtz equation

$$\Delta u_0(\mathbf{r}) + k_0^2 \bar{\varepsilon} u(\mathbf{r}) = 0$$

and $G(\mathbf{r}, \mathbf{r}')$ be the non-perturbed Green function

$$\Delta G(\mathbf{r}, \mathbf{r}') + k_0^2 \bar{\varepsilon} G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}').$$

Assuming that k_0 and G satisfy the necessary boundary conditions, we derive from (14.4.2) the integral equation for the wave field

$$u(\mathbf{r}) = u_0(\mathbf{r}) - k_0^2 \int G(\mathbf{r}, \mathbf{r}') \tilde{\varepsilon}(\mathbf{r}') u(\mathbf{r}') dr'$$

which possesses the solution in the form of the Neumann series

$$u(\mathbf{r}) = \sum_{n=0}^{\infty} u_n(\mathbf{r}),$$

where the terms satisfy the recurrence relation

$$u_{n+1}(\mathbf{r}) = -k_0^2 \int G(\mathbf{r}, \mathbf{r}') \bar{\varepsilon}(\mathbf{r}') u_n(\mathbf{r}')\, d\mathbf{r}'.$$

Here $u_0(\mathbf{r})$ is the non-perturbed (non-scattered) field, $u_1(\mathbf{r})$ is the scattered field, $u_2(\mathbf{r})$ is the double-scattered field, etc.

In the case of transparent medium, i.e., where $\varepsilon(\mathbf{r})$ is real-valued, the relation

$$\operatorname{div}(u^*\nabla u - u\nabla u^*) = 0$$

follows from (14.4.1). Consequently, under an appropriate choice of the normalization factor a, the vector

$$j = \frac{a}{2ik_0}(u^*\nabla u - u\nabla u^*) = \frac{a}{k_0}\Im(u^*\nabla u)$$

can be interpreted as the energy flux density.

Let us introduce the cross-section as the characteristics of single scattering:

$$w(\theta) = \frac{dP_1/d\mathbf{\Omega}}{Vj_0},$$

where $dP_1/d\mathbf{\Omega}$ is the mean power scattered within the unit spherical angle at angle θ with the initial direction, V is the scattering volume, and

$$j_0 = \frac{a}{k_0}\left|\Im(u_0^*\nabla u_0)\right|$$

is the absolute value of energy flux.

The random field of fluctuations $\tilde{\varepsilon}(\mathbf{r})$ is called statistically homogeneous and isotropic if the correlation function

$$\psi_\varepsilon(\mathbf{r}_1, \mathbf{r}_2) \equiv \langle\tilde{\varepsilon}(\mathbf{r}_1)\tilde{\varepsilon}(\mathbf{r}_2)\rangle$$

depends only on the distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$. The spectral density of such fluctuations

$$\Phi_\varepsilon(q) = \frac{1}{8\pi^3}\int \Psi_\varepsilon(r)e^{-i\mathbf{qr}}d\mathbf{r}$$

is also a function of a single variable $q = |\mathbf{q}|$. The differential cross-section is expressed through the spectral density by the relation (Rytov *et al.*, 1978, (26.11))

$$w(\theta) = (1/2)\pi k_0^4\Phi_\varepsilon(q).$$

where $q = 2k_0\sin(\theta/2)$.

Within the inertial wave number interval corresponding to the Kolmogorov 2/3 law, the spectral density of turbulent fluctuations is described by the power law (Rytov *et al.*, 1978, (26.31))

$$\Phi_\varepsilon(q) = Cq^{-11/3}, \qquad C = \text{const}, \tag{14.4.3}$$

and, consequently, for small angles

$$w(\theta) = (1/2)\pi k_0^4 C[2k_0\sin\theta/2]^{-11/3} \sim D\theta^{-11/3}, \qquad D = \text{const}. \tag{14.4.4}$$

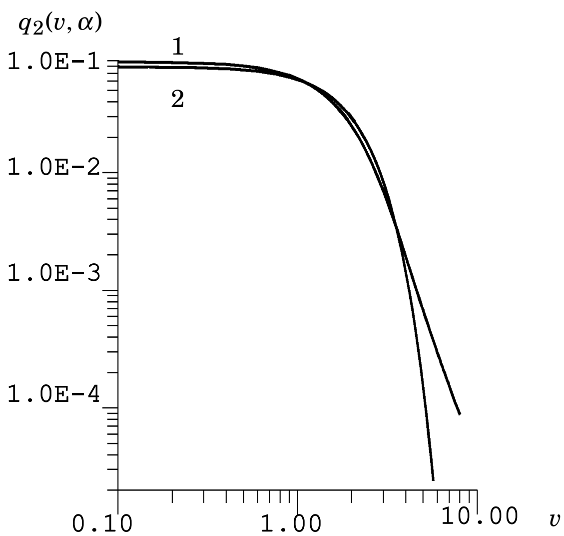


Figure 14.1. The bivariate stable distribution $q_2(v; 5/3)$ with the corresponding Gauss distribution

Assuming the random inhomogeneities to be uniformly distributed in the space, we can use again formula (13.5.1), substituting the differential scattering cross-section (14.4.4)

$$f(\mathbf{k}) = \exp\left\{-2\pi z D \int_0^\infty [1 - J_0(k\theta)]\theta^{-8/3} d\theta\right\}. \tag{14.4.5}$$

Formula (14.4.5) coincides with (33.20) of (Rytov *et al.*, 1978), which is used to analyze the eikonal fluctuations in the turbulent atmosphere, and, at the same time, (14.4.5) agrees with formula (10.6.18) with $\mu = 6/5$. Therefore, the small-angle ($\mathbf{v} = \mathbf{\Omega}_0 - \mathbf{\Omega}$, $|\mathbf{v}| \ll 1$) distribution of the radiation scattered by a layer of turbulent medium is a bivariate axially symmetric stable distribution with $\alpha = 2/\mu = 5/3$. The corresponding density is shown in Fig. 14.1, compared with the normal distribution. The 'width' of the obtained distribution behaves like $z^{3/5}$, i.e., it diffuses noticably faster than in the case of normal two-dimensional diffusion ($z^{1/2}$). We encounter here an example of superdiffusion (Chapter 12).

Of course, the remarks made at the end of the previous section remain true: one should interpret this result as some intermediate asymptotics, the question about its existence and area of applicability should be answered experimentally.

14.5. Chaotic phase screen

Following (West, 1990), we consider the statistical properties of a freely propagating plane wave whose phase is specified on a transverse line at $z = 0$. The wave is propagating in the positive z direction with constant frequency ω and wavelength $\lambda = 2\pi/k$ with amplitude $v(x,z)$ such that

$$v_0(x) = \exp\{ik\Psi(x)\} \qquad z = 0^+, \tag{14.5.1}$$

where the phase shift $\Psi(x)$ is merely a function of the transverse coordinate x. The observed scintillations in the amplitude and phase of the received wave result from interference of phase points along the wave front in the $z = 0$ plane as the wave propagates away from the boundary. The interference pattern is determined by the assumed statistical properties of $\Psi(x)$.

The propagation of the wave field in free space can be determined by solving the wave equation (14.4.1), which is often approximated by a parabolic equation of the form of the one-dimensional Schrödinger equation

$$\left[2ik\frac{\partial}{\partial z} + \frac{\partial^2}{\partial x^2} + k^2\right] u(x,y,z) = 0. \tag{14.5.2}$$

Here

$$v(\mathbf{r}) = u(x,z)e^{ikz}/z^{1/2},$$

and one neglects terms that fall off faster than $z^{-1/2}$. The direction of propagation (z) in (14.5.1) plays the role of time in the actual Schrödinger equation. whereas the boundary condition (14.5.1), the role of initial condition for (14.5.2).

The solution of the parabolic equation (14.5.2), subject to the boundary condition of (14.5.1), is given by the elementary diffraction integral

$$u(x,z) = e^{ikz}\left(\frac{k}{2\pi iz}\right)^{1/2}\int_{-\infty}^{\infty} dx' \exp\left\{i\frac{k}{2z}(x-x')^2\right\} u_0(x'). \tag{14.5.3}$$

Equation (14.5.3) describes the diffraction of the phase front away from the phase-screen point $(x',0)$ to the observation point (x,z). The mean field detected at (x,z) is determined by averaging of (14.5.3) over an ensemble of realizations of the phase fluctuations

$$\langle u(x,z)\rangle = e^{ikz}\left(\frac{k}{2\pi iz}\right)^{1/2}\int_{-\infty}^{\infty} dx' \exp\left\{i\frac{k}{2z}(x-x')^2\right\} \langle u_0(x')\rangle. \tag{14.5.4}$$

Under the integral sign, we have the average field emerging from the phase screen

$$\langle u_0(x)\rangle = \langle u(x,0^+)\rangle = \langle \exp\{ik\Psi(x)\}\rangle. \tag{14.5.5}$$

The coherence between the wave field at the point $(x+\xi, z)$ and at (x, z) is given by

$$\langle u(x+\xi,z)u^*(x,z)\rangle$$
$$= \frac{k}{2\pi z}\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \exp\left\{i\frac{k}{2z}[(x+\xi-x_1)^2-(x-x_2)^2]\right\}\langle u_0(x_1)u_0^*(x_2)\rangle. \tag{14.5.6}$$

If we assume that $\Psi(x)$ is a homogeneous random process with independent increments, then

$$\Psi(x) = \Psi(0) + \int_0^x d\Psi(x'),$$

where $\Psi(0)$ is the reference phase at $x = 0$ taking to be zero, so that (14.5.5) becomes

$$\langle u_0(x)\rangle = \left\langle \exp\left\{ik\int_0^x d\Psi(x')\right\}\right\rangle. \tag{14.5.7}$$

To evaluate the mean, we replace the integral by a sum over n subintervals, each of length $\Delta x = x/n \ll x$, and write

$$I = \lim_{n\to\infty}\left\langle \prod_{j=0}^{n-1}\exp\left\{ik\int_{j\Delta x}^{(j+1)\Delta x} d\Psi(x')\right\}\right\rangle$$
$$= \lim_{n\to\infty}\prod_{j=0}^{n-1}\int_{-\infty}^{\infty}\exp\{ik\Delta\Psi_j(\Delta x)\}\,p(\Delta\Psi_j,\Delta x)d\Delta\Psi_j.$$

This integral is just the characteristic function for small phase shift $\Delta\Psi_j(\Delta x) \equiv \Psi[(j+1)\Delta x] - \Psi(j\Delta x)$:

$$I = \lim_{n\to\infty}\prod_{j=0}^{n-1}\tilde{p}(k,\Delta x), \tag{14.5.8}$$

$$\tilde{p}(k,\Delta x) = \int e^{ik\Delta\Psi_j(\Delta x)}p(\Delta\Psi_j,\Delta x)d\Delta\Psi_j.$$

Setting

$$\tilde{p}(k,\Delta x) \sim 1-\gamma|k|^\alpha\Delta x = 1-\gamma|k|^\alpha x/n,$$

we derive from (14.5.8)

$$I = \exp\{-\gamma|k|^\alpha x\},$$

which is a stable characteristic function.

The two-point correlation function of wave field at the boundary

$$\langle u_0(x_1)u_0^*(x_2)\rangle = \left\langle \exp\left\{ik\left[\int_0^{x_1} d\Psi(x') - \int_0^{x_2} d\Psi(x')\right]\right\}\right\rangle \tag{14.5.9}$$

is calculated in a similar way:

$$\langle u(x+\xi,z)u^*(x,z)\rangle = \exp\left\{-\gamma k^{\alpha}|\xi|\right\}.$$

The discussion of physical corollaries of the above presentation can be found in (West, 1990; Wong & Bray, 1988).

15

Astrophysics and cosmology

15.1. Light of a distant star

Since ancient times, and for many centuries, the only source of information about the Universe was the light. One of the classical effects in general relativity, tested experimentally, is the bending of light beams in the gravitational field. During the recent decades, much attention was paid to this phenomenon due to the discovery of gravitational lens. When the light moves through the intergalaxy space, its deviation from a straight line is noticeable only if circumstances are favorable for it, namely, if the light ray goes sufficiently close to a source of gravitational field and then the gravitational lensing effect appears. In other cases the deviations caused by the galaxy fields are somewhat small. However, as a result of a long distance passed by the light, these small deviations accumulate and also become observable that leads to the displacements of apparent positions of light sources on the celestial sphere. The evaluation of statistical characteristics of a light trajectory in the field of randomly distributed point masses was performed in (Pyragas *et al.*, 1986), whose results are discussed here.

The light ray subjected to the mass m at a distance r from the straight line which it moved along before interaction with the mass undergoes the deviation on a small angle $\mathbf{v} = \mathbf{\Omega} - \mathbf{\Omega}_0$,

$$\mathbf{v} \propto m \left(\mathbf{r}/r^2\right), \tag{15.1.1}$$

where $\mathbf{v}$ and $\mathbf{r}$ are two-dimensional vectors in the plane perpendicular to the primary direction. The gravitational sources distribution is supposed to be homogeneous Poisson ensemble. Performing the calculations, in (Pyragas *et al.*, 1986) the model of sources was essentially used. As a result, they obtain a logarithmically divergent variance of the random deviation angle of a ray passed a certain path. Observing that this divergence is created by the nearest source, they remove a small neighborhood along this ray from consideration, and get a finite mean square angle on the path z proportional to $z \ln z$.

It is easy to see that the problem is identical to the charged particle scattering problem considered in Section 13.5. So, we can assert that the angle distribution of a light ray multiply scattered by gravitational field of stationary randomly distributed point masses is described by the Moliére distribution. We intentionally say here about stationarity. In principle, there exists another possibility connected with changing the position of a light source. It is motion of mass creating the deflecting gravitational field. A more detailed analysis shows that in actual case the contribution of nearest neighbors, though hardly accountable, is more essential and leads not to logarithmic but power-type divergences of mean-square quantities (Pyragas *et al.*, 1986, p. 83).

The power-type divergence of the mean square angle of scattering per unit path length

$$2\pi \int_0^\theta w(\theta)\theta^2\theta \, d\theta \propto \theta^\gamma, \quad 0 < \gamma < 2, \quad \theta \to \infty,$$

makes it evident that the distribution $w(\theta)$ has the power-type tail of the form

$$w(\theta) \propto \theta^{-(4-\gamma)},$$

and consequently, we arrive at the problem of the kind already considered in Section 11.6. Its solution is the two-dimensional axially symmetric stable distribution with $\alpha = 2 - \gamma$ whose width grows as $z^{1/\alpha}$ with the path passed by the light.

A much more interesting situation arises in connection with refusing from the homogeneous Poisson ensemble and introducing the long-distance correlations of fractal type. The combination of power-type free path distribution with power-type scattering angle distribution leads to the model of anomalous diffusion described in fractional derivative equations whose solutions are expressed through stable distributions (Section 12.7).

15.2. Cosmic rays

One of the central questions in astrophysics of cosmic rays is the question about their origin. Now there are weighty reasons to consider that the cosmic rays with energy $E < 10^{17}$ eV have, mainly, the galactic origin. It is also supposed that the most probable cosmic ray sources are supernova bursts.

Using experimental data about cosmic ray fluctuations near the Earth and taking a certain transportation model into account, one can obtain some estimates for space-time source density. In (Lagutin & Nikulin, 1995) devoted to this problem, the random source distribution is assumed to be of homogeneous Poisson type, and the transportation process is described in the framework of the diffusion model. Let us discuss this approach in more detail.

We place the origin at the observation point and consider the field created here at $t = 0$ by all bursts inside the sphere U_R of radius R during the time interval $(-T, 0)$. In diffusion approximation, the angle distribution of particles at some point is the sum of two terms

$$\Phi(\mathbf{\Omega}) = \frac{1}{4\pi}[\Phi_0 + 3\mathbf{\Omega}\mathbf{j}].$$

The former, isotropic component does not depend on the direction and is proportional to the concentration of cosmic ray particles n; the latter, anisotropic component gives the linear dependence on cosine of the angle between the direction of travel of measured particles $\mathbf{\Omega}$ and the vector of diffusion current density $\mathbf{j}$. We denote by $n(\mathbf{r}, t)$ the diffusion propagator giving the concentration of particles at the point $r = 0$ at the time $t = 0$ from the unit source at the point $\mathbf{r}$ at the moment $-t$. Assuming, for the sake of simplicity, the diffusivity $D = 1/4$, we write

$$n(\mathbf{r}, t) = (\pi t)^{-3/2} e^{-r^2/t}. \tag{15.2.1}$$

The vector $\mathbf{j}$ is related to this density by the Fick law

$$\mathbf{j} = D\nabla n(\mathbf{r}, t) = (1/4)n'(r, t)\mathbf{r}/r, \tag{15.2.2}$$

where the prime denotes the derivative with respect to r. This expression differs from the common one by the sign because of the inverse direction of the position vector $\mathbf{r}$ headed to the source from the observation point.

The random sources $\{\mathbf{X}_i, T_i\}$ ($\mathbf{X}_i \in U_R$, $T_i \in (0, T)$) generate the concentration

$$N = \sum_i n(\mathbf{X}_i, T_i), \tag{15.2.3}$$

and the current density

$$\mathbf{J} = \sum_i \mathbf{j}(\mathbf{X}_i, T_i). \tag{15.2.4}$$

It is not hard to see that

$$\begin{aligned} \mathsf{E}N &= \rho \int_0^T dt \int_{U_R} d\mathbf{r}\, n(\mathbf{r}, t) \\ &= \frac{2\rho}{\sqrt{\pi}} \left\{ T \int_0^{R^2/T} e^{-\xi} \xi^{1/2} d\xi + R^2 \int_{R^2/T}^{\infty} e^{-\xi} \xi^{-1/2} d\xi \right\}, \end{aligned}$$

where ρ is the four-dimensional density in $U_R \times (0, T)$; it tends to ∞ as $T \to \infty$ and $R \to \infty$. It is of convenience, therefore, to subtract this from (15.2.3) and pass to the random variable

$$\tilde{N} = N - \mathsf{E}N.$$

Due to the evident symmetry,

$$\mathsf{E}\mathbf{J} = 0.$$

We rewrite the characteristic function for the pair of random variables $\tilde{N}$, $\mathbf{J}$

$$f(q, \mathbf{k}) = \mathsf{E}e^{iq\tilde{N}+i\mathbf{kJ}}$$

in the form

$$\ln f(q, \mathbf{k}) = \rho \int_{\mathbf{R}^3} \int_0^\infty \left\{ e^{i(qn+\mathbf{kj})} - 1 - iqn \right\} d\mathbf{r}\, dt.$$

Integrating with respect to the angular variable $\mathbf{\Omega}_r = \mathbf{r}/r$ and taking (15.2.2) into account. we obtain

$$\ln f(q, \mathbf{k}) = 4\pi\rho \int_0^\infty dt \int_0^\infty dr\, r^2 \left\{ e^{iqn} \frac{\sin kj}{kj} - 1 - iqn \right\}. \qquad (15.2.5)$$

where $n = n(r,t)$ and $j = |\mathbf{j}(\mathbf{r},t)| = (1/4)|n'(r,t)|$ are determined by formulae (15.2.1) and (15.2.2).

To find the characteristic function of isotropic fluctuations $\tilde{N}$, we let $k \to 0$ in (15.2.5); then

$$\ln f(q) \equiv \ln f(q, 0) = 4\pi\rho \int_0^\infty dt \int_0^\infty dr r^2 \left\{ e^{iqn} - 1 - iqn \right\}.$$

Recalling (15.2.1) and introducing

$$\xi = r^2/t, \qquad \tau = |q|(\pi t)^{-3/2},$$

we obtain

$$\ln f(q) = -\left(\frac{4}{3\pi^{3/2}}\right) |q|^{5/3} \int_0^\infty d\tau\, \tau^{-8/3} \int_0^\infty d\xi\, \xi^{1/2} \times \left\{ 1 - \exp[i\tau e^{-\xi} \operatorname{sign} q] + i\tau e^{-\xi} \operatorname{sign} q \right\}.$$

Calculation of the double integral leads us to the conclusion that the isotropic part of fluctuations $\tilde{N}$ is distributed by the stable law with $\alpha = 5/3$, $\beta = 1$. The reader can find the details in (Zolotarev, 1986; Lagutin & Nikulin, 1995; Lifshits, 1956). The distribution of the second (anisotropic) component was not found in the explicit form in (Lagutin & Nikulin, 1995) but on the basis of numerical calculations and dimension reasoning it was concluded that the magnitude $J = |\mathbf{J}|$ is distributed by the one-dimensional stable law with $\alpha = 5/4$ and $\beta = 1$. Let us verify this.

Setting $q = 0$ in (15.2.5), we obtain the expression for the characteristic function of vector $\mathbf{J}$

$$\ln f(\mathbf{k}) \equiv \ln f(0, \mathbf{k}) = -4\pi\rho \int_0^\infty dt \int_0^\infty dr\, r^2 \left\{ 1 - \frac{\sin kj}{kj} \right\}. \qquad (15.2.6)$$

By virtue of (15.2.2)

$$j = (1/4)|n'(r,t)| = (r/2)\pi^{-3/2}t^{-5/2}e^{-r^2/t}.$$

Let us pass from the variables r, t to

$$\xi = r^2/t, \qquad \eta = (1/2)\pi^{-3/2}t^{-2}k$$

respectively. Because in these variables

$$\begin{aligned} jk &= \eta\sqrt{\xi}e^{-\xi}, \\ r^2\,dr &= \frac{(k/\eta)^{3/4}}{2(2\pi^{3/2})^{3/4}}\xi^{1/2}d\xi, \\ dt &= -\frac{k^{1/2}}{2(2\pi^{3/2})^{1/2}}\eta^{-3/2}d\eta, \end{aligned}$$

we reduce characteristic function (15.2.6) to

$$f(\mathbf{k}) = \exp\left\{-Ck^{5/4}\right\}, \tag{15.2.7}$$

where

$$C = \frac{\pi\rho}{(2\pi^{3/2})^{5/4}}\int_0^\infty d\eta\,\eta^{-9/4}\int_0^\infty d\xi\,\xi^{1/2}\left[1 - \frac{\sin(\eta\sqrt{\xi}e^{-\xi})}{\eta\sqrt{\xi}e^{-\xi}}\right].$$

One can easily demonstrate that this improper integral converges. Thus, the characteristic exponent of the stable law describing the distribution of the current density vector $\mathbf{J}$ is equal to 5/4 indeed, but this is a three-dimensional spherically symmetric stable distribution. The distribution of the magnitude $J = |\mathbf{J}|$, in contrast to the statement of (Lagutin & Nikulin, 1995), is not a one-dimensional stable law, and cannot be, due to the mere fact that the magnitude of the sum of two vectors is not equal to the sum of their magnitudes.

Another interesting problem arises in connection with fluctuations of the total energy of all cosmic ray particles in a given volume V of the space. As theoretical calculations show, the energy spectrum of the particles emitted by supernovae has a long power tail (Lang, 1974)

$$S_{\text{th}}(E)\,dE \propto E^{-2.5}dE.$$

The all-particle spectrum arising from measurements shows the same behavior with a somewhat different exponent:

$$S_{\text{exp}}(E)\,dE \propto E^{-2.75}dE$$

(Petrera, 1995). Anyway, we deal here with the distribution of inverse power type

$$S(E)\,dE \propto E^{-\alpha-1}dE, \qquad 1 < \alpha < 2,$$

which stretches for a few orders up to $10^{17} \div 10^{18}$ eV. Consequently, on the assumption that the particles are distributed in the space according to the homogeneous Poisson ensemble we obtain that the total energy

$$E_V = \sum_{i=1}^{N(V)} E_i$$

of all particles in the volume V is a random variable similar to $Y(\alpha, 1)$ (see Section 2.1) with infinite variance. The relative width $\delta = \Delta E_V / \mathsf{E} E_V$ decreases as

$$\delta \propto V^{1/\alpha - 1},$$

i.e., essentially slower than in the case of a finite variance where $\delta \propto V^{-1/2}$.

15.3. Stellar dynamics

The general analysis of the statistical aspects of the fluctuating gravitational field arising from a random distribution of stars was performed by S. Chandrasekhar and J. von Neumann (Chandrasekhar & Neumann, 1941; Chandrasekhar & Neumann, 1943; Chandrasekhar, 1944a; Chandrasekhar, 1944b) and provided the necessary basis for several problems of stellar dynamics. Thus, the notion of the time of relaxation of a stellar system is intimately connected with the influence of such fluctuations in the gravitational field on the motion of stars. Also the dynamic problems presented by star clusters can be treated satisfactorily only in the framework of such a model. The common characteristic of all these problems is that individual stars are subject to the changing influence of a varying local stellar distribution. So, one may formulate the problem in terms of point sources.

Point masses m_i (stars) placed at $\mathbf{r}_i$ generate at the origin the gravitational field of intensity

$$\mathbf{F} = \sum_i \frac{Gm_i}{r_i^3} \mathbf{r}_i. \qquad (15.3.1)$$

Under assumption that the system of stars is a homogeneous Poisson ensemble, Holtsmark's results can be readily adapted to the gravitational case. However, the specification of the Holtsmark distribution does not characterize the essential features of the fluctuating field. An equally important aspect of the problem is the fluctuation rate and the related questions concerning the probability after-effects (Chandrasekhar & Neumann, 1941, p. 490). These later problems are essentially more complicated than the establishment of the stationary distribution. By virtue of (15.3.1),

$$\mathbf{f} = \frac{d\mathbf{F}}{dt} = \sum_i Gm_i \left(\frac{\mathbf{v}_i}{r^3} - \frac{\mathbf{r}_i(\mathbf{r}_i \mathbf{v}_i)}{r_i^5} \right),$$

where $\mathbf{v}_i$ denotes the velocity of the typical field star,

$$\mathbf{v}_i = d\mathbf{r}_i/dt.$$

The fluctuation rate can be specified in terms of the distribution $p(\mathbf{F}, \mathbf{f})$ which gives the joint probability of a given field intensity $\mathbf{F}$ and the associated change rate $\mathbf{f}$. This bivariate distribution was computed in (Chandrasekhar & Neumann, 1941) under the additional assumption that random velocities $\mathbf{v}_i$ are independent of each other and of other random variables and distributed by Maxwell law. We briefly review some results.

Taking a sphere of radius R and setting then $R \to \infty$ under the condition $\rho = \text{const}$, we obtain

$$p(\mathbf{F}, \mathbf{f}) = \frac{1}{(2\pi)^6} \int d\mathbf{K} \int d\mathbf{k} e^{i(\mathbf{KF}+\mathbf{kf})} \varphi(\mathbf{K}, \mathbf{k}),$$

$$\varphi(\mathbf{K}, \mathbf{k}) = \exp\left\{-\rho\psi(\mathbf{K}, \mathbf{k})\right\},$$

$$\psi(\mathbf{K}, \mathbf{k}) = \left\langle \int \left\{1 - \exp\left[iGmr^{-3}\mathbf{rK} + iGm\left(r^{-3}\mathbf{v} - 3r^{-5}(\mathbf{r}, \mathbf{v})\mathbf{r}\right)\right]\right\} d\mathbf{r} \right\rangle$$

Here the angular brackets stand for averaging over masses and velocities of stars.

It is clear that

$$\int p(\mathbf{F}, \mathbf{f})\, d\mathbf{f} = \frac{1}{(2\pi)^3} \int d\mathbf{K} e^{-i\mathbf{KF}} \varphi(\mathbf{K}, 0)$$

is Holtsmark's distribution, and

$$\int p(\mathbf{F}, \mathbf{f})\, d\mathbf{F} = \frac{1}{(2\pi)^3} \int d\mathbf{k}\, e^{-i\mathbf{kf}} \varphi(0, \mathbf{k})$$

is the trivariate Cauchy's distribution (with the isotropic distribution of the velocity $\mathbf{v}$). Chandrasekhar and von Neumann obtained the asymptotic distributions

$$\psi(\mathbf{K}, \mathbf{k}) = aK^{3/2} + bk^2 K^{-3/2} \sin^2\gamma + O(k^4), \qquad k \to \infty.$$

$$\psi(\mathbf{K}, \mathbf{k}) = ck + d(\lambda\cos^2\gamma + \mu)K^2 k^{-1/3} + O(K^4), \qquad K \to \infty,$$

where γ is the angle between the vectors $\mathbf{K}$ and $\mathbf{k}$,

$$a = (4/15)(2\pi)^{3/2} G^{3/2} \langle m^{3/2} \rangle,$$

$$b = (1/4)(2\pi)^{3/2} G^{1/2} \langle m^{1/2} v^2 \rangle,$$

$$c = (2/3)\pi^2 Q_0 G \langle mv \rangle,$$

$$d = G^{5/3} \langle m^{5/3} v^{-1/3} \rangle,$$

and λ, μ, and Q_0 are absolute constants (Chandrasekhar & Neumann, 1941, (281)). The characteristic function obtained was used for evaluation of the second conditional moments of the longitudinal (with respect to **F**) and transversal component of $\bar{\mathbf{f}}$

$$\overline{\mathbf{f}^2_{\|}}(F) = \frac{8ab}{\pi}\frac{\varepsilon^{1/2}}{H(\varepsilon)}\int_0^\infty e^{-(x/\varepsilon)^{3/2}}(\sin x - x\cos x)x^{-5/2}dx,$$

$$\overline{\mathbf{f}^2_{\perp}}(F) = \frac{4ab}{\pi}\frac{\varepsilon^{1/2}}{H(\varepsilon)}\int_0^\infty e^{-(x/\varepsilon)^{3/2}}(x^2\sin x + x\cos x - \sin x)x^{-5/2}dx,$$

and also for its absolute value:

$$\overline{\mathbf{f}^2}(F) = \frac{4ab\varepsilon^{1/2}}{H(\varepsilon)}G(\varepsilon), \qquad \varepsilon = a^{-2/3}F,$$

$$G(\varepsilon) = \frac{2}{\pi}\int_0^\infty e^{-(x/\varepsilon)^{3/2}}x^{-1/2}\sin x\,dx,$$

It was shown, in particular, that

$$\overline{\mathbf{f}^2_{\|}}(F) \sim \overline{\mathbf{f}^2_{\perp}}(F), \qquad F \to 0,$$

$$\overline{\mathbf{f}^2_{\|}}(F) \sim 4\overline{\mathbf{f}^2_{\perp}}(F), \qquad F \to \infty.$$

From here the following conclusion was made in (Chandrasekhar & Neumann, 1941, p. 507): for weak fields the probability of a change's occurring in the field acting at a given instant of time is independent of the direction and magnitude of the initial field, while for strong fields the probability of a change's occurring in the direction of the initial field is twice as great as in a direction at right angles to it.

The physical sense of this is quite clear. A weak field results from a fluctuation of a symmetric configuration of stars about the point considered. We should therefore expect the changes to follow, to be equally likely in all directions. On the other hand, a strong field acting at a point implies a highly asymmetric configuration of stars about the point, and consequently changes are more likely to occur in the direction of the initial field than in other direction.

The magnitude $\overline{\mathbf{f}^2}(F)$ plays the important role while estimating the average life time of the given state **F**. The state **F** means that at a given instant t ($\mathbf{F} = \mathbf{F}_0$ with $t = 0$) at some fixed point the intensity of the gravitational field is equal to **F**. As a result of star movements, the change of state happens (Fig. 15.1). The average life time of the state introduced by Chandrasekhar with exponential representation of falling down correlations is of the form

$$T = [\overline{\mathbf{f}^2}(F)]^{-1/2}F = \sqrt{\frac{(\rho a)^{1/3}}{4b}\frac{\varepsilon^{3/2}H(\varepsilon)}{G(\varepsilon)}}.$$

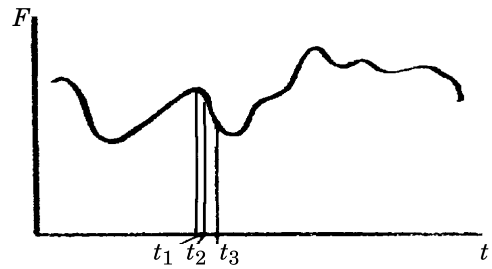


Figure 15.1. Due to the motion of the stars, the force applied to a star observed gradually changes with time (Chandrasekhar & Neumann, 1941)

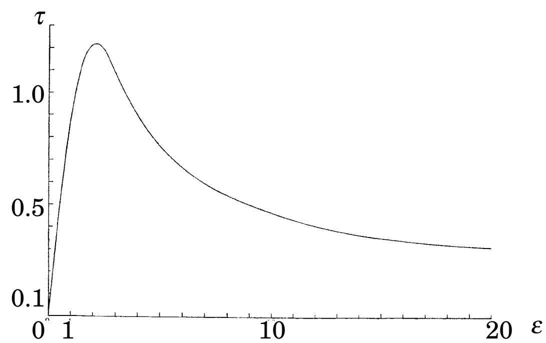


Figure 15.2.

Assuming

$$T_0 = 0.32\langle m^{3/2}\rangle^{1/6}\langle m^{1/2}v^2\rangle^{-1/2}\rho^{-1/3},$$

we introduce the dimensionless life time

$$\tau(\varepsilon) = T/T_0 = \sqrt{\frac{\varepsilon^{3/2}H(\varepsilon)}{G(\varepsilon)}}.$$

Using the asymptotic expressions for $H(\varepsilon)$ and $G(\varepsilon)$ it is possible to show that

$$\begin{aligned} \tau(\varepsilon) &\sim \varepsilon, && \varepsilon \to 0, \\ \tau(\varepsilon) &\sim \sqrt{15/8}\,\varepsilon^{-1/2}, && \varepsilon \to \infty. \end{aligned}$$

The results of numerical computations of the function $\tau(\varepsilon)$ are shown in Fig. 15.2 (Chandrasekhar & Neumann, 1941).

15.4. Cosmological monopole and dipole

The contemporaneous average density of the Universe is written in the form

$$\rho_0 = 1.88 \cdot 10^{-29} \Omega_0 h^2 \text{ g cm}^{-3},$$

where h is the dimensionless parameter connected with the uncertainty of Hubble's constant H,

$$0.5 \le h \le 1,$$

and Ω_0 is the cosmological density parameter. It is accepted to consider that the modern value of the parameter lies in the interval

$$0.03 \le \Omega_0 \le 1.$$

As a first approximation, the Universe looks like a uniform Hubble flow with the relative velocity

$$\mathbf{v} = H(t)\mathbf{r},$$

forming the background for observations of peculiar motions of galaxies. Assuming that the gravitational instability is the cause of the observed peculiar motions, these local deviations from a uniform Hubble flow provide a powerful tool for studying the local mass distribution and hence estimating the cosmological parameter Ω_0.

Using linear perturbation theory, the peculiar velocity $\mathbf{v}_p$ can be related to the peculiar acceleration $\mathbf{g}$ via

$$\mathbf{v}_p \propto w(\Omega_0)\mathbf{g},$$

where $w(\Omega_0)$ measures the logarithmic rate of growth of the mass fluctuation at the present epoch (Peebles, 1980). Calculations yield (Plionis *et al.*, 1993)

$$\mathbf{v}_p = d_{\text{conv}} \frac{\Omega_0^{0.6}}{3b} \frac{|\mathbf{D}|}{M} (\le d_{\text{conv}}),$$

where M and $\mathbf{D}$ are the monopole and dipole moments obtained from observations via the relations

$$M = \frac{1}{4\pi} \sum_{i=1}^{n} \frac{1}{\phi(r_i)} \frac{1}{r_i^2},$$

$$\mathbf{D} = \frac{3}{4\pi} \sum_{i=1}^{n} \frac{1}{\phi(r_i)} \frac{\mathbf{r}_i}{r_i^3}.$$

Here $\mathbf{r}_i$ are the positions of galaxies, $\phi(r_i)$ is a selection function to take into account the fact that at different distances we sample different portions of the luminosity function, d_{conv} is the depth at which the dipole converges to its final value and b is the bias factor that relates galaxy to mass overdensities. The factor $\Omega_0^{0.6}$ arises when one uses the theory of linear gravitational instability to relate the peculiar velocity to the gravitational acceleration (Peebles, 1980).

Sometimes, Monte-Carlo simulation of the values is performed on the assumption that the positions $\mathbf{r}_i$ are distributed by the homogeneous Poisson law. Supposing, for the sake of simplicity,

$$M_R = \frac{1}{4\pi}\sum_i \frac{1}{r^2}1(\mathbf{r}_i; U_R),$$

$$\mathbf{D}_R = \frac{3}{4\pi}\sum_i \frac{\mathbf{r}_i}{r_i^3}1(\mathbf{r}_i; U_R),$$

where U_R is a sphere of radius R centered at the observation point $\mathbf{r} = 0$, one can write the characteristic function

$$f(q, \mathbf{k}) = \mathsf{E}e^{iqM_R + \mathbf{k}\mathbf{D}_R}$$

for the bivariate density $p(M_R, \mathbf{D}_R)$

$$\ln f(q, \mathbf{k}) = \rho \int_{U_R} \left\{e^{i(qM_R + \mathbf{k}\mathbf{D}_R)} - 1\right\} d\mathbf{r}.$$

Setting here $q = 0$ and then letting $R \to \infty$ we arrive at the Holtsmark distribution for $\mathbf{D}$, and in the case where $\mathbf{k} = 0$ we obtain the one-dimensional stable distribution with $\alpha = 3/2$ and $\beta = 1$ for $M_R - \mathsf{E}M_R$. The fact

$$\lim_{R\to\infty} \mathsf{E}M_R = \infty$$

is known as the Olbers paradox, and unfortunately its solution is obtained without using stable laws (see (Harrison, 1990)).

A number of authors analyze the dependence of observed cosmological dipole $\mathbf{D}_R$ on R interpreting its saturation as an evidence for homogeneity of the Universe; some authors interpret the saturation as an evidence for isotropy of the Universe. For a complete review of this discussion we refer the reader to (Coleman & Pietronero, 1992; Sylos Labini, 1994; Baryshev *et al.*, 1994; Borgani *et al.*, 1994; Martinez & Jones, 1990). In any case, this quantity bears a direct relation to the Cosmological Principle which can be formulated in the following way: Except for local irregularities, the Universe presents the same aspect, from whatever point it is observed (Kourganoff, 1980).

The dependence of $\mathbf{D}_R$ on R for some fractal models of the Universe was investigated in (Sylos Labini, 1994; Uchaikin & Korobko, 1997a).

15.5. The Universe as a rippled water

The local constituents of the Universe (from stars to clusters of galaxies) correspond to discrete particles, such as atoms or molecules of statistical physics.

Nevertheless, considering (as in hydrodynamics) only large-scale motions one can ignore the discrete nature of these 'particles' and treat the constituents of the Universe as a continuous fluid.

The density distribution arising at the non-linear stage of gravitational instability is similar to intermittence phenomena in acoustic turbulence. Initially small-amplitude density fluctuations transform into thin dense pancakes, filaments, and compact clumps of matter. A similar process is the distribution of light reflected or refracted from rippled water (see also Section 11.7). The similarity of gravitational instability to acoustic turbulence is highlighted by the fact that the late non-linear stages of density perturbation growth can be described by Burger's equation (Shandarin & Zeldovich, 1989) which is well known in the theory of turbulence. We cite here the paper (Hu & Woyczynski, 1995) where stable distributions are used.

In this work, the one-dimensional Burgers equation

$$\partial v/\partial t + v\partial v/\partial x = \nu\partial^2 v/\partial x^2$$

is considered, where $v(t,x)$ is the velocity and the small constant ν is the viscosity which is supposed to simulate the gravitational adhesion. The initial velocity potential

$$V(x) = \int_0^x v(0,y)\,dy$$

is chosen in the form

$$V(x) = \sum_{i=-\infty}^{\infty} e^{Z_i}\,\delta(x - X_i),$$

where $\{X_i\}$ is the standard Poisson point process on the real line, and random variables

$$Z_i = \sum_{j=0}^{n} c_j Y_{i+1-j}$$

form a strictly stationary sequence of moving averages of independent and identically distributed symmetric stable variables Y_i with $1/2 < \alpha < 2$. The solution random field of the Burgers equation with these initial data can be written in the form

$$v(t,x) = \frac{\sum_i (x - X_i) g(t,x,X_i) e^{Z_i}}{t\sum_i g(t,x,X_i) e^{Z_i}}$$

where

$$g(t,x,y) = (4\pi\nu t)^{-1/2} \exp\left\{-(x-y)^2/(4\nu t)\right\}$$

is the usual Gaussian kernel.

The following theorem is proved in the paper (Hu & Woyczynski, 1995).

THEOREM 15.5.1. *Let the initial velocity potential be*

$$\int_{-\infty}^{x} v(0,y)\,dy = \sum_i e^{Z_i}\,\delta(x - X_i),$$

where $\{X_i\}$ *is the standard Poisson process on* R *and* $\{Z_i\} = \{c_1 Y_i + c_0 Y_{i+1}\}$ *with* $\{Y_i\}$ *being independent identically distributed symmetric stable random variables* $(1/2 < \alpha < 2)$ *and independent of the Poisson ensemble. Then, for each* $x \in R$*, the random field* v *satisfying the Burger equation*

$$\partial v/\partial t + v\partial v/\partial x = \nu\partial^2 v/\partial x^2$$

satisfies the asymptotic condition

$$v(t,x) \overset{\mathsf{P}}{\sim} -\frac{x - X_{i*}}{t}, \qquad t \to \infty,$$

where $X_{i*} = X_{i*}(x,t)$ *is the point where the random field* $g(t,x,X_i)e^{Z_i}$ *attains its maximum, i.e.,*

$$g(t,x,X_{i*})e^{Z_{i*}} = \max_i g(t,x,X_i)e^{Z_i}.$$

This establishes analytically the existence of a one-dimensional version of the net-like structures in the fluid model of matter distribution in the Universe.

15.6. The power spectrum analysis

It is often useful to analyze the statistics of the galaxy distribution in Fourier space, instead of in configuration space, as done by correlation functions. For this purpose, the Fourier transform $\tilde{\Delta}(k)$ of the relative fluctuation density field $\Delta(\mathbf{r})$ is introduced, which leads to the power spectrum $P(k)$ being connected to the correlation function $\xi(r)$ via the relation

$$P(k) \propto \langle|\tilde{\Delta}(\mathbf{k})|^2\rangle \propto \int e^{i\mathbf{kr}}\xi(\mathbf{r})\,d\mathbf{r} \tag{15.6.1}$$

(we omit here the inessential for understanding normalizing constant) There are several reasons for the importance of the power spectrum $P(k)$ in statistics. First, it (or its Fourier transform) completely specifies a homogeneous and isotropic Gaussian random field. The fields are very popular among cosmologists (Bertschinger, 1992) because the inflation theory predicts that the initial density fluctuation field is a Gaussian random field, and naturally, because calculations are somewhat easy. Another reason for the importance of the power spectrum is that it measures the mean-square amplitude of density fluctuations as a function of wavelength $\lambda = 2\pi/k$.

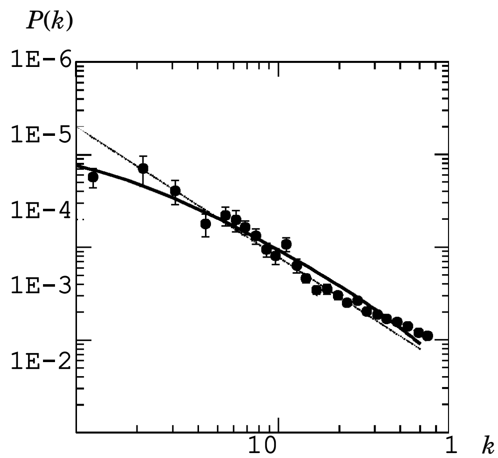


Figure 15.3. Power spectrum $P(k)$ from the Lick Observatory catalogue (taken from (Fry, 1994)) (the dashed line shows $P(k) \sim k^{-1.41}$; the solid line shows approximation (15.6.3))

A particular simple model often used for approximation of observed data is given by the power law shape

$$P(k) \propto k^n, \tag{15.6.2}$$

with $n > -3$ in order to allow for the convergence of the integral of $P(k)$ at large wavelength (Borgani, 1995). The value $n = 1$ for the spectral index corresponds to the scale-free Harrison–Zeldovich spectrum that describes the fluctuations generated in the framework of the canonical inflationary scenario. Inverting (15.6.1), for $\xi(\mathbf{r})$ we obtain

$$\xi \propto \int k^n e^{i\mathbf{kr}} d\mathbf{k} = 4\pi \frac{\Gamma(n+3)}{n+2} \sin[(n+2)\pi/2] r^{-(n+3)}.$$

Thus, the detected power law shape for the 2-point function,

$$\xi(\mathbf{r}) \propto r^{-1.8},$$

turns into a constant logarithmic slope of the power spectrum, with spectral index $n = -1.2$, at least at scales $r \leq 10h^{-1}$ Mpc.

The power spectrum $P(k)$ (from the Lick Observatory catalogue (Fry, 1994)) plotted in Fig. 15.3, goes as $P(k) \sim k^n$ with index $n \approx -1.4$ for $5 \leq k \leq 30$ (k in units of 'waves per box', physical wavelengths are $\lambda = 260h^{-1}$ Mpc/k). Evidently, this approximation is not very accurate and is used mainly because of a simple form. With the same or even better result one can use other approximation formulae.

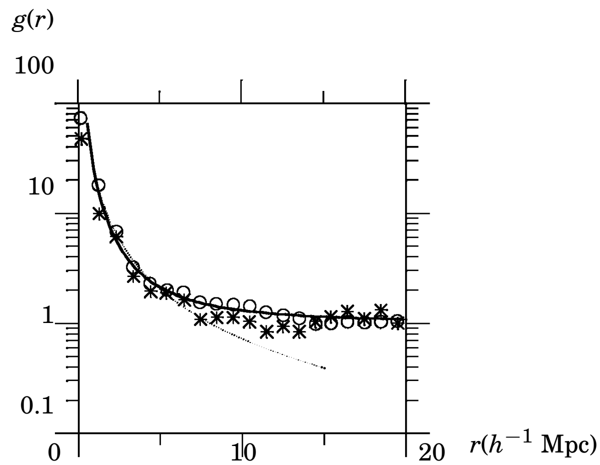


Figure 15.4. The structure function $g(r) = 1 + \xi(r)$ obtained from CfA galaxy redshift survey, sample S65 (the open circles and asterisks represent the results of two different procedures for corrections (Martinez & Jones, 1990); the dotted line is the best-fit power law and the solid line is a result of the use of (15.6.4) (Uchaikin, 1997))

Let us consider the formula

$$P(k) = A\frac{e^{-(bk)^\alpha}}{1 - ce^{-(bk)^\alpha}}, \tag{15.6.3}$$

where A, b, c, and α are positive constants. In spite of its explicit difference from (15.6.2) the formula gives a good fit to the experimental data by choosing appropriate values of the parameters (in this case $\alpha = 1.5$, $c = 0.99$, $b = 0.018$, and $A = 10^{-5}$).

To show why we take such a way of approximation, we rewrite (15.6.3) as

$$P(k) = ce^{-(bk)^\alpha}P(k) + Ae^{-(bk)^\alpha}.$$

Inverting it, we obtain, in view of (15.6.1),

$$\xi(\mathbf{r}) = cb^{-3}\int q_3(\mathbf{r}'/b;\alpha)\xi(\mathbf{r}-\mathbf{r}')d\mathbf{r}' + Ab^{-3}q_3(\mathbf{r}'/b;\alpha) \tag{15.6.4}$$

where $q_3(\mathbf{r};\alpha)$ is the three-dimensional spherically symmetric stable distribution. Thus we immediately arrive at the scheme described in Chapter 11 (see (11.4.15)).

The application of (15.6.4) to fitting the CfA (Center for Astrophysics) data is carried out in (Uchaikin, 1997), which is the source of Fig. 15.4.

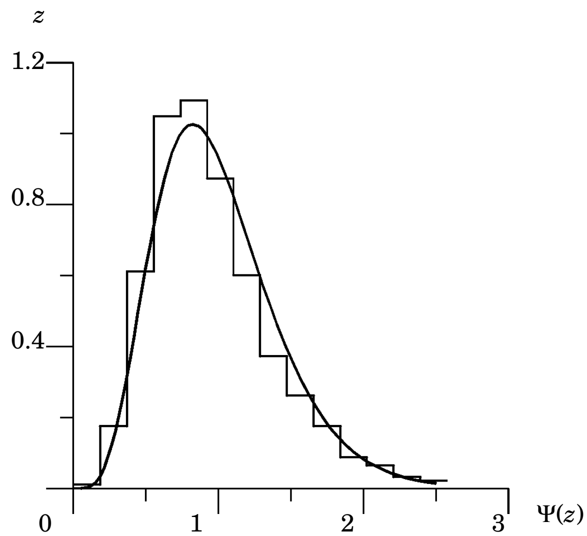


Figure 15.5. The observed distribution of $Z = N(R)/\langle N(R)\rangle$ taken from (Coles *et al.*, 1993) and its approximation by the gamma distribution $\Psi(z)$ (15.7.1)

15.7. Cell-count distribution for the fractal Universe

The numerical calculations performed in Section 11.7 yield the distribution of $N(R)$ for the stochastic fractal model

$$\mathsf{P}\{N(R) = n\} \sim \frac{1}{\langle N(R)\rangle}\Psi\left(\frac{n}{\langle N(R)\rangle}\right), \qquad R \to \infty, \tag{15.7.1}$$

where

$$\Psi(z) = \frac{1}{\Gamma(\lambda)}\lambda^{\lambda} z^{\lambda-1} e^{-\lambda z}$$

is the gamma distribution.

Distributions of $N(R)$ called the cell-count distributions are obtained from galaxy catalogs by means of a not very reliable procedure. Nevertheless, it is interesting to compare the fractal cell-count distribution (15.7.1) with the data observed. To do this, we take the Lick sample data presented in (Coles *et al.*, 1993). Computing $\langle N(R)\rangle$, we find $\Psi^{\text{obs}}(z)$, and calculation of $\langle N^2(R)\rangle$ gives us a possibility to find the parameter λ in approximation formula (15.7.1). The two distributions presented in Fig. 15.5 have turned out to be very close to each other.

15.8. Global mass density for the fractal Universe

One of important parameters characterizing models of the Universe is the global mass density

$$\rho = \lim_{R\to\infty} [M(R)/V(R)], \tag{15.8.1}$$

where $M(R)$ is the total mass within a sphere of radius R and $V(R)$ is the volume of the sphere. For models being homogeneous (at least on large scales) this limit exists and is not zero. Let us see what kind of results one can get for fractal models.

We begin with the deterministic fractal described in (Coleman & Pietronero, 1992). Starting from a point occupied by an object and counting how many objects are within a volume characterized by a certain length scale, we get N_0 point-like objects within a radius R_0, $N_1 = qN_0$ objects within a radius $R_1 = kR_0$, $N_2 = qN_1 = q^2N_0$ objects within $R_2 = kR_1 = k^2R_0$, and so on. In general, we have

$$N_n = q^n N_0, \tag{15.8.2}$$

$$R_n = k^n R_0, \tag{15.8.3}$$

where q and k are some constants. By taking the logarithm of (15.8.2) and (15.8.3), and dividing one by the other, we obtain

$$N_n = CR_n^D \tag{15.8.4}$$

with

$$C = N_0 R_0^{-D}, \qquad D = \frac{\ln q}{\ln k},$$

where C is the proportionality coefficient related to the lower cutoffs N_0 and R_0 of the fractal system, that is, the inner limit where the fractal system ends, and D is the fractal dimension ($D < 3$). If we smooth out the point structure, we obtain

$$N(R) = CR^D. \tag{15.8.5}$$

Assuming that all objects have the same mass m, we get

$$\rho(R) \equiv M(R)/V(R) = [3mC/(4\pi)]R^{-\gamma}, \quad \gamma = 3 - D.$$

Hence,

$$\rho = \lim_{R\to\infty} \rho(R) = 0 \tag{15.8.6}$$

for the fractal structure. This fact is known as the third postulate of the pure hierarchy fractal conception: for a pure hierarchy the global mass density is zero everywhere.

The situation is the same if the masses m_i are independent identically distributed random variables with $\mathsf{E}m_i = \langle m \rangle < \infty$:

$$\langle \rho(R) \rangle \equiv \langle M(R)/V(R) \rangle = [3\langle m \rangle C/(4\pi)]R^{-\gamma}.$$

However, astronomical observations show that in a very large range of masses the density $p(m)$ is of the form

$$p(m) = \alpha A m^{-\alpha-1}, \qquad 0 < m_0 < m, \tag{15.8.7}$$

where α is smaller than 1 (Press & Schechter, 1974). We assume, moreover, that (15.8.1) holds true for all $m > m_0$. Introducing

$$\rho(R) = \sum_{i=1}^{N(R)} m_i/V(R) = [3/(4\pi)]R^{-3} \sum_{i=1}^{N(R)} m_i, \tag{15.8.8}$$

we are able to reduce the problem of finding $\langle \rho \rangle$, which is infinite, to the problem of investigation of the distribution of random variable (15.8.8). In the case where $\alpha < 1$, distribution (15.8.7) belongs to the domain of normal attraction of a one-dimensional standardized stable law $q_A(x; \alpha, 1)$; thus

$$\mathsf{P}\left\{ \sum_{i=1}^{N} m_i/b_N < x \right\} \Rightarrow G_A(x; \alpha, 1), \qquad N \to \infty, \tag{15.8.9}$$

where

$$b_N = b_1 N^{1/\alpha}, \tag{15.8.10}$$

$$b_1 = [A\Gamma(1-\alpha)\cos(\alpha\pi/2)]^{1/\alpha}. \tag{15.8.11}$$

Denoting the probability density of random variable (15.8.8) by $p_\rho(x; R)$ and recalling (15.8.9), we obtain the asymptotic expression for large values of $N(R)$

$$p_\rho(x; R) \sim [4\pi R^3/(3b_{N(R)})]q_A(4\pi R^3 x/[3b_{N(R)}]; \alpha, 1), \qquad N(R) \to \infty. \tag{15.8.12}$$

Substituting (15.8.5) into (15.8.10) and inserting the result into (15.8.11), we obtain

$$p_\rho(x; R) \sim QR^{3-D/\alpha} q^A(QR^{3-D/\alpha}x; \alpha, 1), \qquad R \to \infty, \tag{15.8.13}$$

where

$$Q = 4\pi/(3b_1 C^{1/\alpha}).$$

In view of (15.8.13), the probability density of the random conditional mass density possesses a non-degenerate limit as $R \to \infty$ for $\alpha = D/3$:

$$p_\rho(x;R) \underset{R\to\infty}{\longrightarrow} p_\rho(x) \equiv Qq(Qx;\alpha,1), \qquad (15.8.14)$$

The aggregate considered above seems too artificial to be used as a model of mass distribution in the Universe. The stochastic fractal described in Section 11.6 is more appropriate for this purpose. In this case

$$p_\rho(x;R) = \langle p_\rho(x;R,N(R))\rangle,$$

where $p_\rho(x;R,N(R))$ is the density conditioned by a fixed value $N(R)$ and $\langle\dots\rangle$ stands for averaging over the random variable distributed by the law

$$\mathsf{P}\{N(R)=n\} \sim \frac{1}{\langle N(R)\rangle}\Psi_D\left(\frac{n}{\langle N(R)\rangle}\right), \qquad R\to\infty$$

with

$$\Psi_D(z) = \frac{\lambda^\lambda z^{\lambda-1}}{\Gamma(\lambda)}e^{-\lambda z}. \qquad (15.8.15)$$

Taking (15.8.13) into account, we obtain

$$p_\rho(x;R) \sim QR^{3-D/\alpha}\int_0^\infty z^{-1/\alpha}q_A(QR^{3-D/\alpha}xz^{-1/\alpha};\alpha,1)\Psi_D(z)\,dz.$$

Therefore, the non-degenerate limit of the distribution of ρ exists under the condition $\alpha = D/3$ again:

$$p_\rho(x) = Q\int_0^\infty z^{-1/\alpha}q_A(Qxz^{-1/\alpha};\alpha,1)\Psi_D(z)\,dz.$$

Re-scaling the independent variable $Qx \to x$ and recalling (15.8.15), we arrive at the expression

$$p_\rho(x) = [\Gamma(\lambda)]^{-1}\lambda^\lambda\int_0^\infty z^{\lambda-1-1/\alpha}q_A(xz^{-1/\alpha};\alpha,1)e^{-\lambda z}dz. \qquad (15.8.16)$$

It is easy to verify that this result satisfies the normalization $\int_0^\infty p_\rho(x)\,dx = 1$.

Astronomical observations show that $D \approx 1.16 \div 1.40$ (Martinez & Jones, 1990). Taking, for the sake of simplicity, $D = 1.5$, we obtain $\alpha = 0.5$, which coincides approximately with $\alpha_{\text{obs}} = 0.5 \div 0.6$ obtained from the luminosity observations (Borgani, 1995). In this case, one can use an explicit form of the stable density

$$q^A(x;1/2,1) = \frac{1}{\sqrt{2\pi}}x^{-3/2}e^{-\frac{1}{2x}};$$

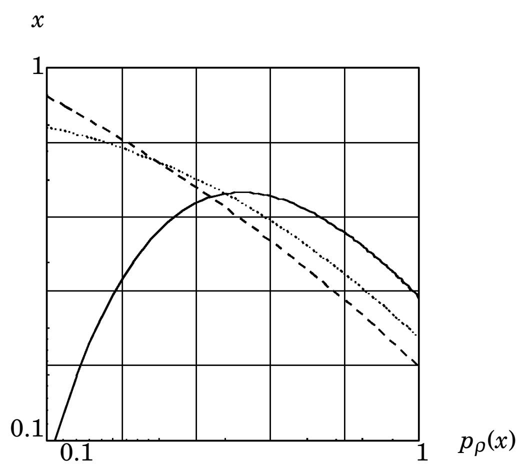


Figure 15.6. The graphs of probability distribution density $p_\rho(x)$ for the deterministic fractal—solid line, the single LM fractal—dashed line, and the coupled LM fractal—dotted line (Uchaikin & Korobko, 1997a)

(15.8.16) thus takes the form

$$p_\rho(x) = \frac{\lambda^\lambda x^{\lambda/2-1}}{\sqrt{2\pi}\Gamma(\lambda)} \int_0^\infty t^{(\lambda-3)/2} e^{-\lambda(x/t)^{1/2}} e^{-\frac{1}{2t}} dt.$$

Using formula (3.462) from (Gradshtein & Ryzhik, 1963), we can rewrite the result as

$$p_\rho(x) = \frac{2\lambda^\lambda x^{\lambda/2-1}}{\sqrt{\pi}\Gamma(\lambda)} \Gamma(\lambda+1) e^{(\lambda^2 x)/4} D_{-(\lambda+1)}(\lambda\sqrt{x})$$

where $D(x)$ is the parabolic cylinder function and $\lambda = 1.5$ for a single fractal and $\lambda = 3$ for a coupled fractal. The graphs of $p_\rho(x)$ for deterministic fractal, single and coupled stochastic fractals are represented in Fig. 15.6. As we can see, the distributions are broad enough with the same asymptotics $x^{-3/2}$.

We have a ground to suppose that the model can be used for description of the property of the Universe, but how can the random behavior of GMD be interpreted? In any case, the following assertion seems to be highly plausible: fractal cosmology should be stochastic. It follows from the main attribute of a fractal, namely, from its self-similarity. If a fractal is stochastic at some scale, then it should be stochastic at all scales. In other words, there is no scale at which the Universe could be described in terms of deterministic continuous medium.

However, it is impossible not to admit that the fractal model of the Universe is a very extreme kind of possible models requiring revision of not only the method of usual analysis of observation data but the Cosmological Principle itself.

16

Stochastic algorithms

16.1. Monte-Carlo estimators with infinite variance

The key idea of Monte-Carlo method is based on the law of large numbers and can be formulated as follows. Let us have a need to calculate, even though somewhat approximately, some value J (say, an integral). We do not know the value but we can generate a sequence of independent identically distributed random variables $Z^{(1)}, Z^{(2)}, \dots$ about which it is known that their mathematical expectation is finite and strictly equal to the unknown value J:

$$\mathsf{E}Z = J.$$

The law of large numbers says that in this case the value J can be estimated by the arithmetic mean as an unbiased estimator

$$J_n = \frac{1}{n}\sum_{j=1}^{n} Z^{(j)}$$

with the error

$$\Delta J_n = J_n - J$$

being random and decreasing in the probabilistic sense as n grows. The point is an estimation of the error characteristics. Of course, if

$$\mathsf{E}Z^2 = \sigma^2 + J^2 < \infty$$

exists, we have an ordinary statistical problem with well known methods of its solution, but if

$$\mathsf{E}Z^2 = \infty$$

we can find ourselves in the domain of attraction of some stable law.

Following (Ermakov & Mikhailov, 1976), we consider the summation of the series

$$J = \sum_{k=1}^{\infty} k^{-1-\gamma}, \qquad \gamma > 0. \tag{16.1.1}$$

Although the Monte-Carlo technique is not the best one for such simple problems, it is useful to consider the algorithm which can serve as a prototype for similar algorithms in more complicated problems and to see how stable laws arise here.

Let K be an integer-valued random variable with probability distribution

$$\mathsf{P}\{K = k\} = \frac{1}{k(k+1)}, \qquad k = 1, 2, \dots,$$

and

$$Z = (K+1)K^{-\gamma}.$$

It is not hard to see that the mathematical expectation of this random variable

$$\mathsf{E}Z = \sum_{n=1}^{\infty} \frac{(k+1)k^{-\gamma}}{k(k+1)} = \sum_{n=1}^{\infty} k^{-1-\gamma} = J$$

coincides with (16.1.1) sought for, therefore

$$J_n = \frac{1}{n} \sum_{j=1}^{n} Z^{(j)}$$

is an unbiased estimator of sum (16.1.1) if the terms are independent. The variance of Z

$$\operatorname{Var} Z = \sum_{k=1}^{\infty} (k+1)k^{-2\gamma-1} - J^2$$

for $\gamma \le 1/2$ appears to be infinite. As concerns the probability

$$\mathsf{P}\{Z > z\} = \mathsf{P}\{(K+1)K^{-\gamma} > z\} \sim \mathsf{P}\left\{K > z^{1/(1-\gamma)}\right\} \sim z^{-1/(1-\gamma)}$$

we conclude that the random variable Z belongs to the domain of attraction of the stable law with

$$\alpha = \frac{1}{1-\gamma} \in (1, 2]$$

and $\beta = 1$. In this case, the measure of the statistical error is not a variance being infinite but the width of an interval containing the given probability of the stable distribution.

16.2. Flux at a point

The problem of particle transport through a medium, in general, is formulated as follows. A source given emits N independent particles, each distributed in the space with the three-dimensional distribution density $s(\mathbf{r})$

$$\int s(\mathbf{r})d\mathbf{r} = 1.$$

The trajectory of the particle is a broken line whose nodes form a homogeneous terminating Markov chain with transition probability $p(\mathbf{r} \to \mathbf{r}')$ and survival probability under collision $c(\mathbf{r})$. In actual reality these characteristics depend on the energy and the direction of movement of the particle, but we omit these details in order not to over-complicate the formulae. In this simple case, the collision density $f(\mathbf{r})$ satisfies the equation of type (10.3.11)

$$f(\mathbf{r}) = f_1(\mathbf{r}) + \int c(\mathbf{r}')f(\mathbf{r}')p(\mathbf{r}' \to \mathbf{r})\,d\mathbf{r}', \tag{16.2.1}$$

where $f_1(\mathbf{r})$ is the first collision density defined by the source distribution with normalization

$$\int f_1(\mathbf{r})\,d\mathbf{r} = \int s(\mathbf{r})\,d\mathbf{r} = N. \tag{16.2.2}$$

By virtue of linearity of (16.2.1), its solution for the source emitting N particles is obtained by multiplying the solution for a unit source by N, so in what follows we assume $N = 1$.

Let us have a need to calculate some linear functional of the solution of equation (16.2.1)

$$J = \int h(\mathbf{r})f(\mathbf{r})\,d\mathbf{r}. \tag{16.2.3}$$

The problem can be solved in the following way: by the probabilities $f_1(\mathbf{r})$, $c(\mathbf{r})$, and $p(\mathbf{r} \to \mathbf{r}')$, the random trajectory $\{\mathbf{X}_1, \dots, \mathbf{X}_N\}$ is simulated, and the function

$$Z = \sum_{i=1}^{N} h(\mathbf{X}_i), \tag{16.2.4}$$

is evaluated, whose value is then averaged over ensemble of N independent trajectories:

$$J_n = \frac{1}{n}\sum_{j=1}^{n} Z^{(j)} = \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{N_j} h(\mathbf{X}_i^{(j)}). \tag{16.2.5}$$

If the interaction cross-section is Σ, the scattering cross-section is Σ_s and the scattering is isotropic; then

$$c(\mathbf{r}) = \Sigma_s/\Sigma,$$
$$p(\mathbf{r} \to \mathbf{r}') = \frac{\Sigma}{4\pi|\mathbf{r} - \mathbf{r}'|} e^{-\Sigma|\mathbf{r}-\mathbf{r}'|^2}.$$

Let us assume that it is necessary to find the particle flux at a certain point, which, for convenience, is chosen to be the origin. This flux can be divided into two parts: the non-scattered flux evaluation, which for a given source is easy to perform, and the scattered flux J, expressed through the collision density $f(\mathbf{r})$ by formulae (16.2.3) with estimator

$$h(\mathbf{r}) = \frac{c}{4\pi r^2} e^{-\Sigma r} \tag{16.2.6}$$

entering into unbiased estimator (16.2.4). It is easy to see that $\mathsf{E}J_n = J$ but

$$\mathsf{E}J_n^2 = \infty,$$

i.e., the variance is infinite. To see this, the second moment of only one of the terms of (16.2.4) should be calculated, because all terms are positive and accounting for the others cannot compensate its divergence. Let us consider the first term of sum (16.2.4), provided that $f_1(\mathbf{r})$ does not vanish at the point $\mathbf{r} = 0$. Then

$$\begin{aligned}\mathsf{E}h^2(\mathbf{X}_1) &= \frac{c^2}{(4\pi)^2} \int e^{-2\Sigma r} r^{-4} f_1(\mathbf{r})\, d\mathbf{r} \\ &= \frac{c^2}{(4\pi)^2} \int_0^\infty e^{-2\Sigma r} \phi_1(r) r^{-2} dr = \infty,\end{aligned} \tag{16.2.7}$$

where

$$\phi_1(r) = \int_{4\pi} f_1(r\mathbf{\Omega})\, d\mathbf{\Omega}.$$

We consider now the asymptotic behavior of the probability

$$\begin{aligned}\mathsf{P}\left\{h(\mathbf{X}_1) > z\right\} &= \int_{h(\mathbf{r})>z} f_1(\mathbf{r}) d\mathbf{r} \sim f_1(0) \int_{4\pi r^2 < c/z} d\mathbf{r} \\ &= \frac{c^{3/2}}{6\sqrt{\pi}} f_1(0) z^{-3/2}, \qquad z \to \infty.\end{aligned} \tag{16.2.8}$$

Thus, if the estimator of (16.2.4) contained only the first term, i.e., were the estimator of the single-scattered flux, then all terms in sum (16.2.5) would be independent and, by virtue of the generalized limit theorem, it could be stated that the random variable

$$Y_n = \frac{J_n - J}{\Delta_n}$$

with

$$\Delta_n = c[f_1(0)/3]^{2/3}(2n)^{-1/3} \tag{16.2.9}$$

for large n is distributed by the stable law with $\alpha = 3/2$ and $\beta = 1$:

$$Y_n \to Y_A(3/2, 1), \qquad n \to \infty. \tag{16.2.10}$$

The situation with estimator (16.2.4) including dependent random variables is a more complicated problem. This problem was stated in (Kalos, 1963) and solved in (Uchaikin & Lappa, 1976a; Uchaikin & Lappa, 1976b; Uchaikin & Lappa, 1978); the result is formulated as follows:

$$\mathsf{P}\left\{\sum_{i=1}^{N} h(\mathbf{X}_i) > z\right\} \sim \int_{h(\mathbf{r})>z} f(\mathbf{r})\,d\mathbf{r} \sim \frac{c^{3/2}}{6\sqrt{\pi}} f(0) z^{-3/2}, \qquad z \to \infty. \tag{16.2.11}$$

The only difference between relations (16.2.11) and (16.2.8) consists of replacement of the first collision density $f_1(\mathbf{r})$ with the density of all collisions $f(\mathbf{r})$. Formula (16.2.10) remains valid if the same change is made in Δ_n:

$$\Delta_n = c[f(0)/3]^{2/3}(2n)^{-1/3}. \tag{16.2.12}$$

The result given by (16.2.11) and (16.2.12) means that the probability to find two or more nodes of the trajectory in a close vicinity of the observation point is asymptotically small as compared with the single-node probability. In other words, the outliers of sum (16.2.4), which determine the limit distribution of sample mean (16.2.5), are generated by only one of the nodes of trajectory and thus the large terms in sum (16.2.5) are statistically independent. The probability for any node of the trajectory to appear in the volume element $d\mathbf{r}$ about $\mathbf{r}$ is equal to $f(\mathbf{r})\,d\mathbf{r}$, which explains the replacement indicated above.

16.3. Examples

The following questions arise from the foregoing results.

(1) How fast is the limiting stable distribution reached in typical problems of transport theory?

(2) How much is the difference between distributions of the random variable

$$Y_n = \frac{J_n - J}{\Delta_n}$$

and of the random variable

$$Y_n' = \frac{J_n - J}{\Delta_n'},$$

where the prime means that Δ_n' includes the approximate estimate $f'(0)$ instead of the exact value $f(0)$ unknown in a real situation?

Let $\Phi(0)$ be the flux at the origin of coordinates; then

$$f(0) = \Sigma\Phi(0) = \Sigma\Phi_0(0) + \Sigma J. \tag{16.3.1}$$

Here we partition the flux into the non-scattered component $\Phi_0(0)$ supposed to be known and the scattered one J whose finding is the goal of Monte-Carlo calculations. Changing in (16.3.1) the scattered component for its estimator J_n, we obtain

$$f'(0) = \Sigma\Phi_0(0) + \Sigma J_n.$$

This value enters into Δ'_n instead of $f(0)$:

$$\Delta'_n = c[f'(0)/3]^{2/3}(2n)^{-1/3}.$$

The answers to questions above depend on the problems under consideration and are hard to be formulated in a general form. Here we will be satisfied with considering some simple example. A non-absorption sphere of radius R is lighted on by plane radiation flux with the density s_0 (the number of particles falling on the sphere is equal to $\pi R^2 s_0$); it is necessary to find the radiation flux at the center of the sphere. This problem posed by G.A. Mikhailov for pedagogical purposes, has a curious solution (Ermakov & Mikhailov, 1976). It turns out that, independently of interaction cross-section Σ ($\Sigma = \Sigma_s$ and $c = 1$ due to the absence of absorption) and of the radius of the sphere, the flux at its center is the same as if the sphere is absent at all:

$$\Phi(0) = s_0.$$

This does not mean, however, that the sphere does not have any effect on the radiation. The particles falling on the sphere are scattered inside it, and the partition of the flux $\Phi(0)$ into the non-scattered $\Phi_0(0)$ and the scattered J components depends on ΣR. Assuming $\Sigma = 1$, we write these formulae:

$$\Phi_0(0) = s_0 e^{-R},$$
$$J = s_0[1 - e^{-R}].$$

In what follows we need the density of collisions and it is, of course, also equal to s_0:

$$f(0) = \Sigma\Phi(0) = s_0.$$

This problem was used in (Uchaikin & Lappa, 1976b; Uchaikin & Lappa, 1978) for experimental investigation of the questions posed.

For the sake of convenience, the quantity s_0 is set equal to $(\pi R^2)^{-1}$, which corresponds to the falling of one particle on the sphere. The distribution of random variable Y_n is estimated over series of 400 independent samplings of n trajectories each. The comparison with the limit distribution is performed with the χ^2 test used. Fig. 16.1 shows the values of χ^2 against the number of

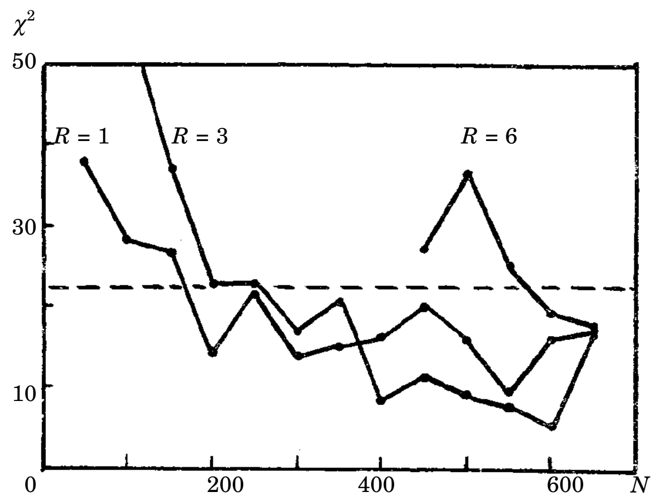


Figure 16.1. The χ^2 test shows that the results of Monte-Carlo calculations agree with (16.2.10) (- - - indicates the critical value $\chi^2_{0.95}$ = 22.4 for 13 degrees of freedom)

trajectories n for the spheres of three different radii (scattering is assumed to be isotropic). The number of degrees of freedom equals 13, and the corresponding value $\chi^2_{0.95} = 22.4$ is depicted by dashed line. One can see from the figure that for $n > 200$ ($R = 1$ and $R = 3$) and for $n > 600$ ($R = 6$) the goodness-of-fit test does not distinguish the distribution of estimator Y_n from the limiting one. In this case, the true value $f(0) = (4\pi R)^{-2}$ is used in the parameter of distribution width Δ_n.

The distribution of the estimator Y'_n is also obtained with the use of the same trajectories for calculating $f'(0)$. It turns out that the χ^2 test can distinguish the distributions of random variables Y'_n and $Y(3/2, 1)$ even for $n = 1000$. However, this difference is caused, mainly, by the different behavior of distribution tails, and has little effect on the width of distributions. The typical result is presented in Fig. 16.2 for $R = 3$, $n = 700$. The value χ^2 is equal to 21.8 for distribution of Y_n and is equal to 129 for distribution of Y'_n (the number of degrees of freedom is still 13), but the distribution functions itself within the interval, say, from -1 to 4, differ inessentially.

16.4. Estimation of a linear functional of a solution of integral equation

In the preceding sections, we considered the Markov model for random walks whose trajectory is a homogeneous terminating Markov chain $X_1, X_2, \dots, X_N$, where the first point X_1 is distributed with the entry density $f_1(x)$; the

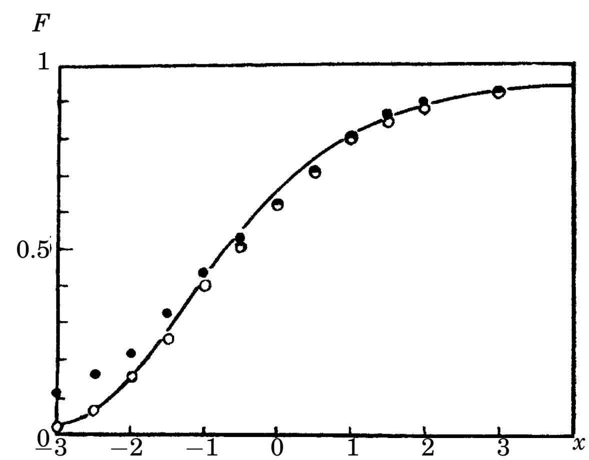


Figure 16.2. The distribution function of Y_n (open circles) and of Y'_n (filled circles) for $n = 700$, obtained by the Monte-Carlo method; the solid curve represents $G^A(x; 3/2, 1)$

transition probability density $p(x' \to x)$ and the termination probability $p_0(x) = 1 - c(x)$. The unbiased estimator for the functional

$$J = \int h(x)f(x)\,dx \tag{16.4.1}$$

of the solution of the non-homogeneous integral equation

$$f(x) = f_1(x) + \int dx' k(x' \to x) f(x') \tag{16.4.2}$$

with the kernel

$$k(x' \to x) = c(x')p(x' \to x)$$

is

$$Z = \sum_{i=1}^{N} h(X_i). \tag{16.4.3}$$

The algorithm can be, in principle, considered as a numerical (statistical) method for solving the integral equations. However, this equation is of a rather specific form constrained by the conditions

$$f_1(x) \ge 0, \qquad \int f_1(x)\,dx = 1,$$

$$k(x' \to x) \ge 0, \qquad \int k(x' \to x)\,dx = c(x') \le 1. \tag{16.4.4}$$

To get rid of these constraints, we consider an arbitrary function of the Markov chain instead of (16.4.3)

$$Z = h_N(X_1, \ldots, X_N). \tag{16.4.5}$$

The probability for the particle to undergo the first collision in the element dx_1, the second in the element dx_2, …, and to be absorbed in the last nth collision in the element dx_n is given by the product

$$f_1(x_1)\,dx_1\,k(x_1 \to x_2)\,dx_2 \ldots k(x_{n-1} \to x_n)\,dx_n\,p_0(x_n).$$

As a consequence, the mathematical expectation of r.v. (16.4.5) is

$$\begin{aligned}\bar{Z} \equiv \mathsf{E}Z = &\int dx_1 f_1(x_1) h_1(x_1) p_0(x_1) \\ &+ \sum_{n=2}^{\infty} \int dx_1 \ldots \int dx_n f_1(x_1) k(x_1 \to x_2) \ldots k(x_{n-1} \to x_n) p_0(x_n) h_n(x_1, \ldots, x_n).\end{aligned} \tag{16.4.6}$$

Each term of this sum contains the factor $f_1(x_1)$ under the sign of integral with respect to variable x_1, so the result can be represented as

$$\bar{Z} = \int dx_1 f_1(x_1) \bar{Z}_1(x_1) \tag{16.4.7}$$

where

$$\begin{aligned}\bar{Z}_1(x_1) = &\, p_0(x_1) h_1(x_1) \\ &+ \sum_{n=2}^{\infty} \int dx_2 \ldots \int dx_n k(x_1 \to x_2) \ldots k(x_{n-1} \to x_n) p_0(x_n) h_n(x_1, \ldots, x_n).\end{aligned}$$

This value is, evidently, estimator (16.4.5) averaged over the ensemble of trajectories with a fixed coordinate of the first collision. Factoring out $k(x_1 \to x_2)$ from the integral, we obtain

$$\bar{Z}_1(x_1) = p_0(x_1) h_1(x_1) + \int dx_2 k(x_1 \to x_2) \bar{Z}_2(x_1, x_2), \tag{16.4.8}$$

where

$$\begin{aligned}\bar{Z}_2(x_1, x_2) = &\, p_0(x_2) h_2(x_1, x_2) \\ &+ \sum_{n=3}^{\infty} \int dx_3 \ldots \int dx_n k(x_2 \to x_3) \ldots k(x_{n-1} \to x_n) p_0(x_n) h_n(x_1, \ldots, x_n)\end{aligned}$$

is the value of Z averaged over the ensemble of trajectories with fixed coordinates x_1 and x_2 of the two first collisions. Proceeding similarly, we arrive at the recurrence relation

$$\bar{Z}_n(x_1, \ldots, x_n) = h_n(x_1, \ldots, x_n) p_0(x_n) + \int dx_{n+1} k(x_n \to x_{n+1}) \bar{Z}_{n+1}(x_1, \ldots, x_n, x_{n+1})$$

which, together with formula (16.4.6) allows us to write the mathematical expectation $\bar{Z}$ in the form of an infinite series equivalent to (16.4.6).

Under some conditions, the derived recurrence relation for Z_n can be transformed into simultaneous equations in a limited number of functions. Let $h_n(x_1, \ldots, x_n)$ be of the form

$$h_n(x_1, \ldots, x_n) = \sum_{i=1}^{n} W_i(x_1, \ldots, x_i)\varphi_1(x_i), \tag{16.4.9}$$

where

$$W_i(x_1, \ldots, x_i) = v(x_1)w(x_1, x_2)\ldots w(x_{i-1}, x_i) = W_{i-1}(x_1, \ldots, x_{i-1})w(x_{i-1}, x_i) \tag{16.4.10}$$

for $i > 1$, and $W_1(x_1) = v(x_1)$. Functions (16.4.9) generalize the simple additive estimator (16.4.3) but, in contrast to the general case (16.4.5), have a rather specific structure permitting, by minimal means, to formulate Monte-Carlo algorithm to solve integral equations.

Representing expression (16.4.9) as

$$\begin{aligned} h_n(x_1, \ldots, x_n) &= v(x_1)\varphi_1(x_1) + v(x_1)[w(x_1, x_2)/v(x_2)] \sum_{i=2}^{n} W_{i-1}(x_2, \ldots, x_i)\varphi_1(x_i) \\ &= v(x_1)\left\{\varphi_1(x_1) + [w(x_1, x_2)/v(x_2)]h_{n-1}(x_2, \ldots, x_n)\right\}, \end{aligned}$$

fixing the two first arguments and changing $x_3 \to X_3, \ldots, x_n \to X_n$, we average the estimator over the remaining variables, and obtain

$$\bar{Z}_2(x_1, x_2) = v(x_1)\left\{\varphi_1(x_1) + [w(x_1, x_2)/v(x_2)]\bar{Z}_1(x_2)\right\} \tag{16.4.11}$$

System (16.4.8) and (16.4.11) completely determines both $\bar{Z}_1(x)$ and $\bar{Z}_2(x_1, x_2)$. Substituting (16.4.11) into (16.4.7) and introducing

$$\varphi(x) = \bar{Z}_1(x)/v(x), \tag{16.4.12}$$

we obtain

$$\varphi(x) = \varphi_1(x) + \int dx' k(x \to x')w(x_1, x')\varphi(x'). \tag{16.4.13}$$

In view of (16.4.7) and (16.4.12), the mathematical expectation of the estimator

$$Z = \sum_{i=1}^{N} W_i(X_1, \ldots, X_i)\varphi_1(X_i) \tag{16.4.14}$$

is

$$J = \int dx\, g(x)\varphi(x), \tag{16.4.15}$$

where

$$g(x) = f_1(x)v(x). \tag{16.4.16}$$

Thus, we arrive at the following conclusion. Functional (16.4.15) of the solution of the non-homogeneous integral equation

$$\varphi(x) = \varphi_1(x) + \int dx' K(x, x')\varphi(x') \tag{16.4.17}$$

is equal to the mathematical expectation of function (16.4.14) of the Markov chain trajectory with the entry density $f_1(x)$, the transition density $p(x \to x')$, and the termination probability $p_0(x) = 1 - c(x)$, provided that the weight factors $W_i(x_1, \dots, x_i)$ are

$$W_i(x_1, \dots, x_i) = v(x_1)w(x_1, x_2)\dots w(x_{i-1}, x_i)$$

with

$$v(x) = g(x)/f_1(x),$$
$$w(x, x') = K(x, x')/[c(x)p(x \to x')].$$

Thus, we obtain an algorithm to estimate functional (16.4.15) of the solution of integral equation (16.4.7) by Monte-Carlo method. A more detailed information can be found in (Ermakov, 1975; Ermakov & Mikhailov, 1976; Sobol, 1973; Spanier & Gelbard, 1968).

We give here only the following remarks. If the integral equation is such that

$$K(x, x') \geq 0,$$
$$\int K(x, x')dx' = c(x) < 1$$

for any x, the factor $g(x)$ is non-negative with the normalization

$$\int g(x)dx = 1,$$

then we can take the function $g(x)$ as the entry density $f_1(x)$, $1 - c(x)$ as the termination probability, and $[c(x)]^{-1}K(x, x')$ as the transition probability density. Then

$$W_i(x_1, \dots, x_i) = 1,$$

and we obtain the additive estimator

$$Z = \sum_{i=1}^{N} \varphi_1(X_i) \tag{16.4.18}$$

of the functional

$$J = \int g(x)\varphi(x)\,dx = \int f_1(x)\varphi(x)\,dx \tag{16.4.19}$$

of the integral equation solution

$$\varphi(x) = \varphi_1(x) + \int dx' k(x \to x')\varphi(x'), \tag{16.4.20}$$

with

$$k(x \to x') = K(x, x').$$

Although this case resembles that considered in Section 15.2, an attentive reader can reveal some difference: estimator (16.4.3) uses the function $h(x)$ determining functional (16.4.1), and the non-homogeneous term of (16.4.2) is used as the source density of trajectories, while estimator (16.4.18) uses the non-homogeneous term of equation (16.4.20), and the function $g(x) = f_1(x)$ determining the functional is used as the source. Moreover, the kernels of integral equations (16.4.2) and (16.4.20) are normalized by integration with respect to the second argument x, but in (16.4.20) it is an integration variable, while in the integral term of equation (16.4.2) x is a parameter.

These distinctions result from the duality principle mentioned in Section 10.3: integral equation (16.4.20) is adjoint to (16.4.2). Expanding their solutions into the Neumann series and substituting the expansions into functionals J, we can easily see that these two approaches are equivalent indeed. Let us demonstrate that the above-described approach to estimating a functional of an integral equation solution can yield an infinite variance and stable distribution as $n \to \infty$ even in the case where the estimating function $h(x)$ is limited. For this purpose we consider the degenerate case again, which allows us to obtain analytical results by simple means but has, of course, only pedagogical value (Uchaikin & Lagutin, 1993).

Let

$$\varphi_1(x) = 1,$$

and let non-negative kernel depend only on the difference of arguments

$$K(x, x') = K_0(x' - x)$$

and satisfy the condition

$$\int K_0(x)dx = q < 1.$$

In this case, $\varphi(x)$ does not depend on x, too, and the integral equation (16.4.17) takes the algebraic form

$$\varphi = 1 + q\varphi; \tag{16.4.21}$$

hence

$$\varphi = \frac{1}{1-q}.$$

Let, moreover, $g(x) \geq 0$ and

$$\int g(x)\,dx = 1.$$

Taking this function as the first collision density

$$f_1(x) = g(x)$$

we obtain

$$v(x) = 1,$$
$$J = \int dx f_1(x)\varphi(x) = \varphi = \frac{1}{1-q}. \tag{16.4.22}$$

The survival probability c on the trajectory is supposed to be constant and the transition probability density $p(x \to x')$ is accepted to be proportional to the equation kernel:

$$p(x \to x') = K(x, x')/q.$$

Under these assumptions, $w(x, x')$ takes the simple form

$$w(x, x') = q/c,$$

as well as the estimator itself:

$$Z = \sum_{i=1}^{N} (q/c)^{i-1} = \frac{1-(q/c)^N}{1-q/c}. \tag{16.4.23}$$

A clear physical sense can be assigned to the problem. A particle emitted by a source moves through an infinite homogeneous medium undergoing collisions with survival probability q. In this case, expression (16.4.22) together with equation (16.4.21) yields the mean number of collisions (including the final one) along such a trajectory. This process can be directly simulated using the probability q. The probability distribution of the number of collisions N is

$$\mathsf{P}\{N = n\} = (1-q)q^{n-1}, \tag{16.4.24}$$

and the mathematical expectation and variance of estimator are

$$\mathsf{E}N = \sum_{n=1}^{\infty} (1-q)q^{n-1}n = (1-q)^{-1},$$
$$\operatorname{Var} N = (1-q)^{-2}q \tag{16.4.25}$$

respectively.

Passage to (16.4.23) means that we are solving the same problem with the use of not the 'natural' but 'artificial' trajectories, where the survival probability c is different from q and, as a result, estimator (16.4.23) also differs from the simple estimator $Z = N$ in the case of natural trajectories or, as they say, in the case of imitation.

The distribution of N for artificial trajectories is obtained from (16.4.24) by replacement of q with c:

$$\mathsf{P}\left\{N = n\right\} = (1-c)c^{n-1};$$

thus,

$$\mathsf{E}Z^2 = \sum_{n=1}^{\infty} \frac{1-(q/c)^n}{1-q/c}(1-c)c^{n-1},$$

$$\mathsf{E}Z^2 = \sum_{n=1}^{\infty} \frac{[1-(q/c)^n]^2}{(1-q/c)^2}(1-c)c^{n-1}.$$

These sums can be easily computed. The former

$$\mathsf{E}Z = (1-q)^{-1}$$

means unbiasedness of the estimator: despite the use of artificial trajectories, we obtain the correct result corresponding to the natural trajectories. Evaluation of the latter sum yields

$$\operatorname{Var} Z = \frac{(1-c)q^2}{(1-q)^2(c-q^2)}. \tag{16.4.26}$$

Taking the ratio of variances

$$\frac{\operatorname{Var} Z}{\operatorname{Var} N} = \frac{(1-c)q}{c-q^2},$$

we see that it is smaller than one if $c > q$: the artificial elongation of trajectories gives a decrease of variance per a trajectory. In the domain $q^2 < c < q$, the variance of Z exceeds the natural variance, and for $c = q^2$ it becomes infinite.

Let us look at the domain $c < q^2$. Rewriting the estimator Z as

$$Z = \frac{(q/c)^N - 1}{q/c - 1},$$

we find the asymptotic expression for the probability of large deviations

$$\mathsf{P}\left\{Z > z\right\} = \mathsf{P}\left\{(q/c)^N - 1 > z(q/c-1)\right\} \sim \mathsf{P}\left\{N > n_z\right\}, \qquad z \to \infty,$$

where

$$n_z \approx \ln[z(q/c-1)]/\ln(q/c).$$

Since

$$\mathsf{P}\{N > n_z\} = \sum_{n=n_z}^{\infty}(1-c)c^{n-1} = c^{n_z-1},$$

we obtain

$$\mathsf{P}\{Z > z\} \sim bz^{-\alpha}, \qquad z \to \infty, \tag{16.4.27}$$

where

$$b = (q/c-1)^{-\alpha}/c,$$
$$\alpha = \frac{|\ln c|}{|\ln c| - |\ln q|}. \tag{16.4.28}$$

In the domain $0 < c < q^2$, α takes values from the interval $(1,2)$; therefore, (16.4.27) means that we deal with the domain of attraction of the stable law with $\alpha \in (1,2)$ given by (16.4.28) and $\beta = 1$.

Note that in this case the outlier of Z arises not from a single collision of the trajectory but is generated by all collisions of the trajectory.

16.5. Random matrices

In conclusion, we consider an example taken from the theory of random matrices (see (Girko, 1990)). Let a system of linear algebraic equations be

$$\Xi_n \mathbf{x}_n = \mathbf{W}, \qquad n = 1, 2, \dots, \tag{16.5.1}$$

where $\Xi = (X_{ij}^{(n)})$ is an $n \times n$ matrix with random elements $X_{ij}^{(n)}$, and $\mathbf{W}_n = (W_i^{(n)})$ is a random vector. As we know, there is a unique solution of (16.5.1) if $\det \Xi_n \neq 0$, and

$$\mathbf{x}_n = (x_j^{(n)}) = \Xi_n^{-1}\mathbf{W}_n.$$

In the case where $\det \Xi_n = 0$, a solution of (16.5.1) can fail to exist, and in this situation we agree to assign $\mathbf{x}_n = 0$. For large values of n, solving linear equations becomes a very laborious computational problem, and the following limit approximation can provide us with a certain amount of information.

We assume that for each n the random variables $X_{ij}^{(n)}$ and $W_j^{(n)}$, $i,j = 1,\dots,n$ are independent, $\mathsf{E}X_{ij}^{(n)} = \mathsf{E}W_j^{(n)} = 0$, $\operatorname{Var} X_{ij}^{(n)} = \operatorname{Var} W_j^{(n)} = 1$, and $\sup_{n,i,j} \mathsf{E}(|X_{ij}^{(n)}|^5 + |W_j^{(n)}|^5)$ are finite. Then the following limit relations hold true for any $1 \le i,j \le n$, $i \neq j$:

$$\lim \mathsf{P}\left\{x_i^{(n)} < \xi\right\} = \lim_{n\to\infty} \mathsf{P}\left\{x_i^{(n)}/x_j^{(n)} < \xi\right\}$$
$$= \frac{1}{2} + \frac{1}{\pi}\arctan\xi = G_A(\xi, 1, 0). \tag{16.5.2}$$

It turns out that a similar limit distribution arises if we consider the joint distributions of any finite number of components of the solution x_n. Namely, under the same assumptions

$$\lim_{n\to\infty} \mathsf{P}\left\{x_{i_1}^{(n)} < \xi_1, \dots, x_{i_k}^{(n)} < \xi_k\right\} = \pi^{-(k+1)/2}\Gamma\left(\frac{k+1}{2}\right)\int_{u_j<\xi_j}(1+|\mathbf{u}|^2)^{-(k+1)/2}d\mathbf{u}$$

for any fixed k, i.e., the limiting law turns out to be a k-dimensional Cauchy distribution.

It is worth mentioning that the Cauchy distribution (both one-dimensional and multidimensional) occurs in various problems more frequently than other stable laws, and compares favorably with the normal law only. The special position of the Cauchy law, like that of the normal law, can be observed also in the analytic setting.

To these laws we could also add the Laplace distribution, assigning $\alpha = 0$ to it by convention (there are well-known grounds for this; see, for example, relation (3.6.1)). The symmetric laws with integer α thereby distinguish themselves from other stable laws by their significance.

The Lévy law with $\alpha = 1/2$, $\beta = 1$, and $\gamma = 0$ can also occur fairly often in actual problems. Both the Cauchy law and the Lévy law are closely connected with the normal law. This becomes obvious from the fact that the ratio N_1/N_2 of independent random variables distributed by the standard normal law has a Cauchy distribution, while the random variable N_1^{-2} has a Lévy distribution.

16.6. Random symmetric polynomials

We conclude this chapter with the results of (Zolotarev & Szeidl, 1992) devoted to the investigation of limit distributions of random symmetric polynomials.

Let $P_n(\mathbf{x})$, $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, $n \ge 1$, be a sequence of symmetric polynomials of degree $k_n \ge k \ge 2$. Let $X_1, X_2, \dots$ be any sequence of independent identically distributed random variables with a common distribution function F. We assume that F possesses the following asymptotic property:

$$1 - F(\xi) = (a + o(1))\xi^{-\alpha}L(\xi), \qquad F(-\xi) = (b + o(1))\xi^{-\alpha}L(\xi) \quad (16.6.1)$$

as $\xi \to \infty$, where $\alpha > 0$, $a, b \ge 0$, $\max(a, b) > 0$, and $L(\xi)$ is a non-negative slowly varying function at infinity. We form the sequence of random polynomials $T_n = P(\mathbf{X})$, $\mathbf{X} = (X_1, \dots, X_n)$, $n \ge 1$, which we center and rescale by a sequence of real-valued constants A_n and $B_n \ne 0$ as follows:

$$\breve{T}_n = (T_n - A_n)/B_n, \qquad n \ge 1.$$

Assume that for some choice of the constants, $\breve{T}_n$ converges in distribution to some non-degenerate random variable T, that is,

$$\breve{T}_n \xrightarrow{d} T, \qquad n \to \infty. \quad (16.6.2)$$

In (Zolotarev & Szeidl, 1992), an analytic representation of the density $p_T(y)$ is obtained. It arises from a special class of symmetric polynomials P_n and the corresponding family of distributions F.

Let us add the following assumption on P_n and X_j to those made above:

(P) P_n are homogeneous polynomials;

(F) the distribution function F belongs to the domain of attraction of a symmetric stable law with parameter $0 < \alpha < 2$, i.e., we have $a = b$ in (16.6.1); without loss of generality, we may set $a = b = 1/2$.

The consideration is based on the so-called canonical representation of symmetric polynomials, which, in the case of a homogeneous symmetric polynomial P_n of degree $k_n = s$, is of the form

$$P_n(\mathbf{x}) = Q_n(\mathbf{y}), \qquad \mathbf{y} = (y_1, \dots, y_s), \tag{16.6.3}$$

where

$$y_\nu = x_1^\nu + \dots + x_n^\nu$$

and

$$Q_n(y) = \sum_{i_1+2i_2+\dots+si_s=s} a_n(i_1, \dots, i_s) y_1^{i_1} \dots y_s^{i_s}. \tag{16.6.4}$$

It is clear that the domain of admissible values of the vector $\mathbf{y}$ in (16.6.3) coincides with the set

$$\mathbb{R}_0^k = \{\mathbf{y} = (y_1, \dots, y_s) \colon y_{2j} \in \mathbb{R}^+ = [0, \infty),\ y_{2j+1} \in \mathbb{R}^1\}.$$

We assume that there exists a non-zero function $Q(\mathbf{y})$ defined on $\mathbb{R}_0^k$, $k \ge 2$, such that for any $\mathbf{y} \in \mathbb{R}_0^k$

(Q) $Q_n(\mathbf{y}) \to Q(\mathbf{y})$ as $n \to \infty$

The limit function Q must be, of course, a polynomial of degree no less than k and with the same structure (16.6.4). We assume that it is of degree k.

In the general case, the polynomials Q_n and their limit Q are neither homogeneous nor symmetric, but they possess a specific 'Λ-homogeneity' property as follows:

$$\lambda Q_n(\mathbf{y}) = Q_n(\Lambda_n \mathbf{y}), \qquad \lambda Q(\mathbf{z}) = Q(\Lambda \mathbf{z}) \tag{16.6.5}$$

for any $\lambda > 0$ and $\mathbf{y} \in \mathbb{R}_0^{k_n}$, $n \ge 1$, $\mathbf{z} \in \mathbb{R}_0^k$, where $\Lambda_n = (\lambda^{\nu/k_n} \delta_{\nu\mu})$ and $\Lambda = (\lambda^{\nu/k} \delta_{\nu\mu})$ are diagonal matrices of order k_n and k respectively.

In fact, the polynomial Q may depend on a subset of the variables y_ν. Denote by $\mathcal{N}$ the set of indices of y_ν, $1 \le \nu \le k$, which Q depends on,

$$\mathbf{y}' = \{y_\nu \colon \nu \in \mathcal{N}\} \in \mathbb{R}', \qquad \mathbf{y}'' = \{y_\nu \colon \nu \notin \mathcal{N}\} \in \mathbb{R}'',$$

and let k' denote the dimension of $\mathbb{R}'$.

By using this notation, we represent Q as

$$Q(\mathbf{y}) = Q^*(\mathbf{y}') + (\mathbf{0}'', \mathbf{y}''), \qquad \mathbf{y} = \mathbf{y}' \times \mathbf{y}'' \in \mathbb{R}_0^k,$$

where $\mathbf{0}''$ is zero vector from $\mathbb{R}''$, and $(\cdot, \cdot)$ stands for the usual scalar product.

We assume that two additional properties of the polynomial Q hold:

(Q1) the set $\mathscr{N}$ contains odd numbers only;

(Q2) $B^* = \int_{\mathbb{R}'} \exp(-Q^*(\mathbf{y}'))\, d\mathbf{y}' < \infty.$

As corollary of these assumptions, we obtain the following properties of Q:

(Q3) the degree of the polynomial Q is an even number, i.e., $k = 2m$, $m \geq 1$; consequently, the dimension k' of the space $\mathbb{R}'$ equals to m;

(Q4) as the polynomial $Q(\mathbf{y})$ may depend on components y_{2j+1} only, we can consider in $Q(\mathbf{y})$ vectors $\mathbf{y} \in \mathbb{R}^k$ instead of vectors $\mathbf{y} \in \mathbb{R}_0^k$; moreover, in this case the space $\mathbb{R}'$ is the space $\mathbb{R}^m$.

Properties (16.6.5) and (Q3) imply that for any $\lambda > 0$ the function $\exp(-\lambda Q^*(\mathbf{y}'))$ is integrable over $\mathbb{R}'$ as well. Another corollary of (Q2) states that, if $W = W(\mathbf{y}')$ is a polynomial, then the function $W \exp(-\lambda Q^*)$ is absolutely integrable over $\mathbb{R}'$.

In view of these corollaries we conclude that for any $\lambda > 0$ the function $\exp(-\lambda Q(\mathbf{y})) = \exp(-Q(\Lambda \mathbf{y}))$ can be represented in the form of the Fourier transform

$$\exp(-\lambda Q(\mathbf{y})) = \int \exp(i(\Lambda \mathbf{y}, \mathbf{x})) G(\mathbf{x})\, d\mathbf{x}, \qquad \mathbf{y} \in \mathbb{R}^k, \tag{16.6.6}$$

where $G(\mathbf{x}) = G^*(\mathbf{x}')\delta(\mathbf{x}'')$, δ is the Dirac function, and $G^*(\mathbf{x}')$ is a bounded and absolutely integrable function on $\mathbb{R}^m$

In view of (16.6.3), the random polynomials T_n can be represented in the form

$$T_n = P_n(\mathbf{X}) = Q_n(\mathbf{S}_n), \qquad n \geq 1,$$

where

$$\mathbf{S}_n = (S_{n\nu} \colon 1 \leq \nu \leq k),$$
$$S_{n\nu} = X_1^\nu + \ldots + X_n^\nu.$$

It is not difficult to see that the asymptotic property (16.6.1) carries over to the distribution F_ν of X_1^ν, for each $2 \leq \nu \leq k$, but with different parameters, and different L. Namely, taking (F) into account, we obtain, as $\xi \to \infty$,

$$1 - F_\nu(\xi) \sim F_\nu(-\xi) \sim \xi^{-\alpha/\nu} L(\xi^{1/\nu})$$

for odd ν, and

$$1 - F_\nu(\xi) \sim 2\xi^{-\alpha/\nu}L(\xi^{1/\nu}), \qquad F_\nu(-\xi) = 0$$

for even ν.

Thus, F_ν belongs to the domain of attraction of a stable law with parameter $\alpha_\nu = \alpha/\nu < 1$, $2 \le \nu \le k$.

Therefore, under the additional assumption (F) on F made above, we can assert the existence of positive constants $d_{n\nu}$, $n \ge 1$, such that

$$\check{S}_{n\nu} = d_{n\nu}^{-1}S_{n\nu} \xrightarrow{d} Y_\nu \tag{16.6.7}$$

as $n \to \infty$ for each $1 \le \nu \le k$, where Y_ν has a stable distribution with parameter α/ν. In fact, methods of the classical theory of limit theorems (see, e.g. (Gnedenko & Kolmogorov, 1954)) enable us to choose the constants to be of the form $d_{n\nu} = (d_n)^\nu$, where $d_n = (nL(n^{1/\alpha}))^{1/\alpha}$.

We denote by D_n, $n \ge 1$, the sequence of constant diagonal matrices $(d_n^\nu \delta\nu\mu)$ of order k defined by the above constants d_n. It is clear that random vectors $\mathbf{S}_n$, $n \ge 1$, are in fact the cumulative sums of independent identically distributed random vectors:

$$\mathbf{S}_n = \sum_{1 \le j \le n} \mathbf{v}_j, \qquad \mathbf{v}_j = (X_j^\nu : 1 \le \nu \le k).$$

Therefore, we can consider the random vectors $\check{\mathbf{S}}_n = D_n^{-1}\mathbf{S}_n$, $n \ge 1$, as a sequence of matrix normalized sums of independent identically distributed random vectors; hence. as $n \to \infty$,

$$\check{\mathbf{S}}_n \xrightarrow{d} \mathbf{Y} = (Y_\nu : 1 \le \nu \le k). \tag{16.6.8}$$

Consequently, the limit random vector $\mathbf{Y}$ has an operator stable distribution in $\mathbb{R}^k$. The characteristic functions corresponding to the operator stable distributions have usually a very complicated analytic form, but for the case we consider the characterisctic function $f_\mathbf{Y}$ of the limit vector $\mathbf{Y}$ can be expressed by essentially simpler formulae (Szeidl, 1986)

$$f_\mathbf{Y}(\mathbf{t}) = \exp\left\{\frac{1}{2}\int (e^{iq} - 1)|\xi|^{-1-\alpha} d\xi\right\}$$

where

$$\mathbf{t} = (t_1, \dots, t_k) \in \mathbb{R}^k, \qquad q = t_1\xi + t_2\xi^2 + \dots + t_k\xi^k.$$

Denote by $\mathbf{t}^*$ a vector from $\mathbb{R}^k$ with components $t_{2j} = 0$, $2j \le k$. Obviously, $\Im \ln f_\mathbf{Y}(\mathbf{t}^*) = 0$; hence $u_\mathbf{Y}(\mathbf{t}) = -\Re \ln f_\mathbf{Y}(\mathbf{t}) = -\ln f_\mathbf{Y}(\mathbf{t})$ for any $\mathbf{t} = \mathbf{t}^*$.

The function $u_\mathbf{Y}(\mathbf{t})$ is strictly positive for any $\mathbf{t} \ne 0$. Moreover, for any $\lambda > 0$ and $\mathbf{t} \in \mathbb{R}^k$ the following equalities are true:

$$u_{\Lambda\mathbf{Y}}(\mathbf{t}) = u_\mathbf{Y}(\Lambda t) = \lambda^\gamma u_\mathbf{Y}(\mathbf{t}), \tag{16.6.9}$$

where Λ is the matrix defined in (16.6.5) and $\gamma = \alpha/k < 1$ because $\alpha < 2$, $k \geq 2$.

We will need also the following lower bound for the function $u_{\mathbf{Y}}$: there exists a constant $\rho > 0$ such that for any $\mathbf{t} \in \mathbb{R}^k$

$$u_Y(\mathbf{t}) \geq \rho \min\left(|\mathbf{t}|^{\alpha/(k+1)}, |\mathbf{t}|^{\alpha/(2k)}\right). \tag{16.6.10}$$

Equalities (16.6.9) are elementary. Equality (16.6.10) has a more complicated proof, but it can be obtained as a corollary of Lemma 3 from (Szeidl, 1990). We consider the random variables

$$\breve{T}_n = d_n^{-k} T_n = d_n^{-k} Q_n(\mathbf{S}_n), \qquad n \geq 1.$$

In view of (16.6.5), (16.6.8), and assumption (Q), we can assert that, as $n \to \infty$,

$$\breve{T}_n = Q_n(D_n^{-1}\mathbf{S}_n) = Q_n(\breve{\mathbf{S}}_n) \stackrel{d}{\to} T = Q(\mathbf{Y}).$$

Thus, condition (16.6.2) is satisfied, and we may consider the problem of analytic representation of the distribution of the random variable T.

We consider the two-sided Laplace transformation

$$\varphi_T(\lambda) = \mathsf{E}\exp(-\lambda T) = \mathsf{E}\exp(-\lambda Q(\mathbf{Y})), \qquad \lambda > 0.$$

By using (16.6.5), (16.6.6), and (16.6.9), we can transform $\varphi_T(\lambda)$ as follows:

$$\begin{aligned}\varphi_T(\lambda) &= \mathsf{E}\exp(-Q(\Lambda\mathbf{Y})) = \mathsf{E}\int \exp(i(\Lambda Y, \mathbf{x}))G(\mathbf{x})\,d\mathbf{x} \\ &= \int f_{\Lambda\mathbf{Y}}(\mathbf{x})G(\mathbf{x})\,d\mathbf{x} = \int_{\mathbb{R}^{\prime}} \exp(-\lambda^\gamma u_{\mathbf{Y}}(\mathbf{x}^*))G^*(\mathbf{x}')\,d\mathbf{x}'.\end{aligned}$$

The function

$$L(\lambda, \mathbf{x}') = \exp(-\lambda^\gamma u_{\mathbf{Y}}(\mathbf{x}^*)), \quad u_{\mathbf{Y}}(\mathbf{x}^*) = u^*_{\mathbf{Y}}(\mathbf{x}'), \gamma = \alpha/k > 1,$$

considered as a function of $\lambda > 0$, is the one-sided Laplace transform of the density $q(\xi; \gamma, 1, u_{\mathbf{Y}})$ of a one-sided stable law which vanishes for $\xi < 0$ (see Section 4.2). For $\xi > 0$ it can be expressed as an absolutely convergent series

$$q(\xi; \gamma, 1, u_{\mathbf{Y}}) = \frac{1}{\pi}\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\Gamma(r\gamma+1)}{\Gamma(r+1)}\sin(\pi\gamma r)u^r_{\mathbf{Y}}\xi^{-\gamma r-1}. \tag{16.6.11}$$

Thus,

$$\varphi_T(\lambda)\iint_{\mathbb{R}^k\times\mathbb{R}^+} q(\xi; \gamma, 1, u_{\mathbf{Y}})G(\mathbf{x})\,d\mathbf{x} = \iint_{\mathbb{R}^{\prime}\times\mathbb{R}^+} e^{-\lambda\xi}q(\xi; \gamma, 1, u^*_{\mathbf{Y}})G^*(\mathbf{x}')\,d\xi\,d\mathbf{x}'. \tag{16.6.12}$$

LEMMA 16.6.1. *The integrand in double integral* (16.6.12) *is an absolutely integrable function on* $\mathbb{R}' \times \mathbb{R}+$.

PROOF. As a corollary to the obvious properties

$$q(\xi;\gamma,1,c) = c^{-1/\gamma} q(c^{-1/\gamma}\xi;\gamma,1,1), \qquad c>0,$$
$$q(\xi;\gamma,1,1) \le h = \frac{1}{\pi}\Gamma\left(1+\frac{1}{\gamma}\right)\left(\cos\frac{\pi\gamma}{2}\right)^{-1/\gamma}$$

we obtain

$$q(\xi;\gamma,1,u^*_{\mathbf{Y}}) \le h(u^*_{\mathbf{Y}})^{-1/\gamma}. \tag{16.6.13}$$

In view of (16.6.10), we obtain the upper bound

$$q(\xi;\gamma,1,u^*_{\mathbf{Y}}) \le cw(\mathbf{x}'), \quad \mathbf{x}' \in \mathbb{R}', \quad c = h\rho^{-1/\gamma},$$

where

$$w(\mathbf{x}') = \max\left(|\mathbf{x}'|^{-k/(k+1)}, |\mathbf{x}'|^{-1/2}\right).$$

Therefore, for integral (16.6.12) the following upper bound holds true:

$$c\iint_{\mathbb{R}'\times\mathbb{R}^+} e^{-\lambda\xi} w(\mathbf{x}')|G^*(\mathbf{x}')|\,d\xi\,d\mathbf{x}' = \frac{c}{\lambda}\int_{\mathbb{R}'} w|G^*|\,d\mathbf{x}'.$$

We divide the last integral into two parts:

$$\int_{\mathbb{R}'} = \int_{|\mathbf{x}'|\le 1} + \int_{|\mathbf{x}'|>1} = I_1 + I_2.$$

Since the function $|G^*|$ is bounded by some constant M (see (Q2) and (16.6.6)), we conclude that

$$I_1 \le M\int_{|\mathbf{x}'|\le 1} |\mathbf{x}'|^{-k/(k-1)} d\mathbf{x}' = c'M.$$

On the other hand, as G^* is absolutely integrable in $\mathbb{R}'$, we see that

$$I_2 = \int_{|\mathbf{x}'|>1} |\mathbf{x}'|^{-1/2}|G^*|\,d\mathbf{x}' \le \int |G^*(\mathbf{x}')|\,d\mathbf{x}' < \infty.$$

These bounds imply the validity of the lemma.

Let us formulate the main result of this section.

THEOREM 16.6.1. *Under the above constraints imposed on the polynomials P_n, their limit Q and the distribution function F, we assert that the limit $F_T(\xi)$*

possesses a bounded and continuous density $p_T(\xi)$ which is equal to zero for $\xi < 0$ and has the analytic representation

$$p_T(\xi) = \int_{\mathbb{R}'} q(\xi;\gamma,1,u^*_{\mathbf{Y}})G(\mathbf{x}')\,d\mathbf{x}', \qquad \xi \in \mathbb{R}^1, \tag{16.6.14}$$

where

$$\mathbb{R}' = \{\mathbf{x}' = (x_1, x_3, \dots, x_{2m-1}),\ x_{2j-1} \in \mathbb{R}^1\},$$

the function $G^*(\mathbf{x}')$ is defined by (16.6.6)*, and q is the density of a one-sided stable law with parameters $\gamma = \alpha/k < 1$,*

$$u^*_{\mathbf{Y}} = \int_0^\infty (1 - \cos(x_1\xi + x_3\xi^3 + \dots + x_{2m-1}\xi^{2m-1}))\xi^{-1-\alpha}d\xi.$$

PROOF. Let us turn back to representation (16.6.12). By virtue of Lemma 16.6.1, the integrand in that integral is an absolutely integrable function. Hence we can change the order of integration, that is,

$$\varphi_T(\lambda) = \int_0^\infty e^{-\lambda\xi} \int_{\mathbb{R}'} q(\xi;\gamma,1,u^*_{\mathbf{Y}})G^*(\mathbf{x}')\,d\mathbf{x}'\,d\xi.$$

The inner integral exists for any $\xi \in \mathbb{R}^1$ and vanishes for $\xi \le 0$, because the function ξ possesses the same property. Uniqueness of the Laplace integral representation implies that the density $p_T(\xi)$ exists and can be represented in form (16.6.14).

Boundedness of $p_T(\xi)$ in fact was established in the proof of Lemma 16.6.1. The uniform convergence of the integral in (16.6.14) is implied by the inequality (see (16.6.13) and thereafter)

$$\int_{|\mathbf{x}'|\ge N} q(\xi;\gamma,1,u^*_{\mathbf{Y}})G^*(\mathbf{x}')\,d\mathbf{x}' \le c\int_{|\mathbf{x}'|\ge N} G^*(\mathbf{x}')\,d\mathbf{x}', \qquad N \ge 1.$$

Because q is a continuous function of ξ, the uniform convergence of the integral in (16.6.14) implies the same property for p_T.

REMARK 16.6.1. We have proved that $P_T(\xi) = 0$ for $\xi \le 0$. This implies, obviously, almost-sure non-negativity of the random variable $T = Q(\mathbf{Y})$. As a corollary to this, we obtain that assumptions (Q1) and (Q2) imply non-negativity of the polynomial $Q(\mathbf{Y})$, $Y \in \mathbb{R}^k$. Moreover, non-negativity of Q implies that its degree k should be even.

REMARK 16.6.2. In representation (16.6.14), the integrand qG^* is the product of two factors which play quite different roles. Indeed, q is defined by the distribution function F and by the value of the degree k. Conversely, the factor G^* is completely defined by the polynomial Q alone. The phenomenon of such a separation of roles of the factors in the representation of p_T was observed in (Szeidl, 1990) for the representation of the characteristic function f_T in essentially more general cases.

REMARK 16.6.3. By substituting formally representation (16.6.11) of the density q into (16.6.14), we can write for $\xi > 0$

$$p_T(\xi) = \frac{1}{\pi}\sum_{r\ge 1}(-1)^{r-1}\frac{\Gamma(r\gamma+1)}{\Gamma(r+1)}\sin(\pi\gamma r)H_r\xi^{-r\gamma-1}, \qquad (16.6.15)$$

where

$$H_r = \int (u^*_{\mathbf{Y}})^r G^*(\mathbf{x}')\,d\mathbf{x}', \qquad r = 1, 2, \dots$$

For the case where conditions (Q1), (Q2) are satisfied, the function $G^*(\mathbf{x}')$ tends to zero at an exponential rate as $|x'| \to \infty$. Because the function $u^*_{\mathbf{Y}}(\mathbf{x}')$ increases at a polynomial rate, the integrals H_r should be finite. In this case we can consider (16.6.15) as a representation of p_T by a series which may be convergent or asymptotic.

To illustrate Theorem 16.6.1, we consider an example. In view of (Q1), (Q2), and non-negativity of the polynomial Q^* following from them, we can write the general form of the polynomials $Q^*(\mathbf{y}')$, $\mathbf{y}' \in \mathbb{R}'$, for $k = 2, 4, 6$:

$$k = 2\ (m = 1,\ \mathbf{y}' = (y_1)): \quad Q^*(\mathbf{y}') = a_1(y_1), \quad a_1 > 0;$$
$$k = 4\ (m = 2,\ \mathbf{y}' = (y_1, y_3)): \quad Q^*(\mathbf{y}') = a_1(y_1)^4, \quad a_1 > 0;$$
$$k = 6\ (m = 3,\ \mathbf{y}' = (y_1, y_3, y_5)): \quad Q^*(\mathbf{y}') = a_1(y_1)^6 + a_2(y_3)^2,\ a_1 > 0,\ a_2 > 0.$$

The cases $k = 2$ and $k = 4$ are not very interesting, because we can calculate the distribution function F_T for these cases by elementary methods.

Consider the case $k = 6$, $a_1 = a_2 = 1$, and $0 < \alpha < 2$. In this case,

$$T = Y_1^6 + Y_3^2,$$

and

$$u^*_{\mathbf{Y}}(x_1, x_3) = \int_0^\infty (1 - \cos(x_1\xi + x_3\xi^3))\xi^{-1-\alpha}d\xi.$$

The function $G^*(x_1, x_3)$ corresponding to the polynomial

$$Q^*(y_1, y_3) = y_1^6 + y_3^2$$

can be expressed by using the density $q(\xi; 2, 0)$ of the normal law with mean 0 and variance 2, and the transstable function $q(\xi; 6, 0)$ (Section 6.10) as follows:

$$G^*(x_1, x_3) = q(x_1; 6, 0)q(x_3; 2, 0).$$

The function $q(z; 6.0)$ is an analytic entire function with the series expression

$$q(z; 6, 0) = \frac{1}{6\pi}\sum_{n\ge 0}(-1)^n\frac{\Gamma((2n+1)/6)}{\Gamma(2n+1)}z^{2n}.$$

Thus,

$$p_T(\xi) = \iint_{\mathbb{R}^2} q(\xi; \alpha/6, 1, u^*_{\mathbf{Y}}) q(x_1; 6, 0) q(x_3; 2, 0)\, dx_1\, dx_3,$$

where

$$q(\xi; \alpha/6, 1, u^*_{\mathbf{Y}}) = (u^*_{\mathbf{Y}})^{-6/\alpha} q((u^*_{\mathbf{Y}})^{-6/\alpha}\xi; \alpha/6, 1).$$

17

Financial applications

17.1. Introduction

Explanation of Brownian motion in terms of random thermal motions of fluid molecules striking the microscopic particle and causing it to undergo a random walk was made by Einstein and many famous physicists and mathematicians began systematic investigations in this field. A less known fact noted in (Klafter *et al.*, 1996) is that Bachelier, a student of Poincaré, developed a theory of Brownian motion in his 1900 thesis (Bachelier, 1900). Because Bachelier's work was in the context of price and stock market fluctuations, it did not attract the attention of physicists. Having derived a diffusion equation for random processes, he said that probability could diffuse in the same manner as heat.

A more specific assertion by Bachelier is that any competitive price follows, in the first approximation, a one-dimensional Brownian motion. Mandelbrot (Mandelbrot, 1983) brought an anti-Brown arguments based on experimental observations and following reasons: we know that Brownian motion's sample functions are almost surely, almost everywhere continuous. But prices on competitive markets need not be continuous, and they are conspicuously discontinuous. The only reason for assuming continuity is that many sciences tend, knowingly or not, to copy the procedures that prove successful in Newtonian physics. Continuity should prove a reasonable assumption for diverse quantities and rates that enter into economics but are defined in purely physical terms. But prices are different: mechanics involves nothing comparable, and gives no guidance on this account.

The typical mechanism of price formation involves both knowledge of the present and anticipation of the future. Even when the exogenous physical determinants of a price vary continuously, anticipations change drastically, 'in a flash'.

Combining his favorite scaling principle with the walking model having large increments with infinite variance, Mandelbrot (Mandelbrot, 1963c) arrived at the Lévy process (see Chapter 12) for the price changes. This approach developed also in (Fama, 1963; Roll, 1970), etc. In (Stanley *et al.*, 1996), where, among others, the behavior of industrial firm sales and their employment was studied, it was emphasized that the diagram of the process can be mapped exactly onto the diagram of the DNA walk model considered above.

The following presentation draws heavily on (McCulloch, 1996).

Financial asset returns are the cumulative outcome of a vast number of bits of information arriving continuously in time. According to the central limit theorem, if the sum of a large number of independent identically distributed random variables has a limiting distribution after appropriate shifting and scaling, the limiting distribution must belong to the stable class (Lévy, 1925; Feller, 1966). It is therefore natural to assume that asset returns are at least approximately governed by a stable distribution if the accumulation is additive, or by a log-stable distribution if the accumulation is believed to be multiplicative.

The normal or Gaussian distribution is the most familiar and computationally tractable stable distribution, and therefore either it or the log-normal has routinely been postulated to govern the actual distribution. However, asset returns often appear to be much more leptokurtic or thick-tailed than is consistent with a Gaussian distribution. This naturally leads one to consider also the non-Gaussian stable distributions, as proposed by Mandelbrot (Mandelbrot, 1963b; Mandelbrot, 1963c) and Fama (Fama, 1963).

If asset returns are truly governed by the infinite-variance stable distributions, life is fundamentally riskier than in a purely Gaussian world. Sudden price movements like the stock market crash turn into real-world possibilities, and the risk immunization promised by 'programmed trading' becomes mere wishful thinking, at best. These price discontinuities render the arbitrage argument of the celebrated Black–Scholes (1973) option pricing model inapplicable, so that we must look elsewhere in order to value options.

Nevertheless, we will see that the capital asset pricing model works as well in the infinite-variance stable cases as it does in the normal case. Furthermore, the Black–Scholes formula may be extended to the non-Gaussian stable cases by means of a utility maximization argument. Two serious empirical objections that have been raised against the stable hypothesis are shown to be inconclusive.

17.2. More on stable processes

Because stable distributions are infinitely divisible, they are particularly attractive for continuous time modeling, as emphasized by Samuelson (Samuelson, 1965); see also (McCulloch, 1978a). The stable generalization of the familiar Brownian motion or Wiener process considered in Section 10.4 is often called an α-stable Lévy motion, and is the subject of two recent monographs

(Samorodnitsky & Taqqu, 1994; Janicki & Weron, 1994).

Recall that it is a continuous time stochastic process $X(t) \equiv X(t; \alpha, \beta)$ whose increments $X(t + \Delta t) - X(t)$ are distributed by the stable law with parameters α, β, zero drift and scale $\lambda = \Delta t^{1/\alpha}$, and whose non-overlapping increments are independently distributed. Such a process may be thought of as having infinitesimal increments $dX(t) = X(t + dt) - X(t)$, with infinitesimal scale $dt^{1/\alpha}$. The process itself may then be reconstructed as the integral of these increments:

$$X(t) = X(0) + \int_0^t dX(\tau).$$

Such a process may easily be generalized to have scale c_0 over unit time increments and drift δ per unit time, as $Y(t) = c_0 X(t) + \delta t$. The scale of dY is then $c_{dt} = c_0 dt^{1/\alpha}$.

Unlike a Brownian motion, which is almost surely everywhere continuous, an α-stable Lévy motion is almost surely dense with discontinuities. It is not hard to see that the probability that $dY > x$ is (in form A)

$$\mathsf{P}\{dY > x\} \sim k_{\alpha\beta}(x/c_{dt})^{-\alpha} = k_{\alpha\beta} c_0^{\alpha} x^{-\alpha} dt, \qquad x \to \infty, \tag{17.2.1}$$

where

$$k_{\alpha\beta} = \pi^{-1}(1 + \beta)\Gamma(\alpha)\sin(\alpha\pi)/2. \tag{17.2.2}$$

Equation (17.2.1), in turn, implies that values of dY greater than any threshold $x_0 > 0$ occur with frequency

$$\lambda = k_{\alpha\beta}(c_0/x_0)^{\alpha}, \tag{17.2.3}$$

and that conditional on their occurrence, they have a Pareto distribution:

$$\mathsf{P}\{dY < x \mid dY > x_0\} \sim 1 - (x_0/x)^{\alpha}, \qquad x \gg x_0. \tag{17.2.4}$$

Likewise, negative discontinuities $dY < -x_0$ also have a conditional Pareto distribution, and occur with frequency determined by (17.2.3), but with $k_{\alpha\beta}$ replaced by $k_{\alpha,-\beta}$.

In the Brownian motion case $\alpha = 2$, $k_{\alpha\beta} = 0$, so that discontinuities almost surely never occur. With α just under 2, large discontinuities are very rare, but the frequency of discontinuities greater than x_0 in absolute value always approaches infinity as $x_0 \to 0$ for $\alpha < 2$. If $\beta = \pm 1$, discontinuities almost surely occur only in the direction of the Paretian tail.

Because $c_{\Delta t} \to 0$ as $\Delta t \to 0$, an α-stable Lévy motion is everywhere almost surely continuous, despite the fact that it is not almost surely everywhere continuous. That is to say, every individual point t is almost surely a point of continuity, even though on any finite interval, there will almost surely be an infinite number of points for which this is not true. Even though they are almost surely dense, the points of discontinuity almost surely constitute only a set of measure zero, so that with probability one any point chosen at random will in fact be a point of continuity. Such a point of continuity will almost surely be a limit point of discontinuity points, but whose discontinuities approach zero in size as the t in question is approached.

The scale of $\Delta X/\Delta t$ is $(\Delta t)^{1/\alpha-1}$, so that if $\alpha > 1$, $X(t)$ is everywhere almost surely not differentiable, just as in the case of a Brownian motion. If $\alpha < 1$, $X(t)$ is everywhere almost surely differentiable, though of course there will be an infinite number of points (the discontinuities) for which this will not be true.

As noted in (McCulloch, 1978a), the discontinuities in an α-stable Lévy motion imply that the bottom may occasionally fall out of the market faster than trades can be executed, as occurred, most spectacularly, in October of 1987. When such events have a finite rate of occurrence, the portfolio risk insulation promised by 'programmed trading' becomes wishful thinking, at best. Furthermore, the arbitrage argument of the Black–Scholes model (1973) cannot be rigorously used to price options, and options will no longer be the redundant assets they are in the absence of price discontinuities.

17.3. Multivariate stable processes

Multivariate stable distributions are, in general, much richer than multivariate normal distributions. This is because the sets of independent identically distributed and of spherically symmetric random vectors are not equivalent for $\alpha < 2$, and because multivariate stable distributions are not in general completely characterized by a simple covariation matrix as are multivariate normal distributions. If Y_1 and Y_2 are independent identically distributed stable with $\alpha < 2$, their joint distribution will not have circular density contours. Near the center of the distribution the contours will be approximately circular, but as we move away from the center, the density will have bulges in the directions of the axes.

Great strides have been made in the past few years in our understanding of multivariate stable distributions (Hardin *et al.*, 1991; Cambanis & Wu, 1991). The newly developed 'spectral representation' of multivariate stable distributions provides a general and unique method of specifying these distributions not. available to early researchers.

Let $\mathbf{Y} = (Y_1(\alpha, 1), \dots, Y_m(\alpha, 1))$ be an m-vector of independent identically distributed random variables, each of whose components is distributed by a stable law with parameters α, $\beta = 1$, unit scale and zero drift, and let $A = (a_{ij})$ be a $d \times m$ matrix of rank $d \leq m$. The d-vector $\mathbf{Z} = A\mathbf{Y}$ then has a 'truly d-dimensional' multivariate stable distribution with 'atoms' in the directions of each of the columns $\mathbf{a}_j$ of A. If any two of these columns are of the same direction, say $\mathbf{a}_2 = \lambda \mathbf{a}_1$ for some $\lambda > 0$, they may, with no loss of generality, be merged into a single column equal to $(1+\lambda^\alpha)^{1/\alpha}\mathbf{a}_1$. If the columns come in pairs that have opposite directions but equal norms, $\mathbf{Z}$ will be symmetric stable, but this is only a special case. Each atom will create a bulge in the joint density in the direction of $\mathbf{a}_j$.

The (discrete) spectral representation represents $\mathbf{a}_j$ as $c_j\mathbf{s}_j$, where $c_j = |\mathbf{a}_j|$ and $\mathbf{s}_j = a_j/c_j$ is the point on the unit sphere $S_d \subset \mathbb{R}^d$ in the direction of $\mathbf{a}_j$. Then $\mathbf{Z}$ may be written as

$$\mathbf{Z} = \sum_{j=1}^{m} c_j \mathbf{s}_j Y_j, \tag{17.3.1}$$

and

$$\ln \mathsf{E} e^{i\mathbf{k}\mathbf{Z}'} = \sum_{j=1}^{m} \gamma_j \psi(\mathbf{s}_j'\mathbf{k}; \alpha, 1), \tag{17.3.2}$$

where the prime stands for transposition, $\gamma_j = c_j^\alpha$, and $\psi(k;\alpha,1) = \ln q^A(k;\alpha,1)$ is the second characteristic of the reduced stable distribution introduced in Section 3.6.

The most general multivariate stable distributions (abstracting from location, which may easily be added on as a vector $\boldsymbol{\delta}$) may be generated by contributions coming from all conceivable directions, with most or perhaps all of c_j in (17.3.1) becoming infinitesimal. In this case,

$$\ln \mathsf{E} e^{i\mathbf{k}\mathbf{Z}'} = \int_{S_d} \psi(\mathbf{s}'\mathbf{k}; \alpha, 1)\Gamma(d\mathbf{s}), \tag{17.3.3}$$

where Γ is a finite spectral measure defined on the Borel subsets of S_d.

In the case $d = 2$, (17.3.3) may be simplified to

$$\ln \mathsf{E} e^{i\mathbf{k}\mathbf{Z}'} = \int_0^{2\pi} \psi(\mathbf{s}_\theta'\mathbf{k}; \alpha, 1) d\Gamma(\theta), \tag{17.3.4}$$

where $\mathbf{s}_\theta = (\cos\theta, \sin\theta)'$ is the point on the unit circle at angle θ and Γ is a non-decreasing, left-continuous function with $\Gamma(0) = 0$ and $\Gamma(2\pi) < \infty$.

Such a random vector $\mathbf{Z} = (Z_1, Z_2)'$ may be constructed from a maximally positively skewed ($\beta = 1$) α-stable Lévy motion $X(\theta)$, whose independent identically distributed increments $dX(\theta)$ have zero drift and scale $(d\theta)^{1/\alpha}$, by

$$\mathbf{Z} = \int_0^{2\pi} \mathbf{s}_\theta \frac{(d\Gamma(\theta))^{1/\alpha} dX(\theta)}{(d\theta)^{1/\alpha}}. \tag{17.3.5}$$

This integrand has the following interpretation. If $\Gamma'(\theta)$ exists, θ contributes $\mathbf{s}_\theta\Gamma'(\theta))^{1/\alpha}dX(\theta)$ to the integral. If Γ instead jumps by $\Delta\Gamma$ at θ, θ contributes an atom $\mathbf{s}_\theta(\Delta\Gamma)^{1/\alpha} Z_\theta$ to the integral, where $Z_\theta = (d\theta)^{-1/\alpha} dX(\theta)$, distributed by the stable law with A-parameters α, 1, 1, 0, is independent of $dX(\theta')$ for all $\theta' \neq \theta$.

If $\mathbf{Z}$ has such a bivariate stable distribution, and $\mathbf{a} = (a_1, a_2)'$ is a vector of constants,

$$\mathbf{a}'\mathbf{Z} = \int_0^{2\pi} (a_1 \cos\theta + a_2 \sin\theta) \frac{(d\Gamma(\theta))^{1/\alpha} dX(\theta)}{(d\theta)^{1/\alpha}} \tag{17.3.6}$$

is univariate stable. In particular,

$$Z_1 = \int_0^{2\pi} \cos\theta \frac{(d\Gamma(\theta))^{1/\alpha} dX(\theta)}{(d\theta)^{1/\alpha}}, \tag{17.3.7}$$

$$Z_2 = \int_0^{2\pi} \sin\theta \frac{(d\Gamma(\theta))^{1/\alpha} dX(\theta)}{(d\theta)^{1/\alpha}}. \tag{17.3.8}$$

Then, $\mathbf{a}'\mathbf{Z}$ will have scale determined by

$$c^\alpha(\mathbf{a}'\mathbf{Z}) = \int_0^{2\pi} |a_1 \cos\theta + a_2 \sin\theta|^\alpha d\Gamma(\theta). \tag{17.3.9}$$

In (Kanter, 1972) it was shown that if $d\Gamma$ is symmetric and $\alpha > 1$, then

$$\mathsf{E}(Z_2, \mid Z_1) = \kappa_{21} Z_1, \tag{17.3.10}$$

where

$$\kappa_{21} = \frac{1}{c^\alpha(Z_1)} \int_0^{2\pi} \sin\theta \,\mathrm{sign}(\cos\theta) |\cos\theta|^{\alpha-1} d\Gamma(\theta), \tag{17.3.11}$$

$$c^\alpha(Z_1) = \int_0^{2\pi} |\cos\theta|^\alpha d\Gamma(\theta). \tag{17.3.12}$$

The integral in this equation is called the covariation of Z_2 on Z_1. In (Hardin *et al.*, 1991), it was demonstrated that if $d\Gamma$ is asymmetric, then $\mathsf{E}(Z_2 \mid Z_1)$ is non-linear in Z_1, but still is a simple function involving κ_{21}. They note that (17.3.10) may be valid in the symmetric cases even for $\alpha < 1$.

If $d\Gamma$, and therefore the distribution of $\mathbf{Z}$, is symmetric, $\psi(\mathbf{s}'\mathbf{k}; \alpha, 1)$ in (17.3.3) and (17.3.4) may be replaced by $\psi(\mathbf{s}'\mathbf{k}; \alpha, 0) = -|\mathbf{s}'\mathbf{k}|^\alpha$, and $X(\theta)$ in (17.3.5) taken to be symmetric. In this case, the integrals may, if desired, be taken over any half of the unit sphere, provided that Γ is replaced by $\Gamma^* = 2\Gamma$.

One particularly important special case of multivariate stable distributions is the elliptical class proposed by Press (Press, 1982). The particular case presented here is Press' order $m = 1$. His higher order cases (with his $m > 1$) are probably not so useful, not even as an ad hoc approximation to a smooth spectral density. In (Press, 1972), Press asserted that these were the most general multivariate symmetric stable distributions, but later concedes that this is not the case.

If $d\Gamma(\mathbf{s})$ in (17.3.3) simply equals a constant times $d\mathbf{s}$, all directions will make equal contributions to $\mathbf{Z}$. Such a distribution will, after appropriate scaling to give the marginal distribution of each component the desired scale, have spherically symmetric joint density $p(x) = f_d(r;\alpha)$, for some function $f_d(r;\alpha)$ depending only on $r = |\mathbf{Z}|$, α, and the dimensionality d of $\mathbf{Z}$. The logarithm of the characteristic function of such a distribution must be proportional to $\psi(|\mathbf{k}|;\alpha,0) = -(\mathbf{k}'\mathbf{k})^{\alpha/2}$.

Press prefers to select the scale factor for his spherical multivariate stable distributions in such a way that in the standard spherical normal case, the variance of each component is unity:

$$\ln \mathsf{E}e^{i\mathbf{kZ}} = -(\mathbf{k}'\mathbf{k})^{\alpha/2}/2. \tag{17.3.13}$$

In the case $d = 2$ of (17.3.4) and (17.3.5), the requisite constant value of $d\Gamma$ is, by (17.3.12),

$$d\Gamma(\theta) = \left(2\int_0^{2\pi} |\cos\varphi|^\alpha d\varphi\right)^{-1} d\theta. \tag{17.3.14}$$

If $\mathbf{Y}$ has such a d-dimensional spherical stable distribution, and $\mathbf{Z} = H\mathbf{Y}$ for some non-singular $d \times d$ matrix H, then $\mathbf{Z}$ will have a d-dimensional (normalized) elliptical stable distribution with

$$\ln \mathsf{E}\exp(i\mathbf{kZ}') = -(\mathbf{k}'\Sigma\mathbf{k})^{\alpha/2}/2 \tag{17.3.15}$$

and joint density

$$p(x) = |\Sigma|^{-1/2} f_d((\mathbf{Z}'\Sigma^{-1}\mathbf{Z})^{1/2};\alpha), \tag{17.3.16}$$

where $\Sigma = (\sigma_{ij}) = HH'$. The component Z_i of $\mathbf{Z}$ will then have the normalized scale $\sigma(x_i) = \sigma_{ii}^{1/2} = 2^{1/\alpha}c(Z_i)$. This Σ thus acts very much like the multivariate normal covariance matrix, which indeed it is for $\alpha = 2$. For $\alpha > 1$, $\mathsf{E}(Z_i \mid Z_j)$ exists and equals $(\sigma_{ij}/\sigma_{jj})Z_j$, just as with normality. In (Cambanis & Wu, 1991), it was demonstrated that $\mathrm{Var}(Z_i \mid Z_j)$ actually exists in cases like this. If Σ is diagonal, the components of $\mathbf{Z}$ will be uncorrelated, in the sense $\mathsf{E}(Z_i \mid Z_j) = 0$, but not independent unless $\alpha = 2$.

17.4. Stable portfolio theory

Tobin (Tobin, 1958) noted that preferences over probability distributions for random consumable wealth W can be expressed in terms of a two-parameter indirect utility function if all distributions under consideration are indexed by these two parameters. He further demonstrated that if the utility $U(W)$ is a concave function of wealth and this two-parameter class is affine, i.e., indexed by a location and scale parameter like the stable λ and γ, the indirect utility function $V(\gamma, \lambda)$ generated by expected utility maximization must be quasi-concave, while the opportunity sets generated by portfolios of risky assets and a risk-free asset will be straight lines. Furthermore, if such a two-parameter affine class is closed under addition, the returns on convex portfolios of assets will be evaluable by the same quasi-concave indirect utility function. If the class is symmetric, even non-convex portfolios, with short sales (negative holdings) of some assets, may be thus compared.

Originally it was believed that only the normal distribution had this closure property, but Fama (Fama, 1965) quickly pointed out that it was shared by the stable distributions.

Fama and Miller demonstrate that the conclusions of the traditional Sharpe–Lintner–Mossin capital asset pricing model (CAPM) carry over to the special class of multivariate symmetric stable distributions in which the time t relative arithmetic return

$$R_i = (P_i(t+1) - P_i(t))/P_i(t)$$

on asset i is generated by the 'market model':

$$R_i = a_i + b_i M + \varepsilon_i, \tag{17.4.1}$$

where a_i and b_i are asset-specific constants, M, distributed by a symmetric reduced stable law with parameter α, is a market-wide factor affecting the returns on all assets, and ε_i, distributed by a symmetric stable law with parameter α, zero drift and scale c_i, is an asset-specific disturbance independent of M and independent across assets, with $c_i > 0$.

Under this market model, the returns $\mathbf{R} = (R_1, \dots, R_N)'$ on N assets have an $(N+1)$-atom multivariate symmetric stable distribution generated by

$$\mathbf{R} = \mathbf{a} + \begin{pmatrix}\mathbf{b} & \mathbf{I}_N\end{pmatrix} \begin{pmatrix} M \\ \boldsymbol{\varepsilon} \end{pmatrix}, \tag{17.4.2}$$

where $\mathbf{I}_N$ is the $N \times N$ unity matrix, $\mathbf{a} = (a_1, \dots, a_N)'$, etc. This distribution has N (symmetric) atoms in the direction of each axis, as well as in the $(N+1)$st, representing the common market factor M, extending into the positive orthant.

Fama and Miller show that when $\alpha > 1$, diversification will reduce the effect of the firm-specific risks, just as in the normal case, though at a slower rate. They note that if two different portfolios of such assets are mixed in proportions x and $(1-x)$, the scale of the mixed portfolio will be a strictly convex function of x and therefore (providing the two portfolios have different expected returns) of its mean return. On the efficient set of portfolios, where mean is an increasing function of scale, maximized mean return will therefore be a concave function of scale, just as in the normal case. Given Tobin's quasi-concavity of the indirect utility function, a tangency between the efficient frontier and an indirect utility indifference curve then implies a global expected utility maximum for an individual investor.

When trading in an artificial, zero-sum asset paying a riskless real return R_f is introduced, all agents will choose to mix positive or negative quantities of the risk-free asset with the market portfolio, just as in the normal case. Letting θ_i, $i = 1, \dots, N$, be the value-weighted share of asset i in the market portfolio, the market return will be given by

$$R_m = \boldsymbol{\theta}'\mathbf{R} = a_m + b_m M + \varepsilon_m,$$

where

$$a_m = \sum_{i=1}^{N} \theta_i a_i, \quad b_m = \sum_{i=1}^{N} \theta_i b_i, \quad \varepsilon_m = \sum_{i=1}^{N} \theta_i \varepsilon_i. \tag{17.4.3}$$

Thus, $(R_m, R_i)'$ will have a three-atom bivariate symmetric stable distribution generated by

$$\begin{pmatrix} R_m \\ R_i \end{pmatrix} = \begin{pmatrix} b_m & 1 & \theta_i \\ b_i & 0 & 1 \end{pmatrix} \begin{pmatrix} M \\ \varepsilon_{\hat{\imath}} \\ \varepsilon_i \end{pmatrix} \tag{17.4.4}$$

where $\varepsilon_{\hat{\imath}} = \varepsilon_m - \theta_i \varepsilon_i$. The variability of R_m is

$$c^{\alpha}(R_m) = b_m^{\alpha} + c^{\alpha}(\varepsilon_m), \tag{17.4.5}$$

where $c^{\alpha}(\varepsilon_m) = \sum \theta_i^{\alpha} c_i^{\alpha}$ is the contribution of the firm-specific risks to the risk of the market portfolio.

The conventional CAPM predicts that the prices of the N assets, and therefore their mean returns a_i will be determined by the market in such a way that

$$\mathsf{E}R_i - R_f = (\mathsf{E}R_m - R_f)\beta_{\text{CAPM}}, \tag{17.4.6}$$

where the CAPM 'beta' (not to be confused with the stable 'beta') is ordinarily computed as

$$\beta_{\text{CAPM}} = \operatorname{cov}(R_i, R_m) / \operatorname{Var} R_m. \tag{17.4.7}$$

This expression involves a covariance and variance that are both infinite for $\alpha < 1$. However, Fama and Miller point out that the market equilibrium condition in fact only requires that

(a) the market portfolio be an efficient portfolio and therefore minimize its scale given its mean return;

(b) in $(E(R), c(R))$ space, the slope of the efficient set at the market portfolio equal $(\mathsf{E}R_m - R_f)/c(R_m)$.

They note that these in turn imply (17.4.6), with

$$\beta_{\text{CAPM}} = \frac{1}{c(R_m)} \frac{\partial c(R_m)}{\partial \theta_i}. \tag{17.4.8}$$

In the finite variance case, (17.4.8) happens to be equivalent to (17.4.7), but the existence of the variance and covariance are in fact inessential. In the case of the single market factor 'market model' of (17.4.1), Fama and Miller show that (17.4.8) becomes

$$\beta_{\text{CAPM}} = \frac{b_i b_m^{\alpha-1} + \theta_i^{\alpha-1} c_i^{\alpha}}{c^{\alpha}(R_m)}. \tag{17.4.9}$$

As the number of comparably-sized firms becomes large, so that $\theta_i \to 0$, $c(R_m) \to b_m$, and hence

$$\beta_{\text{CAPM}} \to b_i/b_m. \tag{17.4.10}$$

Fama and Miller did not explore more general multivariate stable distributions in detail, other than to note that it might be useful to consider adding industry-specific factors to the single market factor model of (17.4.1). To the extent these cannot be fully diversified away, they might contribute to the systematic risk of the individual stocks in the various industries in an important way.

In (Press, 1982) it is demonstrated that portfolio analysis with his elliptical class of multivariate stable distributions is even simpler than in the multi-atom model of Fama and Miller. Let $\mathbf{R} - \mathsf{E}\mathbf{R}$ have a normalized elliptical stable distribution with characteristic function satisfying (17.3.15) and covariation matrix Σ. Then the covariation matrix Σ^* of $(R_m, R_i)'$ will be

$$\Sigma^* = \begin{pmatrix} \sigma_m^2 & \sigma_{im} \\ \sigma_{im} & \sigma_i^2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\theta}' \\ \mathbf{e}_i' \end{pmatrix} \Sigma \begin{pmatrix} \boldsymbol{\theta} & \mathbf{e}_i \end{pmatrix}, \tag{17.4.11}$$

where $\mathbf{e}_i$ is the ith unit N-vector. Press did not quite solve for the CAPM equilibrium, but it can easily be shown that (17.4.8) simply implies

$$\beta_{\text{CAPM}} = \sigma_{im}/\sigma_m^2. \tag{17.4.12}$$

In the general symmetric multivariate stable case, not considered by either Fama and Miller or Press, $\mathbf{Z} = (R_m - \mathsf{E}R_m, R_i - \mathsf{E}R_i)'$ will have a bivariate symmetric stable distribution of type (17.3.4). It then may easily be shown that the Fama–Miller rule (17.4.8) implies

$$\beta_{\text{CAPM}} = \kappa_{im}, \tag{17.4.13}$$

where $\kappa_{im} = \mathsf{E}(R_i - \mathsf{E}R_i \mid R_m - \mathsf{E}R_m)/(R_m - \mathsf{E}R_m)$ is as given by Kanter's formula (17.3.11) above.

The possibility that $\alpha < 2$ therefore adds no new difficulties to the traditional CAPM. However, we are still left with its original problems. The first of these is that it assumes that there is a single consumption good consumed at a single point in time. If there are several goods with variable relative prices, or several points in time with a non-constant real interest rate structure, there may in effect be different CAPM 'beta's for different types of consumption risk, regardless of whether $\alpha = 2$ or $\alpha < 2$.

The second problem with the CAPM is that if arithmetic returns really have a stable distribution with $\alpha > 1$ and positive scale parameter, there is a positive probability that any individual stock price, or even wealth and therefore consumption as a whole, will go negative. Ziemba considers restrictions on the utility function that will keep expected utility and expected marginal utility finite under these circumstances, but a non-negative distribution would be preferred, given free disposal and limited liability, not to mention the difficulty of negative consumption. A further complication is that it is more reasonable to assume that relative, rather than absolute, arithmetic returns are homoscedastic over time. Yet if relative one-period arithmetic returns have any independent identically distributed distribution, then over multiple time periods they will accumulative multiplicatively, not additively as required to retain a stable distribution.

A normal or stable distribution for logarithmic asset returns, $\ln(P_i(t+1)/P_i(t))$, would keep asset prices and therefore wealth non-negative, and could easily arise from the multiplicative accumulation of returns. However, the log-normal or log-stable is no longer an affine two-parameter class of distributions, and consequently Tobin's demonstration of the quasi-concavity of the indirect utility function may no longer be invoked.

Furthermore, while the closure property of stable distributions under addition implies that log-normal and log-stable distributions are closed under multiplication, as may take place for an individual stock over time, it does not imply that they are closed under addition, such as takes place under portfolio formation. A portfolio of log-normal or log-stable stocks therefore does not necessarily have a distribution in the same class. As a consequence, such portfolios may not be precisely commensurate in terms of any two-parameter indirect utility function, whether quasi-concave or not.

Conceivably, two random variables might have a joint distribution with log-stable marginal distributions, whose contours are somehow deformed in such a way that linear combinations of the two variables are nevertheless still log-stable. However, in (McCulloch & Mityagin, 1991) it is shown that this cannot be the case if the log-stable marginal distributions have finite mean, i.e., $\alpha = 2$ or $\beta = -1$. Mityagin's result makes it highly unlikely that the more general infinite mean cases would have the desired property, although this remains to be rigorously demonstrated.

In the Gaussian case, the latter set of problems has been skirted by focusing on continuous time Wiener processes, for which negative outcomes may be ruled out by a log-normal assumption, but for which instantaneous logarithmic and relative arithmetic returns differ only by a drift term governed by Itô's lemma. With $\alpha < 2$, however, the discontinuities in continuous-time stable processes make even instantaneous logarithmic and relative arithmetic returns behave fundamentally differently.

It therefore appears that the stable CAPM, like the Gaussian CAPM, provides at best only an approximation to the equilibrium pricing of risky assets. There is, after all, nothing in economic theory that guarantees that asset pricing will in fact have the simplicity and precision that was originally sought in the two-parameter asset pricing model, or that actual asset returns will conveniently accommodate whatever distributional assumptions it may require.

17.5. Log-stable option pricing

This section draws heavily on (McCulloch, 1985b).

An option is a derivative financial security that gives its owner the right, but not the obligation, to buy or sell a specified quantity of an underlying asset at a contractual price called the striking price or exercise price, within a specified period of time. An option to buy is a call option, while an option to sell is a put option. If the option may only be exercised on its maturity date it is said to be European, while if it may be exercised at any time prior to its final maturity it is said to be American. In practice, most options, even those traded on European exchanges, are 'American', but 'European' options are easier to evaluate, and under some circumstances the two will be equivalent in value.

Black and Scholes (1973) find a precise formula for the value of a European option on a stock whose price on maturity has a log-normal distribution, by means of an arbitrage argument involving the almost surely everywhere continuous path of the stock price during the life of the option. Merton (Merton, 1976) noted early on that practitioners often claim that deep-in-the money, deep-out-of-the money, and shorter maturity options tend to sell for more than their Black–Scholes predicted value. Furthermore, if the Black–Scholes formula were based on the true distributional assumption, the implicit volatility calculated from it using synchronous prices for otherwise identical options with different striking prices would be a constant across striking prices. Hull (Hull, 1993) notes that in practice the resulting implicit volatility curve instead often bends up at the ends, to form what is commonly referred to as the 'volatility smile'. Rubinstein (Rubinstein, 1985) finds this effect to be highly significant for almost all maturities for out-of-the-money calls during the period 10/77–8/78, as predicted by the stable model. During 8/76–10/77, it is highly significant for maturities under 120 days, but surprisingly, it is significantly reversed for longer maturities. These related phenomena suggest that the market, at least, believes that large price movements have a higher probability, relative to small price movements, than is consistent with the

Black–Scholes formula.

Unfortunately the logic of the Black–Scholes model cannot be adapted to the log-stable case, because of the (infinite number of) discontinuities that appear in the time path of an α-stable Lévy process. Rachev and Samorodnitskii (Rachev & Samorodnitsky, 1993) attempt to price a log-symmetric stable option, using a hedging argument with respect to the directions of the jumps in an underlying α-stable Lévy motion, but not with respect to their magnitudes. Furthermore, their hedge ratio is computed as a function of the still unobserved magnitude of the jumps. These drawbacks render their formula less than satisfactory, even apart from its difficulty of calculation. Jones (Jones, 1984) calculates option values for a compound jump–diffusion process in which the jumps, and therefore the process, have infinite variance, but this is neither a stable nor a log-stable distribution. Furthermore, if the logarithm of the price of a stock is stable with $\alpha < 2$ and $\beta > -1$, the expected payoff on a call option on it is infinite. This infinite expectation left Paul Samuelson (as quoted in (Smith, 1976, p. 19)) inclined to believe in [Robert] Meton's conjecture that a strict Lévy–Pareto [stable] distribution on $\ln(S^*/S)$ would lead, with $1 < \alpha < 2$, to a 5-minute warrant or call being worth 100 percent of the common. Merton further conjectured (Merton, 1976) that an infinite expected future price for stocks would require the risk-free interest rate to be infinite, in order for the current price to be finite. We shall see in the present section that these fears are unfounded, even in the extreme cases $\alpha < 1$.

17.5.1. Spot and forward asset prices

Let there be two assets, A_1 and A_2, and let X be the random price of A_2 in terms of A_1 at future time T. Let U_1 and U_2 be the marginal utilities of A_1 and A_2 for a representative individual. If $\ln U_1$ and $\ln U_2$ are both stable with a common characteristic exponent, then the logarithm of

$$X = U_2/U_1 \tag{17.5.1}$$

will also be stable, with the same exponent.

Let F be the forward price in the market at present time 0 on a contract to deliver 1 unit of A_2 at time T. Unconditional payment of F units of A_1 is to be made entirely at time T, with no 'marking to market' in the interim. The value of F that sets the expected utility gain from a small position in this contract equal to 0 is

$$F = \mathsf{E}U_2/\mathsf{E}U_1. \tag{17.5.2}$$

The expectations in (17.5.2) are both conditional on present (time 0) information.

In order for the expectations in (17.5.2) to be finite when $\ln U_i$ are both stable with $\alpha < 2$, they both must be maximally negatively skewed, i.e., have

$\beta = -1$, and therefore no upper Paretian tail. In order to evaluate log-stable options we presently see no alternative but to make this assumption. Note that this constraint does not prevent $\ln X$ from being intermediately skew-stable, or even symmetric stable.

Let V_1 and V_2 be independent asset-specific maximally positively skewed stable variables contributing negatively to $\ln U_1$ and $\ln U_2$ respectively. In order to add some generality, let V_3 be a common component, contributing negatively and equally to both $\ln U_1$ and $\ln U_2$, and which is independent of V_1 and V_2, so that

$$\ln U_1 = -V_1 - V_3, \tag{17.5.3}$$

$$\ln U_2 = -V_2 - V_3. \tag{17.5.4}$$

Let $(\alpha, \beta, c, \delta)$ be the parameters of

$$\ln X = V_1 - V_2, \tag{17.5.5}$$

and let c_i, δ_i, $i = 1, 2, 3$, be the scale and drift parameters of V_i. We assume that α, β, c, and F are known, but that $\delta, c_1, c_2, c_3, \delta_1, \delta_2, \delta_3$ are not directly observed. We can nevertheless infer all we need to know about these unobserved parameters from the observed ones. We have

$$\delta = \delta_1 - \delta_2, \qquad \alpha \neq 1, \tag{17.5.6}$$

$$c^\alpha = c_1^\alpha + c_2^\alpha, \tag{17.5.7}$$

$$\beta c^\alpha = c_1^\alpha - c_2^\alpha. \tag{17.5.8}$$

We will return to the case $\alpha = 1$ presently, but for the time being we assume $\alpha \neq 1$.

Equations (17.5.7) and (17.5.8) may be solved for

$$c_1 = ((1+\beta)/2)^{1/\alpha} c, \qquad c_2 = ((1-\beta)/2)^{1/\alpha} c. \tag{17.5.9}$$

Using Zolotarev's formula

$$\ln \mathsf{E} e^x = \begin{cases} \delta - c^\alpha \sec(\pi\alpha/2), & \alpha \neq 1, \\ \delta + \frac{2}{\pi} c \ln c, & \alpha = 1, \end{cases} \tag{17.5.10}$$

and setting $\theta = \pi\alpha/2$, we have

$$\mathsf{E} U_i = e^{-\delta_i - \delta_3 - (c_i^\alpha + c_3^\alpha)\sec\theta}, \qquad i = 1, 2, \tag{17.5.11}$$

hence (17.5.2) yields

$$F = e^{\delta + \beta c^\alpha \sec\theta}. \tag{17.5.12}$$

If $\beta = 0$ (because $c_1 = c_2$), (17.5.12) implies Osborne's (1964) special case of a log-random walk: $\ln F = \mathsf{E} \ln X$. Note that this case does not require logarithmic utility, but only that U_1 and U_2 make equal contributions to the uncertainty of X.

17.5.2. Option pricing

Let C be the value, in terms of units of the numeraire asset A_1 to be delivered unconditionally at time 0, of a 'European' call option on 1 unit of asset A_2 to be exercised at (but not before) time T, with striking price X_0. Let r_1 be the default-free interest rate on loans denominated in A_1 with maturity T. A payment of C units of A_1 at time 0 is thus equivalent to an unconditional payment of $C\exp(r_1T)$ units at time T.

If $X > X_0$ at time T, the option will be exercised. Its owner will receive 1 unit of A_2, in exchange for X_0 units of A_1. If $X < X_0$, the option will not be exercised. In either event, its owner will be out the interest-augmented $C\exp(r_1T)$ units of A_1 originally paid for the option. In order for the expected utility gain from a small position in this option to be zero, we must have

$$\int_{X>X_0} (U_2 - X_0U_1)\, d\mathsf{P}(U_1, U_2) - Ce^{r_1T} \int_{\text{all } X} U_1\, d\mathsf{P}(U_1, U_2) = 0, \tag{17.5.13}$$

or, using (17.5.2),

$$C = e^{-r_1T} \left[\frac{F}{\mathsf{E}U_2} \int_{X>X_0} U_2\, d\mathsf{P}(U_1, U_2) - \frac{X_0}{\mathsf{E}U_1} \int_{X>X_0} U_1\, d\mathsf{P}(U_1, U_2) \right]. \tag{17.5.14}$$

In the above two equations, $\mathsf{P}(U_1, U_2)$ represents the joint probability distribution for U_1 and U_2. Relation (17.5.14) is valid for a European call option with any joint distribution, stable or otherwise, for which the expectations exist.

It is not very difficult to see that for $\alpha \neq 1$, (17.5.14) becomes

$$C = Fe^{-r_1T + c_2^\alpha \sec\theta} I_1 - X_0 e^{-r_1T + c_1^\alpha \sec\theta} I_2, \tag{17.5.15}$$

where

$$I_1 = \int_{-\infty}^{\infty} e^{-c_2z} s_{\alpha 1}(z) \left[1 - S_{\alpha 1}\left(\left(c_2z - \ln\frac{F}{X_0} + \beta c^\alpha \sec\theta \right) \Big/ c_1 \right) \right] dz, \tag{17.5.16}$$

$$I_2 = \int_{-\infty}^{\infty} e^{-c_1z} s_{\alpha 1}(z) S_{\alpha 1}\left(\left(c_1z - \ln\frac{F}{X_0} + \beta c^\alpha \sec\theta \right) \Big/ c_2 \right) dz, \tag{17.5.17}$$

$S_{\alpha\beta}$, $s_{\alpha\beta}$ stand for the distribution function and density of the stable law with parameters α, β. Relation (17.5.15) effectively gives C as a function $C(X_0, F, \alpha, \beta, c, r_1, T))$, since c_1 and c_2 are determined by (17.5.9), and $\theta = \pi\alpha/2$. Note that δ is not directly required, since all we need to know about it is contained in F through (17.5.12). Note also that the common component of uncertainty, namely u_3, completely drops out.

In (Rubinstein, 1976) it is demonstrated that (17.5.14) leads to the Black–Scholes formula when $\ln U_1$ and $\ln U_2$ have a general bivariate normal distribution. Relation (17.5.15) thus extends the Black–Scholes model to the case $\alpha < 2$.

If the forward price F is not directly observed, we may use the spot price S to construct a proxy for it if we know the default-free interest rate r_2 on A_2-denominated loans, since arbitrage requires

$$F = Se^{(r_1 - r_2)T}. \tag{17.5.18}$$

The value P of a European put option giving one the right to sell 1 unit of A_2 at striking price X_0 at future time T may be evaluated with (17.5.15), in conjunction with the put-call parity arbitrage condition

$$P = C + (X_0 - F)e^{-r_1 T}. \tag{17.5.19}$$

Relations (17.5.12) and (17.5.15) are valid even for $\alpha < 1$. When $\alpha = 1$, (17.5.12) becomes

$$F = e^{\delta - \frac{2}{\pi}\beta c \ln c}. \tag{17.5.20}$$

and (17.5.15) becomes

$$C = Fe^{-r_1 T - \frac{2}{\pi} c_2 \ln c_2} I_1 - X_0 e^{-r_1 T - \frac{2}{\pi} c_1 \ln c_1} I_2, \tag{17.5.21}$$

where c_1 and c_2 are as above, but

$$I_1 = \int_{-\infty}^{\infty} e^{-c_2 z} s_{\alpha 1}(z) \times \left[1 - S_{\alpha 1}\left(\left(c_2 z + \ln \frac{X_0}{F} + \frac{2}{\pi}(c_2 \ln c_2 - c_1 \ln c_1)\right) \Big/ c_1\right)\right] dz, \tag{17.5.22}$$

$$I_2 = \int_{-\infty}^{\infty} e^{-c_1 z} s_{\alpha 1}(z) \times S_{\alpha 1}\left(\left(c_1 z - \ln \frac{X_0}{F} + \frac{2}{\pi}(c_1 \ln c_1 - c_2 \ln c_2)\right) \Big/ c_2\right) dz. \tag{17.5.23}$$

17.5.3. Applications

The stable option pricing formula (17.5.15) may be applied without modification to options on commodities, stocks, bonds, and foreign exchange rates, simply by appropriately varying the interpretation of the two assets A_1 and A_2.

Commodities. Let A_1 and A_2 be two consumption goods, both of which are available for consumption on some future date T; A_1 could be a broad-based numeraire aggregating all goods other than A_2. Let r_1 be the default-free real interest rate on A_1-denominated loans.

Let U_1 and U_2 be the random future marginal utilities of A_1 and A_2, and suppose that $\ln U_1$ and $\ln U_2$ have both independent (u_1 and u_2) and common (u_3) components, as in (17.5.3) and (17.5.4) above. The price X of A_2 in terms of A_1, as determined by (17.5.1), is then intermediately skewed log-stable as in (17.5.5), with forward price F as in (17.5.12). The shadow price C of a European call option on 1 unit of A_2 at time T that sets the expected marginal utility gain of a small position in the option equal to zero is then given by (17.5.15) above.

Such a scenario might, for example, arise from an additively separable constant relative risk aversion (CRRA) utility function

$$U(A_1, A_2) = \frac{1}{1-\eta}(A_1^{1-\eta} + A_2^{1-\eta}), \quad \eta > 0, \quad \eta \neq 1, \tag{17.5.24}$$

with the physical endowments given by $A_i = e^{v_i + v_3}$, $i = 1, 2$, where v_1, v_2, and v_3 are independent stable variables with a common α and $\beta = \pm 1$.

Stocks. Suppose now that there is a single good G, which will serve as our numeraire, A_1. Let A_2 be a share of stock in a firm that produces a random amount y of G per share. Let r_1 be the default-free interest rate on G-denominated loans with maturity T. The firm pays continuous dividends, in stock, at rate r_2, and its stock has no valuable voting rights before time T, so that one share for spot delivery is equivalent to $\exp(r_2 T)$ shares at T. Let U_G be the random future marginal utility of one unit of G at time T, and assume that

$$\ln U_G = -u_1 - u_3, \tag{17.5.25}$$

$$\ln y = u_1 - u_2, \tag{17.5.26}$$

where u_i are independent stable variables as above.

The marginal utility of one share is then $yU_G = \exp(-u_2 - u_3)$, and the price per share using unconditional claims on G as numeraire, $X = (yU_G)/U_G$, is as given in (17.5.5). The forward price of one share, $F = \mathsf{E}(yU_G)/\mathsf{E}(U_G)$, is as given in (17.5.12). The value of a European call option on 1 share of stock is then given by (17.5.15). If the forward price of the stock is not directly observed, it may be constructed from r_1, r_2, and the spot price S by means of (17.5.18).

Relation (17.5.26) states that to the extent there is firm-specific good news ($-u_2$), it is assumed to have no upper Paretian tail. This means that the firm will produce a fairly predictable amount if successful, but that it may still be highly speculative, in the sense of having a significant probability of producing much less or virtually nothing at all. To the extent there is firm non-specific good news (u_1), the marginal utility of G, given by (17.5.24), is assumed to be correspondingly reduced.

We make no attempt here to generalize (17.5.25) and (17.5.26) to more complicated interactions. Despite this rather restrictive underlying scenario, our stock price X can take on a completely general log-stable distribution.

Bonds. Now suppose that there is a single consumption good, G, that can be available at either or both of two future dates, $T_2 > T_1 > T_0 = 0$. Let A_1 and A_2 be unconditional claims on one unit of G at T_1 and T_2, respectively, and let U_1 and U_2 be the marginal utility of G at these two dates. Let $\mathsf{E}_1 U_2$ be the expectation of U_2 as of T_1. As of present time 0, both U_1 and $\mathsf{E}_1 U_2$ are random variables. Assume $\ln U_1 = -u_1 - u_3$, and $\ln \mathsf{E}_1 U_2 = -u_2 - u_3$, where u_i are independently distributed stable variables as above. The price at T_1 of a zero-coupon price level indexed bond that matures at time T_2, $X = \mathsf{E}_1 U_2 / U_1$ is then given by (17.5.5), and the forward price F of such a bond implicit in the term structure at present time 0, $F = \mathsf{E}_0 U_2 / \mathsf{E}_0 U_1 = \mathsf{E}_0 (E_1 U_2) / \mathsf{E}_0 U_1$ is governed by (17.5.12) above.

This model leads to the log expectation hypothesis $\ln F = \mathsf{E} \ln X$ in the special case $p = 0$. Cox, Ingersoll and Ross (Cox *et al.*, 1981) claim that this necessarily violates a no-arbitrage condition in continuous time with $\alpha = 2$, but McCulloch in (McCulloch, 1993) demonstrates with a counterexample that this claim is incorrect. The requisite forward price F may he computed as $\exp(r_1 T_1 - R_2 T_2)$, where R_2 (not the same as r_2 above) is the real interest rate an loans maturing at T_2.

The shadow price of a European call is then given by (17.5.15), where r_1 is now the real interest rate on indexed loans maturing at time T_1, and T is replaced by T_1.

Foreign exchange rates. The present subsection draws heavily on (McCulloch, 1987).

The log-stable-option pricing model has a particularly appealing interpretation in terms of a purchasing power parity model of foreign exchange rate determination. In practice, purchasing power parity performs rather poorly. However, to the ex- tent that real exchange rates fluctuate, they may simply be modeled as real commodity price fluctuations. Purchasing power parity provides an instructive alternative interpretation of the stable option model, in terms of purely nominal risk.

Let P_1 and P_2 be the price levels in countries 1 and 2, respectively, at future time T. Price level uncertainty itself is generally positively skewed. Astronomical inflations are very easily arranged, simply by throwing the printing presses into high gear, and this policy has considerable fiscal appeal to it. Comparable deflations would be fiscally intolerable, and are in practice unheard of. It is therefore particularly reasonable to assume that $\ln P_1$ and $\ln P_2$ are both maximally positively skewed.

Let u_1 and u_2 be independent 'country specific' components of $\ln P_1$ and $\ln P_2$, respectively, and let u_3 be an 'international component' of both price levels, reflecting the 'herd instincts' of central bankers, that is independent of both u_1 and u_2, so that $\ln P_i = u_i + u_3$, $i = 1, 2$. Under purchasing power parity, the exchange rate giving the value of currency 2 (A_2) in terms of currency 1 (A_1), $X = P_1/P_2$, is then as given in (17.5.5) above.

It can easily be shown that the lower Paretian tail of $\ln X$ will give the density of X itself a mode (with infinite density but no mass) at 0, as well as a second mode (unless c is large as compared to unity) near $\exp(\mathsf{E} \ln X)$. Thus log-stable distributions achieve the bimodality sought by Krasker (Krasker, 1980), all in terms of a single story about the underlying process.

Assume that inflation uncertainty involves no systematic risk. In this case the forward exchange rate F will have to equal $\mathsf{E}(1/P_2)/\mathsf{E}(1/P_1)$ in order to set the expected profit (in terms of purchasing power) from forward speculation equal to zero, and will be determined by (17.5.12).

Let r_1 and r_2 be the default-free nominal interest rates in countries 1 and 2. Then the shadow price of a European call on one unit of currency 2 that sets the expected purchasing power gain from a small position in the option equal to zero is given by (17.5.15) above. The forward price F may, if necessary, be inferred from the spot price S by means of the covered interest arbitrage formula (17.5.18).

McCulloch in (McCulloch, 1985a) uses the results of this section, in the short-lived limit, to evaluate deposit insurance in the presence of interest-rate risk, as faced by traditional banks and thrift institutions who are short-funded.

Pseudo-hedge ratio. Although the arbitrage logic of Black and Scholes cannot be used to evaluate log-stable options, the risk exposure from writing a call option on one unit of an asset can be partially neutralized (to a first-order approximation) by simultaneously taking a long forward position on

$$\frac{\partial(C \exp(r_1 T))}{\partial F} = e^{c_2^\alpha \sec\theta} I_1 \tag{17.5.27}$$

units of the underlying asset. Unfortunately, the discontinuities leave this position imperfectly hedged if $\alpha < 2$. At the same time, this imperfect ability to hedge implies that options are not redundant financial instruments.

17.6. Low probability and short-lived options

This section draws heavily on (McCulloch, 1978b).

Assume $X_0 > F$ and that the scale parameter c is small relative to $\ln(X_0/F)$. Holding p constant, c_1 and c_2 are then small as well. Then (see (McCulloch, 1985b, Appendix B) for details) that the call value C behaves like

$$Fe^{-r_1 T} c^\alpha (1+\beta)\Psi(\alpha, X_0/F), \tag{17.6.1}$$

where

$$\Psi\left(\alpha, \frac{X_0}{F}\right) = \frac{\Gamma(\alpha)\sin\theta}{\pi}\left[\left(\ln\frac{X_0}{F}\right)^{-\alpha} - \alpha\frac{X_0}{F}\int_{\ln\frac{X_0}{F}}^{\infty} e^{-\zeta}\zeta^{-\alpha-1}d\zeta\right]. \quad (17.6.2)$$

This function is tabulated in some detail in Table 17.1. It becomes infinite as $X_0/F \downarrow 1$, and 0 as $\alpha \uparrow 2$. By the put/call inversion formula, P behaves as

$$Se^{-r_1T}c^{\alpha}(1-\beta)\Psi(\alpha, F/X_0). \quad (17.6.3)$$

In an α-stable Lévy motion, the scale that accumulates in T time units is $c_0T^{1/\alpha}$, where c_0 is the scale that accumulates in 1.0 time unit. As $T \downarrow 0$, the forward price F converges on the spot price S. Therefore,

$$\lim_{T\downarrow 0}\frac{C}{T} = (1+\beta)c_0^{\alpha}S\Psi(\alpha, X_0/S), \quad (17.6.4)$$

$$\lim_{T\downarrow 0}\frac{P}{T} = (1-\beta)c_0^{\alpha}X_0\Psi(\alpha, S/X_0), \quad (17.6.5)$$

The last formula has been employed in (McCulloch, 1981; McCulloch, 1985a) to evaluate the put option implicit in deposit insurance for banks or thrift institutions that are exposed to interest rate risk, using empirical estimates of the stable parameters of returns on U.S. Treasury securities (with β assumed to be 0) to quantify pure interest rate risk.

17.7. Parameter estimation and empirical issues

If $\alpha > 1$, ordinary least squares method provides us with a consistent estimator of the stable location parameter δ. However, it has an infinite variance stable distribution with the same α as the observations, and does not tell us anything about the deviation from normality. Furthermore, as Batchelor (Batchelor, 1981) has pointed out, expectations proxies based on a false normal assumption will generate spurious evidence of irrationality if the true distribution is stable with $\alpha < 2$. It is therefore important to tailor estimation methods to the sub-Gaussian cases.

17.7.1. Empirical objections to stable distributions

The initial interest in stable distributions has undeservedly waned, as a result of two groups of statistical tests. The first group of tests is based on the observation that if daily returns are independent identically distributed stable, weekly and monthly returns must, by the basic stability property of stable distributions, be stable with the same characteristic exponent. Blattberg and Gonedes (Blattberg & Gonedes, 1974), and many subsequent investigators,

Table 17.1. $\Psi(\alpha, X_0/F)$

	Striking price/forward price											
α	1.001	1.01	1.02	1.04	1.06	1.10	1.15	1.20	1.40	2.00	4.00	10.00
2.00	0.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.0000	0.0000	0.0000
1.95	18.10	1.962	0.989	0.492	0.324	0.190	0.124	0.091	0.043	0.0168	0.0062	0.0028
1.90	26.43	3.199	1.665	0.854	0.573	0.343	0.227	0.169	0.082	0.0329	0.0126	0.0059
1.80	28.38	4.275	2.369	1.291	0.896	0.560	0.382	0.291	0.149	0.0633	0.0256	0.0125
1.70	23.13	4.319	2.544	1.471	1.056	0.688	0.484	0.376	0.203	0.0914	0.0391	0.0199
1.60	17.01	3.916	2.448	1.498	1.112	0.753	0.547	0.434	0.246	0.1172	0.0531	0.0282
1.50	11.93	3.365	2.227	1.441	1.103	0.777	0.582	0.471	0.280	0.1411	0.0676	0.0375
1.40	8.22	2.812	1.966	1.341	1.059	0.774	0.596	0.492	0.306	0.1634	0.0827	0.0479
1.30	5.65	2.319	1.707	1.225	0.995	0.753	0.597	0.503	0.327	0.1842	0.0985	0.0594
1.20	3.92	1.904	1.471	1.106	0.923	0.724	0.589	0.505	0.343	0.2039	0.1150	0.0723
1.10	2.77	1.567	1.266	0.995	0.852	0.689	0.575	0.502	0.356	0.2227	0.1325	0.0868
1.00	2.02	1.300	1.092	0.894	0.784	0.654	0.558	0.496	0.366	0.2411	0.1511	0.1031
0.90	1.51	1.090	0.949	0.806	0.722	0.619	0.541	0.489	0.375	0.2592	0.1710	0.1215
0.80	1.18	0.928	0.833	0.730	0.667	0.587	0.524	0.481	0.383	0.2776	0.1926	0.1424

notably Akgiray and Booth (Akgiray & Booth, 1988), have found evidence that weekly and monthly returns have significantly higher characteristic exponent estimates than do daily returns. This type of evidence ultimately led even Fama (Fama, 1976) to abandon the stable distribution as a model for stock prices.

However, as Diebold (Diebold, 1993) has pointed out, all that such evidence really rejects is the compound hypothesis of i.i.d. stability. It demonstrates either that returns are not identically distributed, or that they are not independently distributed, or that they are not stable distributed. If returns are not independent identically distributed, then it should come as no surprise that they are not independent identically distributed stable, either. It is now generally acknowledged that most time series on financial returns exhibit strong serially conditional heteroscedasticity of the type characterized by ARCH (autoregressive conditional heteroscedasticity) or GARCH (generalized ARCH) models. The unconditional distribution of such disturbances will be much more leptokurtic, and therefore would tend to generate misleadingly low a estimates under a false i.i.d. stable assumption, than will the underlying conditional distribution.

Baillie in (Baillie, 1993) wrongly characterizes ARCH and GARCH models as 'competing' with the stable hypothesis. In fact, if conditional heteroscedasticity is present, as it often clearly is, it is as desirable to remove it in the infinite variance stable case as in the Gaussian case. And if after removing it there is still leptokurtosis, it is as desirable to model the adjusted residuals correctly as it is in the independent identically distributed case. McCulloch in (McCulloch, 1985b) thus fits GARCH-like and GARCH models, respectively, to monthly bond returns by symmetric stable ML, and find significant evidence both conditional heteroscedasticity and residual non-normality. Liu and Brorsen similarly find, contrary to the findings of Gribbin, Harris and Lau, that a stable model for commodity and foreign exchange futures returns cannot be rejected, once GARCH effects are removed. Their observations apply also to the objections of (Lau *et al.*, 1990) to a stable model for stock price returns.

Day-of-the-week effects are also well known to be present in both stock market (Gibbons & Hess, 1981) and foreign exchange (McFarland*et al.*, 1982) data. Such hebdomodalities may be present in the mean and/or the volatility. Either way, they imply that daily data is not identically distributed, and will therefore also contribute to a rejection of independent identically distributed stability. It is again as important to remove these, along with any annually recurring seasonals that may be present, in the infinite variance stable case as in the normal case.

A second group of tests that purport to reject a stable model of asset returns is based on estimates of the Paretian exponent of the tails, using either the Pareto distribution itself (Hill, 1975), or the generalized Pareto distribution proposed in (DuMouchel, 1983). Numerous investigators have applied this type of test to data that includes interest rate changes, stock returns, and foreign exchange rates, and typically have found an exponent that appears to be significantly greater than 2. A distribution with such a Paretian tail will have a finite variance and lie in the domain of attraction of the Gaussian distribution rather than of a sub-Gaussian stable distribution.

However, in (McCulloch, 1994) it is demonstrated that tail index estimates greater than 2 are to be expected from stable distributions with α greater than approximately 1.65. These estimates may even appear to be be significantly greater than 2, relative to the class of truly Pareto- or generalized Pareto-tailed distributions, for a sufficiently large sample. These tests are therefore in no way inconsistent with a non-normal stable distribution. Mittnik and Rachev (Mittnik & Rachev, 1993b) similarly find that Weibull distribution is consistent with tail index estimators in the range 2.5–5.5, even though the Weibull distribution has no Paretian tail. Wade Brorsen has had some success testing these non-nested hypotheses against one another by means of Monte-Carlo tabulations of the LR statistic under each null.

In view of this apparent evidence against stable distributions, several alternative distributions have been proposed to account for the conspicuously leptokurtic behavior of financial returns. Blattberg and Gonedes (Blattberg & Gonedes, 1974) thus propose the Student's distributions, which may be computed for fractional degrees of freedom, and which, like the stable distributions, include the Cauchy (1DOF) and the normal (∞DOF). Others consider a mixture of normals. In (DuMouchel, 1973b) it is shown that it is often very hard to distinguish between such leptokurtic distributions. To the extent this is true, these alternative models cannot have importantly different practical implications about mean returns, etc. The choice among them may in the end simply depend on whatever desirable properties these distributions may have, including divisibility, parsimony, and central limit attributes.

Mittnik and Rachev (Mittnik & Rachev, 1993a) generalize the concept of 'stability' beyond the stability under summation and multiplication that leads to the stable and log-stable distributions, respectively, to include 'stability' under the maximum and minimum operators, as well as 'stability' under a random repetition of these accumulation and extremum operations, with the number of repetitions governed by a geometric distribution. They find that the Weibull distribution has two of these generalized stability properties. Since it has only positive support, they therefore propose a 'double Weibull' distribution (with two Weibull distributions back-to-back) as a model for asset returns. This distribution has the unfortunate property that its density is, with only one exception, either infinite or zero at the origin. The only exception is the special case of a back-to-back exponential distribution, which still has a cusp at the origin. The stable densities, on the other hand, are finite, unimodal, absolutely differentiable, and have a closed support.

17.7.2. Estimation of multivariate stable distributions

The estimation of multivariate stable distribution parameters is still in its infancy, despite the great importance of these distributions for financial theory and practice. In (Mittnik & Rachev, 1993b), a method is suggested to estimate the general bivariate spectral measure for a vector whose distribution lies in this domain of attraction. Cheng and Rachev apply this method to the $/DM and $/¥ exchange rates, with the interesting result that there is considerable density near the center of the first and third quadrants, as would be expected if a un- specific factor were affecting both exchange rates equally, but very little along the axes. The latter effect seems to indicate that there are negligible DM- or ¥ -specific shocks. However, the Mittnik and Rachev method employs only a small subset of the data, drawn from the extreme tails of the sample.

We propose an alternative method based on likelihood maximization, which uses the entire data set. This method does not necessitate the often arduous task of actually computing the MV stable density, but relies only on the standard univariate stable density. Unlike Mittnik and Rachev, we expressly assume that $\mathbf{x}$ actually has a bivariate stable distribution, rather than that it lies in its domain of attraction. As is customary in the normal case, one may hope that this assumption will approximately capture the important properties of a distribution which in fact merely lies in the domain of attraction.

Let $\mathbf{x}$ have a general bivariate stable distribution with spectral measure $\Gamma(\theta)$, as in (17.3.4) and (17.3.5), but possibly with a non-zero location vector $\boldsymbol{\delta} = (\delta_1, \delta_2)'$. For each $\omega \in [0, \pi)$, let

$$y(\omega) = \mathbf{s}'_\theta \mathbf{x} = \mathbf{s}'_\theta \boldsymbol{\delta} + \int_{\theta=0}^{2\pi} \cos(\theta - \omega) \frac{(d\Gamma(\theta))^{1/\alpha} dz(\theta)}{(d\theta)^{1/\alpha}}. \tag{17.7.1}$$

This has scale $c(\omega)$, where

$$c^\alpha(\omega) = \int_{\theta=0}^{2\pi} |\cos(\theta - \omega)|^\alpha d\Gamma(\theta). \tag{17.7.2}$$

By breaking the integral in (17.7.1) in half at $\omega + \pi/2$ and $\omega - \pi/2$ ($d\Gamma$ is cyclic by definition), $y(\omega)$ may be decomposed into its location parameter plus the sum of two maximally skewed zero-location stable variables, one with $\beta = 1$ and scale $C(\omega)$, and the other with $\beta = -1$ and scale $C(\omega + \pi)$, where

$$C^\alpha(\omega) = \int_{\omega-\pi/2}^{\omega+\pi/2} |\cos(\theta - \omega)|^\alpha d\Gamma(\theta), \qquad \omega \in [0, 2\pi), \tag{17.7.3}$$

so that

$$c^\alpha(\omega) = C^\alpha(\omega) + C^\alpha(\omega + \pi). \tag{17.7.4}$$

Hence the skewness of $y(\omega)$ is given by

$$\beta(\omega) = (C^\alpha(\omega) - C^\alpha(\omega + \pi))/c^\alpha(\omega); \tag{17.7.5}$$

therefore,

$$C^\alpha(\omega) = c^\alpha(\omega)(1 + \beta(\omega))/2,$$
$$C^\alpha(\omega + \pi) = c^\alpha(\omega)(1 - \beta(\omega))/2. \quad (17.7.6)$$

Now let $\mathbf{x}_i = (x_{1i}, x_{2i})'$ be a set of n independent identically distributed observations on $\mathbf{x}$. The components x_1 and x_2 of $\mathbf{x}$ are each univariate stable with a common α. The parameters α, β_1, c_1, and δ_1 of x_1 may therefore be estimated consistently by univariate maximum likelihood from the n observations on x_1. Similarly, α, β_2, c_2, and δ_2 may be consistently estimated from the n observations on x_2. These two estimates of α will ordinarily not precisely agree. However, if the two log likelihoods are pooled by averaging them together, and the resulting function maximized subject to the restriction that x_1 and x_2 have a common exponent, an even more efficient common estimate of α may be obtained. This is not a true ML estimate unless x_1 and x_2 happen to be independent, but it will share the consistency of univariate ML, and will be far more reliable than the Mittnik and Rachev tail estimator of u when the true distribution is bivariate stable.

Next, center the $\mathbf{x}_i$ by subtracting out the pooled ML estimates of δ_i. Set $\theta_j = \omega_j = 2\pi j/m, j = 0, \ldots, m - 1$, for some large integer m divisible by 4. For $j = 0, \ldots, m/2-1$, calculate $y_i(\omega_j)$ by rotating the centered $\mathbf{x}_i$ as in (17.7.1) above, and then use these to estimate $\beta(\omega_j)$ and $c(\omega_j)$ by univariate ML, constraining α to the pooled ML estimate, and δ to 0. Because $c(\omega)$ is continuous, the estimates for each j are good initial values for $j + 1$. Next estimate $C(\omega_j)$, $j = 0, \ldots, m - 1$ using (17.7.6).

Relation (17.7.3) states that $C^\alpha(\omega)$ is a moving average of $d\Gamma(\theta)$. This moving average may be numerically approximated by

$$\gamma_j = C^\alpha(\omega_j) \approx \sum_{h=j-m/4}^{j+m/4} |\cos(\omega_j - \theta_h)|^\alpha \Delta_h, \quad (17.7.7)$$

where

$$\Delta_h = \Gamma(\theta_h + \pi/m) - \Gamma(\theta_h - \pi/m); \quad (17.7.8)$$

(17.7.7) is a system of m equations in m unknowns of the form $\gamma \approx \mathbf{B\Delta}$. It may be solved for $\mathbf{\Delta} \approx \mathbf{B}^{-1}\boldsymbol{\gamma}$ so long as $\mathbf{B}$ is non-singular. The matrix $\mathbf{B}$ is cyclic, as is $\mathbf{B}^{-1}$, i.e., each row is an image of the row above, offset by one, and wrapped. The inverse may therefore be stored as its first row, thus alleviating storage constraints with large m. The same inverse works for every problem involving the same α and m.

Having thus estimated $\mathbf{\Delta}$ from $\boldsymbol{\gamma}$, $\Gamma(\theta_j + \pi/m)$ may be estimated by summing Δ_h from 0 to j. The offset of π/m is desirable, because the axes are ordinarily prime candidates for atoms. With the offset, these will show up uniquely in Δ_0, $\Delta_{m/4}$, $\Delta_{m/2}$, and $\Delta_{3m/4}$, rather than being split in two .

Due to sampling error, some of the Δ_h estimates may be negative, particularly if m is high and/or the true value is zero. This may be prevented, if desired, by instead solving the quadratic program: Find $\mathbf{\Delta} > \mathbf{0}$ such that $(\boldsymbol{\gamma} - \mathbf{B\Delta})'(\boldsymbol{\gamma} - \mathbf{B\Delta}) = \min$.

Because the matrix $\mathbf{B}$ does not depend on the data, m is not limited by the sample size n, and ordinarily may be set as high as desired without $\mathbf{B}$ becoming singular. However, in the Gaussian case $\alpha = 2$, two atoms are always sufficient to generate the joint distribution, and there is an infinite number of ways in which these may be selected. In this case Γ is not identified. In itself this is not a problem, since we then simply have a bivariate normal distribution to estimate. However, it does suggest that Δ_h will behave increasingly erratically as $\alpha \to 2$ with any fixed m and sample size n. In such a case, it may be desirable to impose some prior discreteness or smoothness restrictions on the spectral measure, such as the 'market model' (17.4.1), with or without industry-specific factors, a state-space model such as that considered in (Oh, 1994), or the elliptical restriction (Press, 1982) in (17.3.16). In the elliptical case, a fast numerical approximation to $f_d(r;\alpha)$ would make numerical full information ML quite feasible.

The general bivariate method described above may readily be extended to the general multivariate case (17.3.3) by approximating the spectral measure defined on the unit sphere in n-space by a discrete measure in which a large but finite number of points on the unit sphere represent a small adjacent region. This may be done most efficiently by repeated geodesic triangulation and hexagonal regions, though a system based on rectangular regions in polar coordinates would give essentially the same results with simpler calculations.

18

Miscellany

This chapter is devoted to the use of stable laws in the fields beyond the exact sciences. Being no experts in these fields, we confine ourselves to expounding a few more or less simple examples from different sciences, mainly as demonstration of great potentialities of stable laws in these fields.

18.1. Biology

In the 1960s, stable laws began to attract the attention of scholars working in the area of economics, biology, sociology, and mathematical linguistics, due to a series of publications by the American mathematician Mandelbrot and his disciples. The fact is that statistical principles described by the so-called Zipf–Pareto distributions were empirically discovered fairly long ago in all these areas of knowledge. Discrete distributions of this type are of the form

$$p_k = ck^{-1-\alpha}, \qquad k \ge 1,\ \alpha > 0,$$

while their continuous analogues (densities) are

$$p(x) = cx^{-1-\alpha}, \qquad x \ge a > 0.$$

Mandelbrot called attention to the fact that the use of the extremal stable distributions (corresponding to $\beta = 1$) to describe empirical principles was preferable to the use of Zipf–Pareto distributions for a number of reasons. It can be seen from many publications, both theoretical and applied, that Mandelbrot's ideas received ample recognition of experts. In this way, the hope arose to confirm empirically established principles in the framework of mathematical models and, at the same time, to clear up the mechanism of formation of these principles.

In 1922, the English biologist Willis in (Willis, 1922) studied statistical principles lying in the heart of evolution. One of the basic results in his investigations was the discovery of the following principle.

Biological species are commonly taken as the primary elements in the classification of living organisms. The species are then combined into coarser groups called genera.

We consider a sequence of genera in the animal or plant life, ordering them according to the number of species occurring in them. Then we calculate how many genera in the total number contain a single species, two species, etc.; let these numbers be $M_1, M_2, \dots$, etc., and let $M = M_1 + M_2 + \dots$ be the total number of genera involved. Next, we form the sequence of frequencies $p_k = M_k/M$ ($k = 1, 2, \dots$), which allows us to represent the probability of finding exactly k species in a randomly chosen genus.

In the language of probability theory, the discovery of Willis consisted in seeing that for genera containing sufficiently many species, i.e., for $n \geq n_0$

$$\sum_{k>n} p_k \approx A n^{-1/2}, \tag{18.1.1}$$

where A is a constant.

In other words, the probability of finding in a genus at least n species decreases as $1/\sqrt{n}$, as n increases.

In 1924, the English mathematician Yule (Yule, 1925) was able to find a theoretical basis for the relation (3.3.1) in the framework of a stochastic model that can be included in the theory of branching processes.

We demonstrate below (in a way different from that of (Yule, 1925)) how the Willis–Yule law could be explained. The most interesting point is the occurrence in our model of a stable law and its connection with the principle (18.1.1), which is traditionally associated with the Zipf–Pareto distribution.

A model of a random branching process with two types of particles will be taken as a basis for the reasoning.

Let us consider the reproduction process for particles of two types as time passes. This process goes as follows. At the beginning of the process, we have a single particle of type T_1. During a unit time, this particle produces μ_{10} particles of type T_0 and μ_{11} particles of type T_1. The particles of type T_0 remain unchanged, while the particles of type T_1 can be further transformed. The numbers μ_{10} and μ_{11} are random variables. We impose some conditions on this random process of reproduction of particles.

(1) The transformation of each of the particles of type T_1 takes place independently of its history and independently of what happens with the other particles.

(2) The joint probability distribution of the random variables μ_{10} and μ_{11} at the time of transformation of a particle of type T_1 remains the same for all particles of this type and, does not depend on the time when the transformation happens. Furthermore, the progeny $\mu_{10} + \mu_{11}$ in the course of a single transformation cannot exceed some constant h.

(3) The mean number $\delta = \mathsf{E}\mu_{10}$ of particles of type T_0 produced in a single act of transformation is positive, while the mean number of particles of type T_1 is equal to one. Moreover, $c = \operatorname{Var}\mu_{11} > 0$. The last condition, in particular, does not permit a particle of type T_1 to produce with probability one only a single particle of type T_1 at the time of transformation. Together with the condition $\mathsf{E}\mu_{11} = 1$, this means that with some non-zero probability a particle of type T_1 produces no particles of the same type. Since particles of type T_0 are not transformed, further transformation of the given branch is thereby terminated.

It is well known that transformations of a single particle of type T_1 under the above conditions ultimately (with probability one) terminate. Its transformation process gives a certain random number U of particles of type T_0 called the final particles.

Let us assume that there are n initial particles of type T_1. In the process of their transformation and the transformation of their offspring (this process, as mentioned, stops with probability one) there appear $U_1, \dots, U_n$ final particles produced by the first, second, etc., initial particles of type T_1.

According to conditions 1 and 2, the random variables U_j are independent and identically distributed. Let us form a normalized sum of these random variables

$$V_n = (U_1 + \dots + U_n)(2\delta n^2/c)^{-1}.$$

It can be proved that the distributions of the random variables V_n converge as $n \to \infty$ to the stable distribution with parameters $\alpha = 1/2$, $\beta = 1$, $\gamma = 0$, and $\lambda = 1$ (that is, to the Lévy law).

In the opinion of recognized experts, the diversity of biological species stems from evolution of living beings in the course of strict natural selection. The Earth climate has changed, both globally and in regional parts, and this created new requirements on plants and animals—to survive, they had to acquire new qualities.

This was accomplished due to the variability of characteristics in new generations and to natural selection of the most well-suited of their representatives. If we attempt to formalize this selection, in simplified form the picture reminds us of the random branching process described above. We distinguish some portion of the population united by some important from the viewpoint of survival characteristic, and we understand transformations of parts to be the changes arising in a long series of generations; the fractional parts of type T_0 (the final parts) should be interpreted as the offspring able to secure qualities needed for stable existence, while the parts of type T_1 should be interpreted as the offspring lacking such qualities and thus doomed either to extinction or to relatively rapid change.

Of course, the model with final particles is somewhat idealized. Even well-adapted species are subjected to subsequent changes. But this happens much more rarely and slowly than with the groups which 'feel' discomfort in their condition and are pressed toward further variability. Therefore, a model where fractional parts of both types T_0 and T_1 are transformed is in more close correspondence with the actual situation: at the end of its existence, the T_0-particle produces μ_{00} T_0-particles and μ_{01} T_1-particles, whereas a T_1-particle gives birth to μ_{10} T_0-particles and μ_{11} T_1-particles. It is assumed that

$$\mathsf{E}\mu_{00} = 1, \quad \mathsf{E}\mu_{11} = 1, \quad 0 < c_1 < c = \operatorname{Var}\mu_{11} < c_2, \quad \sum \mu_{ij} < h,$$

as in the original model, and that $\delta = \mathsf{E}\mu_{10}$, $\varepsilon = \mathsf{E}\mu_{01}$, and $\sigma = \operatorname{Var}\mu_{00}$ are positive but small in absolute value.

In this version of the process, the transformations of the distinguished parts do not die with probability one, as was the case in the original process. Changes take place over an arbitrarily long period of time with positive probability. This property of the process corresponds more closely to actual evolutionary processes. At the same time, if the value of ε is small, then the average lifetime of particles of type T_0 becomes large. The total number of particles in this model increases on the average. However, on any bounded interval of time this growth depends on the value of the sum $\eta = \varepsilon + \delta$ and can be arbitrary slow if this sum is small.

All the foregoing agrees well with our ideas about the flow of evolutionary processes.

Before passing to an explanation of the Willis–Yule law we dwell further on a certain feature of the new version of the model.

There are no final distinguished particles. Therefore, one can speak only of the number $V_n(\Delta) = U_1(\Delta) + \dots + U_n(\Delta)$ of long-lived particles of type T_0 at a time Δ sufficiently long after the initial time.

The principal way in which we see the second model as differing from the first is that the T_0-particles are not final but long-lived. The quantity $\varepsilon = \mathsf{E}\mu_{01}$ is regarded as the basic parameter of the second model, because the first model is obtained by fixing h, δ and c, and setting $\varepsilon \to 0$. In other words, in a certain sense the second model exhibits continuity with respect to variation of ε.

Therefore, it seems quite likely (and is actually corroborated by computations) that the distribution of the sum $V_n(\Delta)$ depends continuously on ε as $\varepsilon \to 0$. This can serve as a base for replacing the second model, which better reflects the evolutionary process in progress, by the first model in computations, since the latter is more convenient on the analytic level. The analysis of the behavior of the distribution of the sum $V_n(\Delta)$ in the first model can, in turn, be replaced by an analysis of the behavior of this sum corresponding to $\Delta = \infty$. The last condition means that we are considering the distribution of the total number of final particles generated by the initial n particles of type T_1.

The validity of this replacement in analyzing the asymptotic behavior of the distribution can be justified. Of course, the error is the smaller, the greater Δ is. Thus, for our purposes we can use the limit theorem given for the first model with respect to the distributions of the variables V_n. In refined form, it asserts that, as $n \to \infty$,

$$\mathsf{P}\{V_n > x\} = \mathsf{P}\{Y < x\}(1 + o(1)),$$

where $Y = Y(1/2, 1, 0, 1)$ and $o(1)$ tends to zero uniformly in x.

Considering a set of species (particles of type T_0) within the scope of a single genus, it is natural to assume that they all have a common root (i.e., are generated by a single particle of type T_1). Between species and genus, of course, there were also intermediate forms not constituting independent units of classification. We regard them as $n - 1$ initial particles of type T_1 that in the final analysis produce $U_2 + \ldots + U_n$ particles of type T_0. The initial part of type T_1 also produces U_1 particles of type T_0. Together they produce $W = U_1 + \ldots + U_n$ particles of type T_0. The distribution of this sum is exactly the distribution of the number of species in a genus.

Since δ is small while the quantity c is above-bounded and separated from zero, the ratio $B^2 = 2\delta n^2/c$ can be regarded as both above and below bounded (for an appropriate n). Then the number of species in a genus has a distribution that can be approximated by the stable Lévy law

$$\mathsf{P}\left\{W > xB^2\right\} = \mathsf{P}\{V_n > x\} \approx \mathsf{P}\{Y > x\}.$$

If we assume that for large x

$$\mathsf{P}\{Y > x\} = 1 - G(x; 1/2, 1) \approx \frac{1}{\sqrt{\pi}} x^{-1/2},$$

then we obtain the Willis–Yule principle (18.1.1)

$$\mathsf{P}\{W > x\} \approx \mathsf{P}\left\{Y > xB^{-2}\right\} \approx \frac{B}{\sqrt{\pi}} x^{-1/2}.$$

18.2. Genetics

A few approaches are developed in order to shed light on the following fundamental problems (Allegrini *et al.*, 1995):

- establishing the role of the non-coding regions in DNA sequences (introns) in the hierarchy of biological functions;
- finding simple methods of statistical analysis of such sequences to distinguish the non-coding regions from the coding ones (exons);
- discovering the constraints and regularities beyond the DNA evolution and their connection to Darwin's theory or to modern evolution theories;
- extracting new global information on DNA and its function;

- establishing the roles of chance and determinism in genetic evolution and coding regarded as the 'program' underlying the development and life of every organism.

One of the model we would like to refer here is based on a diffusion process. It is used as a tool of the statistical description of correlations in DNA sequences (West, 1994; Stanley *et al.*, 1994; Stanley *et al.*, 1996). Omitting particular details, we can say that all molecules incoming into DNA are divided into two groups, say A (purine) and B (pyrimidine), and the difference

$$X(t) = N_A(t) - N_B(t)$$

is investigated, where $N_A(t)$ and $N_B(t)$ denote the numbers of A- and B-molecules, respectively, occurring in a segment of length t along a DNA sequence. Thus the random variable $X(t)$ with increase of 'time' t describes a trajectory similar to that of diffusional one-dimensional motion. Assigning integer values to t, i.e., expressing it in terms of a distance between neighboring molecules, we can issue the stochastic relation

$$X(t+1) = X(t) + Y \tag{18.2.1}$$

where Y is a random variable taking the value +1 if a purine occurs and -1 if a pyrimidine occurs at the position $t+1$.

In (Allegrini *et al.*, 1995), (18.2.1) is presented in the form of the differential equation

$$\dot{X}(t) = Y(t). \tag{18.2.2}$$

Under the equilibrium assumption

$$\langle Y(t)\rangle_{eq} = 0, \qquad \langle Y^2(t)\rangle_{eq} = \text{const},$$

the second moment

$$\langle X^2(t)\rangle = \langle X^2(0)\rangle + 2\langle Y^2\rangle_{eq} \int_0^t dt' \int_0^{t'} dt'' \Phi_Y(t''),$$

was introduced, where $\Phi_Y(t)$ denotes the equilibrium correlation function

$$\Phi_Y(t) = \langle Y(0)Y(t)\rangle / \langle Y^2\rangle. \tag{18.2.3}$$

Taking the large-range correlations observed in DNA into account, they choose the function asymptotically behaving as

$$\Phi_Y(t) \propto t^{-\beta}, \qquad t \to \infty, \tag{18.2.4}$$

with

$$0 < \beta < 1. \tag{18.2.5}$$

The correlation function is connected with another important statistical function, namely the waiting time distribution $\psi(t)$

$$\Phi_Y(t) = \int_t^\infty (t - t')\psi(t')\,dt' \Big/ \int_0^\infty t\psi(t)\,dt\,. \tag{18.2.6}$$

The $\psi(t)\,dt$ determines the probability that Y changed its state in the interval $(t, t+dt)$, where t is counted off from the moment of preceding transition. From (18.2.4) and (18.2.5), one can see that condition (18.2.5) is fulfilled, provided that

$$\psi(t) \propto t^{-\mu}, \qquad t \to \infty,$$

with

$$2 < \mu < 3. \tag{18.2.7}$$

This constraint on the index μ arises from (18.2.5), since it follows from (18.2.6) that

$$\beta = \mu - 2.$$

It is easy to prove that in the case where (18.2.7) applies, the asymptotic behavior of the second moment of the diffusing variable X is given by

$$\langle X^2 \rangle \propto t^{2H}, \tag{18.2.8}$$

with

$$H = 2 - \mu/2 \tag{18.2.9}$$

which therefore ranges from 1/2 to 1. The relation between the indices (18.2.9) can be easily obtained by twice differentiating (18.2.8) and (18.2.3) and equating the resulting expressions. Much more exciting is the fact that the distribution of X is not Gaussian and is characterized by long-range tails. These tails cannot result in diverging moments, a fact that would be incompatible with the dynamic realization of the process, where the diffusing particle cannot travel with a velocity faster than that of the limiting trajectory $|X| = t$. However, if this unavoidable truncation is ignored, the distribution is indistinguishable from that of a Lévy process with the Lévy index $\alpha = \mu - 1$.

This means that we are observing an α-stable Lévy process with an index in the interval $1 < \alpha < 2$. In principle, the α-stable Lévy process concerns the wider range $0 < \alpha < 2$. However, the condition $\alpha < 1$ refers to a process faster than the ballistic diffusion and thus is incompatible with the dynamic nature of the process described by (18.2.2).

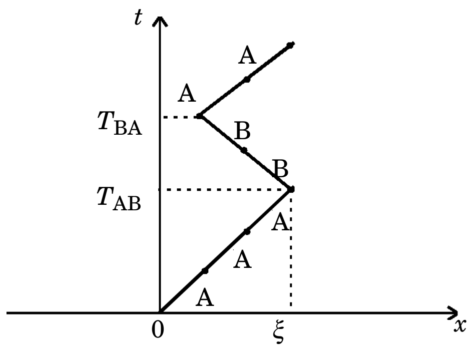


Figure 18.1. Walk model representing DNA structure

The conclusion is certainly valid, but the reasoning can be made more rigorous without the use of the mean square $\langle X^2 \rangle$, which diverges for $\alpha < 2$. To do this, we look at Fig. 18.1. As one can see, the random trajectory of the process under consideration can be represented as a sequence of segments of different length and varying (at times $T_{AB}, T_{BA}, \ldots$) slope. In other words, we deal with a walker performing one-dimensional walk with the unit velocity and the free path distribution density

$$p(\xi) = \psi(\xi) \propto \xi^{-\mu} = \xi^{-\alpha-1}.$$

This process is described by the two-state model considered in Section 12.8 which indeed leads us to symmetric stable distributions.

18.3. Physiology

There exist numerous instances in nature when a system is characterized by a number of states and when a specific state is attained for the first time the system changes its properties or perhaps activates mechanisms to change the properties of another system that is controlled by the first. A neuron's membrane charge under conditions of spontaneous activity is activated in an analogous manner. The membrane charge is subjected to instantaneous changes due to excitatory and inhibitory input of a neuron arriving randomly in time and that can be modeled by a random walk. If its membrane charge attains a barrier of a certain height (threshold), the neuron excites and then returns to its resting state (the 'all and none' law). The excitation distribution is then determined by the distribution of times at which the membrane charge attains this threshold for the first time from its resting state.

Taking the normal diffusion approximation for a random walk between the states, in (Gerstein & Mandelbrot, 1964) they described the spike activity of a single neuron in terms of the Lévy distribution connected with the first

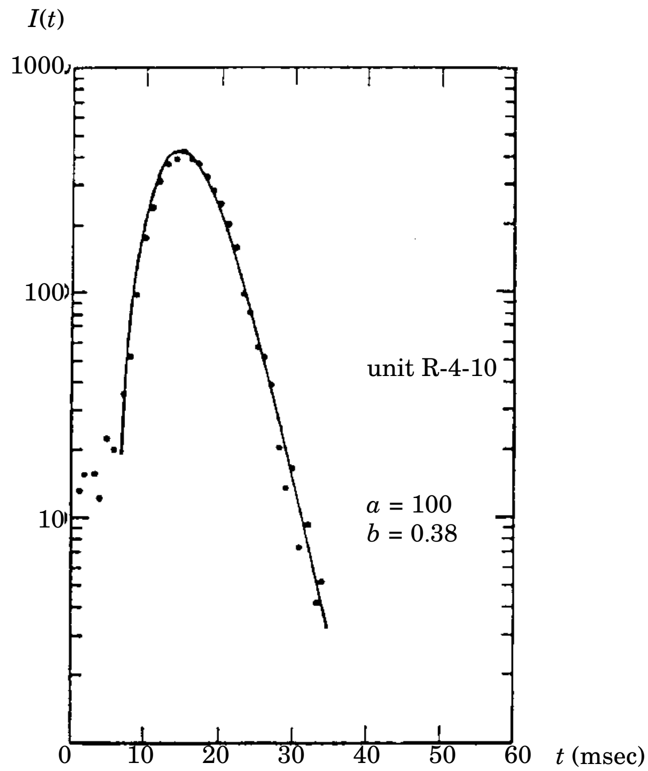


Figure 18.2. A fit to the inter-spike histograms for excitation of a neuron in the cochlear nucleus of a cat under moderate dial-urethane anesthesia using the Gerstein and Mandelbrot model (taken from (West, 1994))

passage time (see Section 8.3). In order to reach the quantitative agreement with the experimental data, they modified the model to allow for difference in the rates of excitation and inhibition. The associated diffusion admits of some drift. In this case, the first passage time distribution becomes

$$p_T(t) = At^{-3/2} \exp\{-bt - a/t\}.$$

Here a is a parameter associated with the difference between the threshold and resting potentials, b is a parameter connected with the difference in the arrival rates of excitatory and inhibitory inputs, and A is the normalization constant. Thus $p_T(t)\,dt$ is the probability that the neuron excites for the first time in the time interval $(t, t + dt)$. It is called the inter-spike interval distribution. The parameters a and b are adjusted to fit the experimental distributions and the fit in some cases is quite good (see Fig. 18.2).

One more model called the fractal shot noise is developed in connection with simulations of functional systems (Lowen & Teich, 1989).

The total mass within a specified volume at limit t is written as the sum of impulse response functions

$$M(t) = \sum_{i=1}^{N} m(t - T_i), \qquad 0 < T_1 < T_2 < \ldots < T_N < t,$$

where

$$m(t) = \begin{cases} m_0 t^{-\mu}, & 0 \le t \le A, \\ 0 & \text{otherwise;} \end{cases} \qquad m_0 = \text{const} > 0, \quad 0 < \mu < 1,$$

and the random times T_i constitute a homogeneous Poisson process of rate ρ. The characteristic function of the random variable $M(t)$ is

$$f(k) = \exp\left\{\rho \int_0^t [e^{ikm(\tau)} - 1] d\tau\right\}, \qquad k > 0.$$

Recalling the known explicit expression for $m(\tau)$, we obtain

$$f(k) = \exp\left\{\rho \int_0^A [e^{ikm_0\tau^{-\mu}} - 1] d\tau\right\}, \qquad k > 0,$$

or, after changing the variables,

$$f(k) = \exp\left\{(\rho/\mu)(m_0 k)^{1/\mu} \int_{m_0 k A^{-\mu}}^{\infty} [e^{ix} - 1] x^{-1/\mu - 1} dx\right\}, \qquad k > 0.$$

Making use of the asymptotic expressions (2.3.17)–(2.3.19) as $A \to \infty$, we arrive at

$$\ln f(k) = -\lambda |k|^{\alpha}[1 - i \tan(\alpha\pi/2) \operatorname{sign} k],$$

where $\alpha = 1/\mu$ and

$$\lambda = \rho C(\alpha) m_0^{\alpha}, \qquad C(\alpha) = \Gamma(1 - \alpha) \cos(\alpha\pi/2).$$

Hence the random mass $M(t)$ in the limit as $A < t \to \infty$ and $A \to \infty$ is distributed by the one-sided stable law with parameters $\alpha = 1/\mu$ and $\beta = 1$. But it is worthwhile to notice that the conclusion about the symmetric form of the distribution made in (West, 1994), which serves as the source of both of the above examples, contradicts the abovesaid reasoning.

18.4. Ecology

The following experiment was performed with `Diomedea exulans` (Stanley *et al.*, 1996). Electronic recording devices were attached to the legs of 19 birds in order to measure durations of uninterrupted flights between consecutive landing points on the water surface. It was established that the probability to find a flight with duration t decays as

$$p(t) \propto t^{-\alpha-1}$$

with α close to one. If we assume that the bird selects the direction of each flight at random and then flies with constant speed which does not depend on the direction of flight, then each flight corresponds to a portion of a straight line that connects two consecutive landing points. The distribution of these segments again follows a power law with the same exponent. The above scheme is consistent with the Lévy flight model described in Chapter 10.

Assuming that the density of landing points is proportional to the density of food, they are able to conclude that the plankton forms a fractal set on the surface of the ocean.

18.5. Geology

In (Vlad, 1994), random distributions of rare minerals, say gold, on Earth crust are studied under the assumption that the spatial concentration of the mineral is described by a field produced by a random number of sources obeying inhomogeneous Poisson statistics. We consider one of the models suggested there.

The contribution of the source placed at a position x_i in a n-dimensional Euclidean space to the concentration field Z at the position $x = 0$ depends on the distance $|x_i|$:

$$n_i = c(|x_i|).$$

The random concentration generated by N points placed at random positions $X_1, \ldots, X_N$ is

$$Z = \sum_{i=1}^{N} c(|X_n|).$$

Assuming that

(1) N is distributed by the Poisson law;

(2) X_i are independent of each other and of N;

(3) each of X_i is distributed in a fractal manner, i.e., with the density

$$p(x) = a|x|^{d_f - n},$$

where $d_f < n$ is the fractal dimension, $d_f < n$;

(4) the contribution function $c(|x|)$ is of the form

$$c(|x|) = Mb|x|^{-\gamma}, \qquad \gamma > 0,$$

where M is a positive random variable with a given distribution;

one studies the Laplace transform

$$\tilde{p}_Z(\lambda) = \int_0^\infty e^{-\lambda z} p_Z(z)\, dz$$

of the distribution density $p_Z(z)$ of the random concentration Z at the observation point; it was presented in the form

$$\tilde{p}_Z(\lambda) = \exp\left\{-(C\lambda)^\alpha\right\} \tag{18.5.1}$$

where

$$C = a\langle M^\alpha \rangle^{1/\alpha} \left[\frac{\pi^{d_f/2}\Gamma(1-\alpha)}{\rho\Gamma(1+d_f/2)}\right]^{1/\alpha},$$

ρ is a constant average density of sources on the fractal set, and positive

$$\alpha = d_f < \gamma \tag{18.5.2}$$

should be less than one. Thus, we arrive at the one-sided stable law.

First we analyze the significance of the Lévy distribution (18.5.1). At first sight it might seem that the occurrence of the positive Lévy law (18.5.1) is due to the geometrical fractal structure with fractal dimension d_f. However, this is not true; (18.5.1) is also valid for an average homogeneous distribution of the sources in the Euclidean space, i.e., for $d_f = n$. In this case, $\alpha = d_f/\gamma = n/\gamma$. For the distribution of minerals $n = 2$, and inequality (18.5.2) shows that the Lévy distribution (18.5.1) is valid for any $\gamma > n = 2$. The true cause for the existence of the Lévy laws is the assumption that the system is approximately infinite, that is, that sources are very dense.

We would like to make here some remarks.

First, this is a particular case of the general model described in Section 9.3, so the restriction $\alpha < 1$ is not necessary to solve the problem.

Second, it seems to us that the model of a deterministic fractal is not a case here, and more appropriate would be a stochastic fractal with non-Poisson fluctuations described in Section 11.6–11.8.

Appendix

A.1. One-dimensional densities $q^A(x; \alpha, \beta)$ (Holt & Crow, 1973)

$\alpha = 0.25$

x	$\beta = 0.00$	0.25	-0.25	0.50	-0.50	0.75	-0.75	1.00
0.000	7.6394	6.8502	6.8502	4.8079	4.8079	2.2737	2.2737	0.0000
.1	.3995	.5117	.2899	.6233	.1854	.7310	.0882	.8309
.2	.2157	.2766	.1567	.3384	.1005	.3997	.0481	.4592
.3	.1477	.1894	.1074	.2320	.0691	.2747	.0331	.3169
.4	.1120	.1437	.0815	.1761	.0525	.2088	.0253	.2413
.5	.0912	.1155	.0656	.1416	.0423	.1680	.0204	.1944
.6	.0752	.0964	.0548	.1181	.0354	.1402	.0171	.1624
.7	.0644	.0826	.0470	.1012	.0304	.1202	.0147	.1393
.8	.0563	.0721	.0411	.0884	.0266	.1050	.0128	.1217
.9	.0500	.0640	.0365	.0784	.0236	.0931	.0114	.1080
1	.0449	.0574	.0328	.0704	.0212	.0836	.0103	.0969
1.1	.0407	.0521	.0297	.0638	.0192	.0757	.0093	.0879
1.2	.0372	.0476	.0272	.0583	.0176	.0692	.0085	.0803
1.3	.0342	.0438	.0250	.0536	.0162	.0637	.0078	.0739
1.4	.0317	.0405	.0231	.0496	.0150	.0589	.0073	.0684
1.5	.0295	.0377	.0215	.0461	.0140	.0548	.0068	.0636
1.6	.0275	.0352	.0201	.0431	.0131	.0512	.0063	.0594
1.7	.0258	.0330	.0189	.0404	.0123	.0480	.0059	.0557
1.8	.0243	.0311	.0178	.0381	.0115	.0452	.0056	.0525
1.9	.0230	.0293	.0168	.0359	.0109	.0427	.0053	.0495
2	.0217	.0278	.0159	.0340	.0103	.0404	.0050	.0469
2	.0217	.0278	.0159	.0340	.0103	.0404	.0050	.0469
2.1	.0206	.0264	.0151	.0323	.0098	.0383	.0048	.0445
2.2	.0197	.0251	.0144	.0307	.0094	.0365	.0045	.0423
2.3	.0187	.0239	.0137	.0293	.0089	.0348	.0043	.0404
2.4	.0176	.0229	.0131	.0280	.0085	.0332	.0041	.0386
2.5	.0171	.0219	.0126	.0268	.0082	.0318	.0040	.0369
2.6	.0164	.0210	.0121	.0257	.0078	.0305	.0038	.0354
2.7	.0158	.0201	.0116	.0246	.0075	.0293	.0037	.0340
2.8	.0152	.0194	.0111	.0237	.0072	.0281	.0035	.0327
2.9	.0146	.0186	.0107	.0228	.0070	.0271	.0034	.0314
3	.0141	.0180	.0103	.0220	.0067	.0261	.0033	.0303
3.1	.0136	.0174	.0100	.0212	.0065	.0252	.0032	.0292
3.2	.0131	.0168	.0096	.0205	.0063	.0243	.0030	.0282
3.3	.0127	.0162	.0093	.0198	.0061	.0235	.0030	.0273
3.4	.0123	.0157	.0090	.0192	.0059	.0228	.0029	.0264
3.5	.0119	.0152	.0087	.0186	.0057	.0221	.0028	.0256
3.6	.0116	.0148	.0085	.0180	.0055	.0214	.0027	.0248
3.7	.0112	.0143	.0082	.0175	.0054	.0208	.0026	.0241
3.8	.0109	.0139	.0080	.0170	.0052	.0202	.0025	.0234
3.9	.0106	.0135	.0078	.0165	.0051	.0196	.0025	.0228
4	.0103	.0131	.0076	.0161	.0049	.0191	.0024	.0221
4.1	.0100	.0128	.0074	.0156	.0048	.0186	.0023	.0215
4.2	.0098	.0125	.0072	.0152	.0047	.0181	.0023	.0210
4.3	.0095	.0121	.0070	.0148	.0046	.0176	.0022	.0204
4.4	.0093	.0118	.0068	.0145	.0045	.0172	.0022	.0199
4.5	.0091	.0115	.0067	.0141	.0043	.0167	.0021	.0194
4.6	.0088	.0113	.0065	.0138	.0042	.0163	.0021	.0190
4.7	.0086	.0110	.0063	.0135	.0041	.0160	.0020	.0185
4.8	.0084	.0108	.0062	.0131	.0040	.0156	.0020	.0181
4.9	.0083	.0105	.0061	.0129	.0040	.0152	.0019	.0177
5	.0081	.0103	.0059	.0126	.0039	.0149	.0019	.0173

$\alpha = 0.50$

x	$\beta = 0.00$	0.25	−0.25	0.50	−0.50	0.75	−0.75	1.00
0.000	.6366	.5287	.5287	.3056	.3056	.1141	.1141	0.0000
.1	.4764	.5841	.3190	.5585	.1743	.3628	.0678	.0850
.2	.3411	.4582	.2218	.5323	.1221	.5126	.0488	.3661
.3	.2597	.3590	.1680	.4461	.0935	.4907	.0381	.4586
.4	.2071	.2895	.1342	.3706	.0753	.4326	.0312	.4518
.5	.1708	.2397	.1110	.3115	.0628	.3753	.0263	.4151
.6	.1443	.2028	.0942	.2658	.0537	.3261	.0227	.3731
.7	.1243	.1746	.0814	.2299	.0467	.2854	.0199	.3335
.8	.1087	.1526	.0715	.2014	.0412	.2518	.0177	.2984
.9	.0963	.1349	.0635	.1782	.0368	.2240	.0159	.2681
1	.0861	.1205	.0570	.1592	.0332	.2007	.0144	.2420
1.1	.0777	.1085	.0516	.1434	.0301	.1812	.0131	.2195
1.2	.0706	.0984	.0470	.1300	.0276	.1645	.0121	.2001
1.3	.0646	.0898	.0431	.1186	.0254	.1502	.0111	.1832
1.4	.0594	.0825	.0397	.1088	.0235	.1379	.0103	.1685
1.5	.0549	.0761	.0368	.1003	.0218	.1271	.0096	.1556
1.6	.0509	.0705	.0342	.0929	.0203	.1177	.0090	.1442
1.7	.0475	.0656	.0319	.0863	.0190	.1094	.0084	.1341
1.8	.0444	.0612	.0299	.0805	.0179	.1020	.0079	.1251
1.9	.0416	.0574	.0281	.0753	.0168	.0954	.0075	.1171
2	.0391	.0539	.0265	.0707	.0159	.0895	.0071	.1098
2.1	.0369	.0507	.0250	.0665	.0150	.0841	.0067	.1033
2.2	.0349	.0479	.0237	.0627	.0142	.0793	.0064	.0974
2.3	.0331	.0453	.0225	.0593	.0135	.0749	.0061	.0920
2.4	.0314	.0430	.0214	.0562	.0129	.0709	.0058	.0871
2.5	.0298	.0408	.0204	.0533	.0123	.0673	.0055	.0826
2.6	.0284	.0389	.0194	.0507	.0117	.0640	.0053	.0785
2.7	.0271	.0370	.0186	.0483	.0112	.0609	.0051	.0747
2.8	.0259	.0354	.0178	.0461	.0108	.0581	.0049	.0712
2.9	.0248	.0338	.0170	.0440	.0103	.0554	.0047	.0680
3	.0238	.0324	.0163	.0421	.0099	.0530	.0045	.0650
3.1	.0228	.0311	.0157	.0404	.0095	.0508	.0044	.0622
3.2	.0219	.0298	.0151	.0387	.0092	.0487	.0042	.0596
3.3	.0211	.0286	.0145	.0372	.0089	.0467	.0040	.0572
3.4	.0203	.0275	.0140	.0357	.0086	.0449	.0039	.0549
3.5	.0196	.0265	.0135	.0344	.0083	.0432	.0038	.0528
3.6	.0189	.0256	.0130	.0331	.0080	.0415	.0036	.0508
3.7	.0182	.0247	.0126	.0319	.0077	.0400	.0035	.0490
3.8	.0176	.0238	.0122	.0308	.0075	.0386	.0034	.0472
3.9	.0170	.0230	.0118	.0298	.0072	.0373	.0033	.0456
4	.0165	.0223	.0114	.0288	.0070	.0360	.0032	.0440
4.1	.0160	.0216	.0111	.0278	.0068	.0348	.0031	.0425
4.2	.0155	.0209	.0107	.0270	.0066	.0337	.0030	.0412
4.3	.0150	.0203	.0104	.0261	.0064	.0326	.0030	.0398
4.4	.0146	.0196	.0101	.0253	.0062	.0316	.0029	.0386
4.5	.0142	.0191	.0098	.0246	.0061	.0307	.0028	.0374
4.6	.0138	.0185	.0096	.0238	.0059	.0298	.0027	.0363
4.7	.0134	.0180	.0093	.0232	.0058	.0289	.0027	.0352
4.8	.0130	.0175	.0091	.0225	.0056	.0281	.0026	.0342
4.9	.0127	.0170	.0088	.0219	.0055	.0273	.0025	.0332
5	.0123	.0166	.0086	.0213	.0053	.0265	.0025	.0323

$\alpha = 0.75$

x	$\beta = 0.00$	0.25	−0.25	0.50	−0.50	0.75	−0.75	1.00
0.00	.3790	.2308	.2308	.0808	.0808	.0214	.0214	0.0000
.1	.3669	.2760	.1929	.0969	.0688	.0244	.0189	0.0000
.2	.3367	.3227	.1627	.1189	.0594	.0283	.0170	0.0000
.3	.2995	.3583	.1388	.1502	.0521	.0335	.0153	0.0000
.4	.2627	.3720	.1197	.1939	.0461	.0409	.0139	0.0000
.5	.2296	.3646	.1045	.2478	.0412	.0520	.0127	0.0000
.6	.2010	.3435	.0920	.2997	.0371	.0709	.0117	0.0000
.7	.1768	.3159	.0817	.3365	.0336	.1046	.0108	0.0000
.8	.1564	.2868	.0731	.3536	.0307	.1553	.0100	.0004
.9	.1391	.2588	.0659	.3537	.0281	.2135	.0093	.0047
1	.1245	.2330	.0597	.3421	.0259	.2660	.0087	.0226
1.1	.1120	.2098	.0544	.3237	.0239	.3042	.0082	.0605
1.2	.1013	.1894	.0499	.3021	.0222	.3263	.0077	.1140
1.3	.0921	.1714	.0459	.2796	.0207	.3343	.0072	.1718
1.4	.0841	.1556	.0423	.2576	.0193	.3318	.0068	.2237
1.5	.0771	.1417	.0392	.2367	.0181	.3221	.0064	.2640
1.6	.0709	.1295	.0365	.2175	.0170	.3080	.0061	.2913
1.7	.0655	.1187	.0340	.1998	.0160	.2915	.0058	.3067
1.8	.0607	.1092	.0318	.1838	.0151	.2740	.0055	.3125
1.9	.0564	.1007	.0298	.1694	.0143	.2565	.0053	.3109
2	.0526	.0932	.0280	.1563	.0135	.2394	.0050	.3042
2.1	.0491	.0865	.0264	.1446	.0129	.2232	.0048	.2941
2.2	.0460	.0804	.0249	.1340	.0122	.2080	.0046	.2818
2.3	.0432	.0750	.0236	.1245	.0116	.1938	.0044	.2683
2.4	.0407	.0701	.0224	.1158	.0111	.1807	.0042	.2544
2.5	.0383	.0657	.0212	.1080	.0106	.1686	.0040	.2405
2.6	.0362	.0617	.0202	.1010	.0101	.1575	.0039	.2270
2.7	.0343	.0580	.0192	.0945	.0097	.1473	.0037	.2139
2.8	.0325	.0547	.0183	.0887	.0093	.1380	.0036	.2014
2.9	.0309	.0517	.0175	.0834	.0089	.1294	.0035	.1897
3	.0293	.0489	.0167	.0785	.0086	.1215	.0033	.1786
3.1	.0279	.0463	.0160	.0740	.0082	.1143	.0032	.1683
3.2	.0266	.0439	.0153	.0699	.0079	.1076	.0031	.1587
3.3	.0254	.0417	.0147	.0661	.0076	.1015	.0030	.1497
3.4	.0243	.0397	.0141	.0626	.0073	.0959	.0029	.1413
3.5	.0232	.0378	.0135	.0594	.0071	.0907	.0028	.1336
3.6	.0223	.0360	.0130	.0564	.0068	.0858	.0027	.1263
3.7	.0213	.0344	.0125	.0537	.0066	.0814	.0027	.1196
3.8	.0205	.0329	.0121	.0511	.0064	.0773	.0026	.1134
3.9	.0197	.0315	.0116	.0487	.0062	.0734	.0025	.1076
4	.0189	.0302	.0112	.0465	.0060	.0699	.0024	.1022
4.1	.0182	.0289	.0108	.0445	.0058	.0665	.0023	.0971
4.2	.0175	.0278	.0105	.0425	.0056	.0635	.0023	.0924
4.3	.0169	.0267	.0101	.0407	.0054	.0606	.0022	.0880
4.4	.0163	.0257	.0098	.0390	.0053	.0579	.0022	.0839
4.5	.0157	.0247	.0095	.0374	.0051	.0554	.0021	.0801
4.6	.0152	.0238	.0092	.0359	.0050	.0530	.0020	.0765
4.7	.0147	.0229	.0089	.0345	.0049	.0508	.0020	.0731
4.8	.0142	.0221	.0086	.0332	.0047	.0487	.0019	.0700
4.9	.0138	.0213	.0084	.0319	.0046	.0467	.0019	.0670
5	.0133	.0206	.0081	.0308	.0045	.0449	.0018	.0642

$\alpha = 1.00$

x	$\beta = 0.00$	−0.25	0.25	−0.50	0.50	−0.75	0.75	−1.00	1.00
0.0	.3183	.3096	.3096	.2925	.2925	.2761	.2761	.2622	.2622
.1	.3152	.3147	.2999	.3011	.2814	.2849	.2657	.2702	.2532
.2	.3061	.3143	.2867	.3061	.2685	.2915	.2543	.2768	.2434
.3	.2920	.3077	.2711	.3069	.2545	.2954	.2423	.2814	.2332
.4	.2744	.2953	.2543	.3025	.2400	.2958	.2301	.2836	.2228
.5	.2547	.2776	.2369	.2926	.2254	.2921	.2178	.2830	.2123
.6	.2340	.2562	.2197	.2772	.2111	.2838	.2057	.2789	.2020
.7	.2136	.2327	.2030	.2568	.1973	.2705	.1940	.2710	.1919
.8	.1941	.2087	.1872	.2325	.1841	.2523	.1827	.2588	.1820
.9	.1759	.1855	.1724	.2061	.1716	.2297	.1719	.2424	.1726
1	.1591	.1640	.1587	.1793	.1599	.2036	.1617	.2218	.1635
1.1	.1440	.1447	.1461	.1537	.1490	.1754	.1520	.1974	.1549
1.2	.1305	.1277	.1346	.1306	.1389	.1469	.1430	.1701	.1467
1.3	.1183	.1130	.1241	.1107	.1295	.1197	.1344	.1411	.1389
1.4	.1075	.1002	.1145	.0939	.1208	.0955	.1265	.1120	.1315
1.5	.0979	.0892	.1058	.0801	.1128	.0751	.1190	.0845	.1246
1.6	.0894	.0798	.0979	.0689	.1054	.0589	.1121	.0599	.1180
1.7	.0818	.0716	.0907	.0597	.0986	.0465	.1056	.0396	.1119
1.8	.0751	.0646	.0842	.0522	.0923	.0374	.0996	.0240	.1061
1.9	.0691	.0585	.0783	.0460	.0865	.0307	.0940	.0132	.1006
2	.0637	.0531	.0729	.0409	.0812	.0258	.0887	.0065	.0955
2.1	.0588	.0485	.0680	.0365	.0763	.0220	.0838	.0028	.0907
2.2	.0545	.0444	.0636	.0329	.0718	.0191	.0793	.0010	.0862
2.3	.0506	.0408	.0595	.0298	.0676	.0168	.0751	.0003	.0820
2.4	.0471	.0376	.0557	.0271	.0637	.0150	.0711		.0780
2.5	.0439	.0348	.0523	.0247	.0602	.0134	.0674		.0742
2.6	.0410	.0322	.0492	.0227	.0568	.0121	.0640		.0707
2.7	.0384	.0300	.0463	.0209	.0538	.0110	.0608		.0674
2.8	.0360	.0279	.0437	.0193	.0509	.0101	.0578		.0643
2.9	.0338	.0261	.0412	.0179	.0483	.0092	.0550		.0614
3	.0318	.0244	.0390	.0166	.0458	.0085	.0523		.0586
3.1	.0300	.0229	.0369	.0155	.0435	.0079	.0499		.0560
3.2	.0283	.0215	.0349	.0145	.0414	.0073	.0476		.0536
3.3	.0268	.0202	.0331	.0136	.0394	.0068	.0454		.0513
3.4	.0253	.0191	.0315	.0128	.0375	.0064	.0434		.0491
3.5	.0240	.0180	.0299	.0120	.0358	.0060	.0415		.0471
3.6	.0228	.0171	.0285	.0113	.0341	.0056	.0397		.0451
3.7	.0217	.0162	.0271	.0107	.0326	.0053	.0380		.0433
3.8	.0206	.0153	.0259	.0101	.0312	.0050	.0364		.0416
3.9	.0196	.0146	.0247	.0096	.0298	.0047	.0349		.0399
4	.0187	.0139	.0236	.0091	.0285	.0045	.0335		.0384
4.1	.0179	.0132	.0226	.0086	.0273	.0042	.0321		.0369
4.2	.0171	.0126	.0216	.0082	.0262	.0040	.0309		.0355
4.3	.0163	.0120	.0207	.0078	.0252	.0038	.0297		.0342
4.4	.0156	.0115	.0199	.0075	.0242	.0036	.0285		.0329
4.5	.0150	.0110	.0191	.0072	.0232	.0035	.0275		.0317
4.6	.0144	.0105	.0183	.0068	.0223	.0033	.0264		.0306
4.7	.0138	.0101	.0176	.0065	.0215	.0032	.0255		.0295
4.8	.0132	.0097	.0169	.0063	.0207	.0030	.0246		.0285
4.9	.0127	.0093	.0163	.0060	.0199	.0029	.0237		.0275
5	.0122	.0089	.0157	.0058	.0192	.0028	.0228		.0266

$\alpha = 1.25$

x	$\beta = 0.00$	0.25	−0.25	0.50	−0.50	0.75	−0.75	1.00	−1.00
0.0	.2965	.2375	.2375	.1578	.1578	.1090	.1090	.0808	.0808
.1	.2949*	.2237	.2507	.1472	.1690	.1022	.1163	.0763	.0856
.2	.2901	.2097	.2629	.1371	.1807	.0958	.1241	.0721	.0907
.3	.2827	.1957	.2738	.1277	.1928	.0899	.1323	.0681	.0962
.4	.2727	.1821	.2828	.1188	.2025	.0843	.1410	.0644	.1020
.5	.2606	.1689	.2896	.1105	.2177	.0791	.1502	.0609	.1082
.6	.2469	.1563	.2937	.1029	.2301	.0743	.1599	.0577	.1147
.7	.2320	.1444	.2950	.0957	.2422	.0699	.1700	.0547	.1216
.8	.2166	.1333	.2933	.0891	.2537	.0657	.1805	.0518	.1290
.9	.2009	.1229	.2884	.0830	.2643	.0618	.1913	.0491	.1367
1	.1854	.1133	.2806	.0774	.2736	.0582	.2023	.0466	.1448
1.1	.1703	.1044	.2700	.0722	.2813	.0549	.2134	.0443	.1533
1.2	.1560	.0962	.2570	.0674	.2869	.0517	.2245	.0421	.1621
1.3	.1424	.0887	.2421	.0630	.2903	.0488	.2354	.0400	.1713
1.4	.1298	.0819	.2259	.0589	.2910	.0461	.2459	.0381	.1808
1.5	.1182	.0757	.2089	.0552	.2889	.0436	.2557	.0362	.1905
1.6	.1075	.0699	.1915	.0517	.2838	.0412	.2646	.0345	.2004
1.7	.0979	.0647	.1744	.0485	.2758	.0391	.2723	.0329	.2104
1.8	.0891	.0600	.1579	.0455	.2650	.0370	.2784	.0314	.2203
1.9	.0811	.0557	.1422	.0428	.2516	.0351	.2828	.0300	.2300
2	.0740	.0517	.1276	.0403	.2361	.0333	.2850	.0286	.2394
2.1	.0676	.0481	.1142	.0379	.2189	.0316	.2849	.0274	.2483
2.2	.0618	.0448	.1021	.0357	.2007	.0301	.2821	.0262	.2565
2.3	.0566	.0418	.0912	.0337	.1820	.0286	.2767	.0250	.2637
2.4	.0519	.0390	.0815	.0319	.1633	.0272	.2684	.0240	.2697
2.5	.0477	.0365	.0729	.0301	.1452	.0260	.2575	.0230	.2743
2.6	.0440	.0342	.0653	.0285	.1281	.0248	.2440	.0220	.2772
2.7	.0406	.0321	.0587	.0270	.1123	.0236	.2282	.0211	.2782
2.8	.0375	.0301	.0528	.0256	.0980	.0226	.2106	.0203	.2770
2.9	.0348	.0283	.0476	.0243	.0852	.0215	.1917	.0195	.2734
3	.0323	.0267	.0431	.0231	.074	.0206	.172	.0187	.2674
3.1	.0300	.0251	.0391	.0220	.0643	.0197	.1522	.0180	.2588
3.2	.0280	.0237	.0356	.0209	.0559	.0189	.1327	.0173	.2476
3.3	.0261	.0224	.0325	.0199	.0488	.0181	.1142	.0166	.2340
3.4	.0244	.0212	.0298	.0190	.0427	.0173	.0971	.0160	.2182
3.5	.0228	.0201	.0273	.0181	.0376	.0166	.0816	.0154	.2005
3.6	.0214	.0190	.0251	.0173	.0333	.0160	.0679	.0149	.1813
3.7	.0201	.0180	.0232	.0165	.0296	.0153	.0562	.0143	.1612
3.8	.0189	.0171	.0215	.0158	.0264	.0147	.0463	.0138	.1407
3.9	.0178	.0163	.0199	.0151	.0237	.0142	.0382	.0133	.1203
4	.0168	.0155	.0185	.0145	.0213	.0136	.0315	.0129	.1007
4.1	.0158	.0148	.0172	.0139	.0193	.0131	.0262	.0124	.0824
4.2	.0150	.0141	.0161	.0133	.0176	.0126	.0219	.0120	.0657
4.3	.0142	.0134	.0150	.0128	.0161	.0122	.0186	.0116	.0510
4.4	.0134	.0128	.0141	.0123	.0147	.0117	.0159	.0113	.0385
4.5	.0128	.0123	.0132	.0118	.0135	.0113	.0137	.0109	.0282
4.6	.0121	.0117	.0124	.0113	.0125	.0109	.0120	.0105	.0200
4.7	.0115	.0112	.0117	.0109	.0116	.0105	.0106	.0102	.0137
4.8	.0110	.0108	.0110	.0105	.0107	.0102	.0094	.0099	.0090
4.9	.0105	.0103	.0104	.0101	.0100	.0098	.0085	.0096	.0057
5	.0100	.0099	.0099	.0097	.0093	.0095	.0077	.0093	.0035

$\alpha = 1.50$

x	$\beta = 0.00$	0.25	−0.25	0.50	−0.50	0.75	−0.75	1.00	−1.00
0.0	.2873	.2778	.2778	.2541	.2541	.2252	.2252	.1975	.1975
.1	.2863	.2712	.2828	.2442	.2630	.2144	.2356	.1872	.2078
.2	.2831	.2629	.2858	.2334	.2707	.2034	.2455	.1770	.2180
.3	.2780	.2533	.2868	.2221	.2769	.1923	.2546	.1669	.2280
.4	.2710	.2426	.2857	.2104	.2815	.1812	.2627	.1572	.2375
.5	.2623	.2310	.2824	.1986	.2843	.1704	.2697	.1477	.2465
.6	.2521	.2189	.2772	.1867	.2851	.1598	.2754	.1386	.2548
.7	.2408	.2063	.2700	.1749	.2839	.1495	.2795	.1299	.2621
.8	.2285	.1936	.2610	.1634	.2806	.1397	.2819	.1216	.2683
.9	.2155	.1808	.2505	.1522	.2753	.1303	.2825	.1137	.2733
1	.2020	.1683	.2386	.1415	.2680	.1214	.2811	.1062	.2769
1.1	.1884	.1561	.2257	.1313	.2589	.1129	.2777	.0992	.2788
1.2	.1748	.1443	.2119	.1216	.2481	.1050	.2724	.0926	.2790
1.3	.1615	.1331	.1977	.1124	.2359	.0975	.2650	.0865	.2774
1.4	.1486	.1224	.1832	.1038	.2225	.0905	.2559	.0807	.2738
1.5	.1361	.1124	.1688	.0958	.2082	.0840	.2450	.0753	.2684
1.6	.1243	.1031	.1546	.0884	.1933	.0780	.2325	.0702	.2610
1.7	.1133	.0944	.1409	.0815	.1781	.0723	.2188	.0655	.2518
1.8	.1029	.0863	.1278	.0751	.1628	.0671	.2041	.0612	.2408
1.9	.0933	.0789	.1154	.0692	.1478	.0623	.1887	.0571	.2283
2	.0845	.0721	.1038	.0638	.1333	.0579	.1729	.0534	.2145
2.1	.0765	.0660	.0931	.0589	.1194	.0538	.1569	.0499	.1995
2.2	.0692	.0603	.0833	.0543	.1064	.0500	.1412	.0467	.1838
2.3	.0625	.0551	.0743	.0501	.0942	.0465	.1259	.0437	.1675
2.4	.0565	.0505	.0663	.0463	.0831	.0433	.1112	.0409	.1510
2.5	.0511	.0462	.0590	.0428	.0729	.0403	.0974	.0383	.1346
2.6	.0463	.0424	.0526	.0396	.0639	.0376	.0847	.0359	.1186
2.7	.0420	.0389	.0468	.0367	.0558	.0351	.0730	.0337	.1033
2.8	.0381	.0358	.0418	.0341	.0486	.0328	.0625	.0317	.0888
2.9	.0346	.0329	.0373	.0316	.0423	.0306	.0531	.0298	.0753
3	.0315	.0303	.0333	.0294	.0369	.0287	.0450	.0280	.0631
3.1	.0287	.0280	.0299	.0274	.0322	.0268	.0378	.0264	.0521
3.2	.0262	.0258	.0268	.0255	.0281	.0252	.0318	.0248	.0424
3.3	.0240	.0239	.0242	.0238	.0246	.0236	.0266	.0234	.0340
3.4	.0220	.0222	.0218	.0222	.0216	.0222	.0223	.0221	.0269
3.5	.0203	.0206	.0197	.0208	.0190	.0209	.0186	.0209	.0209
3.6	.0186	.0191	.0179	.0194	.0168	.0196	.0157	.0197	.0160
3.7	.0172	.0178	.0163	.0182	.0149	.0185	.0132	.0187	.0121
3.8	.0159	.0166	.0149	.0171	.0133	.0174	.0112	.0177	.0089
3.9	.0147	.0155	.0136	.0161	.0120	.0165	.0095	.0167	.0065
4	.0137	.0145	.0125	.0151	.0107	.0156	.0082	.0159	.0047
4.1	.0127	.0136	.0115	.0142	.0097	.0147	.0071	.0151	.0033
4.2	.0118	.0128	.0106	.0134	.0088	.0139	.0062	.0143	.0023
4.3	.0110	.0120	.0098	.0127	.0080	.0132	.0055	.0136	.0015
4.4	.0103	.0113	.0091	.0120	.0073	.0125	.0048	.0130	.0010
4.5	.0097	.0106	.0084	.0113	.0067	.0119	.0043	.0123	.0007
4.6	.0091	.0100	.0078	.0107	.0062	.0113	.0039	.0118	.0004
4.7	.0085	.0094	.0073	.0102	.0057	.0107	.0035	.0112	.0003
4.8	.0080	.0089	.0068	.0096	.0053	.0102	.0032	.0107	.0002
4.9	.0075	.0084	.0064	.0092	.0049	.0097	.0029	.0102	.0001
5	.0071	.0080	.0060	.0087	.0046	.0093	.0027	.0098	

$\alpha = 1.75$

x	$\beta = 0.00$	0.25	−0.25	0.50	−0.50	0.75	−0.75	1.00	−1.00
0.0	.2835	.2821	.2821	.2782	.2782	.2720	.2720	.2642	.2642
.1	.2824	.2793	.2833	.2736	.2813	.2660	.2768	.2569	.2703
.2	.2800	.2750	.2828	.2677	.2828	.2587	.2801	.2486	.2752
.3	.2761	.2692	.2807	.2604	.2827	.2503	.2819	.2395	.2787
.4	.2706	.2621	.2770	.2520	.2809	.2410	.2821	.2297	.2808
.5	.2636	.2537	.2717	.2427	.2775	.2310	.2807	.2193	.2813
.6	.2553	.2443	.2649	.2325	.2725	.2204	.2776	.2086	.2802
.7	.2459	.2340	.2568	.2216	.2660	.2094	.2730	.1976	.2775
.8	.2355	.2229	.2474	.2103	.2581	.1981	.2668	.1865	.2731
.9	.2244	.2114	.2371	.1986	.2489	.1866	.2591	.1754	.2672
1	.2126	.1994	.2258	.1868	.2385	.1751	.2501	.1644	.2599
1.1	.2003	.1872	.2138	.1750	.2272	.1637	.2399	.1536	.2511
1.2	.1878	.1750	.2013	.1632	.2152	.1526	.2286	.1431	.2411
1.3	.1752	.1628	.1885	.1516	.2025	.1417	.2165	.1330	.2299
1.4	.1626	.1508	.1755	.1404	.1894	.1312	.2037	.1232	.2178
1.5	.1502	.1392	.1626	.1295	.1761	.1212	.1904	.1139	.2050
1.6	.1381	.1279	.1498	.1191	.1628	.1116	.1768	.1051	.1915
1.7	.1265	.1172	.1373	.1092	.1495	.1025	.1632	.0968	.1777
1.8	.1153	.1070	.1252	.0999	.1366	.0940	.1495	.0890	.1637
1.9	.1047	.0973	.1136	.0911	.1241	.0860	.1362	.0817	.1498
2	.0948	.0883	.1026	.0830	.1121	.0785	.1232	.0749	.1360
2.1	.0855	.0799	.0923	.0754	.1007	.0716	.1107	.0685	.1225
2.2	.0768	.0721	.0827	.0683	.0899	.0653	.0989	.0627	.1096
2.3	.0689	.0650	.0738	.0619	.0800	.0594	.0877	.0573	.0973
2.4	.0616	.0585	.0656	.0560	.0707	.0540	.0773	.0524	.0857
2.5	.0549	.0525	.0581	.0506	.0623	.0491	.0678	.0478	.0749
2.6	.0489	.0471	.0514	.0457	.0546	.0446	.0590	.0437	.0649
2.7	.0435	.0422	.0453	.0413	.0477	.0405	.0511	.0399	.0558
2.8	.0386	.0378	.0398	.0372	.0415	.0368	.0440	.0364	.0477
2.9	.0343	.0339	.0350	.0336	.0360	.0334	.0377	.0333	.0403
3	.0304	.0303	.0307	.0303	.0312	.0304	.0321	.0305	.0338
3.1	.0270	.0272	.0269	.0274	.0269	.0276	.0273	.0279	.0282
3.2	.0239	.0244	.0235	.0248	.0232	.0252	.0230	.0255	.0232
3.3	.0213	.0219	.0206	.0224	.0199	.0229	.0194	.0234	.0190
3.4	.0189	.0197	.0181	.0203	.0172	.0209	.0163	.0215	.0154
3.5	.0168	.0177	.0159	.0185	.0148	.0191	.0136	.0197	.0124
3.6	.0150	.0159	.0139	.0168	.0127	.0175	.0113	.0181	.0099
3.7	.0134	.0144	.0123	.0153	.0110	.0160	.0095	.0167	.0078
3.8	.0120	.0130	.0108	.0139	.0095	.0147	.0079	.0154	.0061
3.9	.0108	.0118	.0096	.0127	.0082	.0135	.0066	.0142	.0047
4	.0097	.0107	.0085	.0116	.0071	.0124	.0055	.0131	.0036
4.1	.0087	.0098	.0076	.0106	.0062	.0114	.0046	.0121	.0028
4.2	.0079	.0089	.0068	.0098	.0055	.0106	.0039	.0113	.0021
4.3	.0072	.0081	.0061	.0090	.0048	.0098	.0033	.0104	.0016
4.4	.0065	.0075	.0055	.0083	.0042	.0090	.0028	.0097	.0012
4.5	.0060	.0069	.0049	.0077	.0038	.0084	.0024	.0090	.0008
4.6	.0054	.0063	.0045	.0071	.0034	.0078	.0021	.0084	.0006
4.7	.0050	.0058	.0041	.0066	.0030	.0072	.0018	.0079	.0004
4.8	.0046	.0054	.0037	.0061	.0027	.0068	.0016	.0073	.0003
4.9	.0042	.0050	.0034	.0057	.0025	.0063	.0014	.0069	.0002
5	.0039	.0046	.0031	.0053	.0023	.0059	.0013	.0064	.0001

A.2. One-sided distribution functions $G^B(x; \alpha, 1)$ multiplied by 10^4 (Bolshev *et al.*, 1970)

x	α						x	α					
	0.2	0.3	0.4	0.5	0.6	0.7		0.2	0.3	0.4	0.5	0.6	0.7
.01	963	179	0	0	0	0	.51	3594	3522	3409	3221	2893	2264
.02	1328	433	33	0	0	0	.52	3608	3546	3442	3268	2958	2358
.03	1563	658	104	0	0	0	.53	3623	3563	3475	3314	3023	2452
.04	1749	854	203	4	0	0	.54	3637	3591	3508	3359	3087	2544
.05	1895	1026	315	16	0	0	.55	3651	3613	3539	3404	3149	2635
.06	2018	1178	434	39	0	0	.56	3665	3635	3571	3447	3210	2724
.07	2124	1316	555	75	0	0	.57	3679	3656	3601	3490	3270	2812
.08	2217	1440	675	124	0	0	.58	3692	3677	3631	3532	3329	2898
.09	2300	1555	792	184	0	0	.59	3705	3697	3661	3573	3387	2983
.1	2375	1660	906	253	6	0	.6	3718	3718	3690	3613	3443	3067
.11	2444	1757	1046	330	13	0	.61	3739	3737	3718	3653	3499	3148
.12	2507	1848	1122	412	25	0	.62	3743	3757	3746	3692	3554	3229
.13	2565	1933	1224	499	44	0	.63	3755	3776	3774	3730	3607	3307
.14	2619	2013	1323	588	71	0	.64	3767	3795	3801	3768	3660	3385
.15	2670	2089	1417	679	105	0	.65	3779	3813	3827	3805	3712	3461
.16	2718	2160	1509	771	146	0	.66	3790	3832	3854	3841	3762	3535
.17	2763	2228	1596	863	195	0	.67	3802	3850	3879	3877	3812	3608
.18	2805	2292	1681	956	251	1	.68	3813	3867	3905	3912	3861	3680
.19	2845	2354	1762	1048	313	3	.69	3824	3885	3930	3946	3909	3750
.2	2884	2412	1841	1138	381	7	.7	3835	3902	3954	3980	3956	3819
.21	2920	2463	1917	1228	453	14	.71	3846	3919	3978	4014	4003	3837
.22	2955	2522	1990	1317	530	24	.72	3857	3936	4002	4047	4048	3953
.23	2989	2574	2061	1404	610	39	.73	3867	3952	4026	4079	4093	4018
.24	3021	2623	2129	1489	693	59	.74	3877	3968	4049	4111	4137	4081
.25	3052	2671	2196	1573	778	85	.75	3888	3984	4072	4142	4180	4144
.26	3081	2717	2260	1655	865	118	.76	3898	4000	4094	4173	4223	4205
.27	3110	2762	2322	1736	953	158	.77	3908	4016	4116	4203	4265	4265
.28	3137	2804	2382	1815	1042	205	.78	3917	4031	4138	4233	4306	4324
.29	3464	2846	2441	1892	1131	259	.79	3927	4046	4160	4263	4346	4381
.3	3190	2886	2498	1967	1221	319	.8	3937	4061	4181	4292	4386	4438
.31	3214	2925	2553	2041	1310	385	.81	3946	4076	4202	4321	4425	4494
.32	3239	2963	2606	2113	1399	457	.82	3955	4090	4222	4349	4464	4548
.33	3262	3000	2659	2184	1488	534	.83	3965	4105	4243	4377	4502	4601
.34	3285	3035	2709	2253	1576	616	.84	3974	4119	4263	4404	4539	4654
.35	3307	3070	2759	2320	1663	702	.85	3983	4133	4283	4431	4576	4705
.36	3328	3104	2807	2386	1749	791	.86	3991	4147	4302	4458	4612	4756
.37	3349	3136	2854	2450	1834	883	.87	4000	4161	4322	4484	4647	4805
.38	3369	3168	2900	2513	1918	978	.88	4009	4174	4341	4510	4682	4854
.39	3389	3200	2945	2575	2001	1075	.89	4017	4187	4359	4535	4716	4901
.4	3408	3230	2988	2636	2082	1173	.9	4026	4201	4378	4561	4750	4948
.41	3427	3260	3031	2695	2162	1272	.91	4034	4214	4396	4585	4784	4994
.42	3446	3289	3072	2752	2241	1372	.92	4043	4227	4415	4610	4817	5039
.43	3464	3317	3112	2809	2319	1473	.93	4051	4239	4433	4634	4849	5083
.44	3481	3345	3153	2864	2395	1573	.94	4059	4252	4450	4658	4881	5127
.45	3498	3372	3192	2918	2470	1674	.95	4067	4264	4468	4682	4912	5169
.46	3515	3398	3230	2971	2544	1774	.96	4075	4277	4485	4705	4943	5211
.47	3531	3424	3267	3023	2616	1874	.97	4033	4289	4502	4728	4974	5253
.48	3547	3449	3304	3074	2687	1973	.98	4090	4301	4519	4750	5004	5293
.49	3563	3474	3339	3124	2757	2071	.99	4098	4313	4536	4773	5033	5333
.5	3579	3498	3374	3173	2825	2168	1	4106	4324	4552	4795	5063	5372

A.3. One-sided distributions represented by function $F(y;\alpha) = 10^4 G^B(y^{-1/\alpha};\alpha,1)$ ($F(y;0) \equiv 10^4 e^{-y}$) (Bolshev *et al.*, 1970)

y	α							
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
.00	10000	10000	10000	10000	10000	10000	10000	10000
.01	9900	9907	9914	9923	9933	9944	9955	9966
.02	9802	9815	9830	9847	9866	9887	9909	9933
.03	9704	9723	9745	9771	9800	9831	9864	9899
.04	9608	9632	9662	9695	9733	9774	9818	9864
.05	9512	9543	9579	9620	9667	9718	9772	9829
.06	9418	9454	9497	9546	9601	9662	9726	9795
.07	9324	9366	9415	9472	9535	9605	9680	9759
.08	9231	9278	9334	9398	9470	9549	9634	9724
.09	9139	9192	9254	9325	9405	9493	9588	9688
.10	9048	9106	9174	9252	9340	9436	9541	9652
.11	8958	9021	9095	9180	9275	9380	9494	9616
.12	8869	8937	9016	9108	9210	9324	9447	9579
.13	8781	8853	8938	9036	9146	9268	9400	9542
.14	8694	8771	8861	8965	9082	9211	9353	9505
.15	8607	8689	8785	8895	9018	9155	9305	9468
.16	8521	8608	8709	8824	8955	9099	9258	9430
.17	8437	8527	8633	8755	8891	9043	9210	9392
.18	8353	8448	8558	8685	8828	8987	9163	9354
.19	8270	8369	8484	8616	8765	8931	9115	9315
.20	8187	8290	8411	8548	8703	8875	9067	9277
.21	8106	8213	8338	8480	8640	8820	9018	9237
.22	8025	8136	8265	8412	8578	8764	8970	9198
.23	7945	8060	8193	8345	8516	8708	8922	9158
.24	7866	7985	8122	8278	8455	8652	8873	9119
.25	7788	7910	8051	8212	8393	8597	8824	9078
.26	7711	7836	7981	8146	8332	8541	8776	9038
.27	7634	7763	7911	8080	8271	8486	8727	8997
.28	7558	7690	7842	8015	8211	8431	8678	8956
.29	7483	7618	7774	7951	8150	8375	8629	8915
.30	7408	7547	7706	7886	8090	8320	8579	8873
.31	7334	7476	7638	7823	8030	8265	8530	8831
.32	7261	7406	7572	7759	7971	8210	8481	8789
.33	7189	7337	7505	7696	7912	8155	8431	8746
.34	7118	7268	7439	7634	7853	8100	8381	8704
.35	7047	7200	7374	7571	7794	8045	8332	8661
.36	6977	7132	7309	7510	7735	7991	8282	8617
.37	6907	7065	7245	7448	7677	7936	8232	8574
.38	6839	6999	7182	7387	7619	7882	8182	8530
.39	6771	6934	7118	7327	7516	7827	8132	8486
.40	6703	6869	7056	7267	7504	7773	8082	8441
.41	6637	6804	6994	7207	7447	7719	8032	8397
.42	6570	6740	6932	4147	7390	7665	7981	8352
.43	6505	6677	6871	7088	7333	7611	7931	8307
.44	6440	6614	6810	7029	7276	7557	7881	8261
.45	6376	6552	6750	6971	7220	7503	7830	8215
.46	6313	6491	6690	6913	7164	7450	7780	8169
.47	6250	6430	6631	6856	7109	7396	7729	8123
.48	6188	6369	6572	6799	7053	7343	7678	8076
.49	6126	6310	6514	6742	6998	7290	7628	8030

y	α							
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
.50	6065	6250	6456	6686	6944	7237	7577	7983
.51	6005	6192	6399	6630	6889	7184	7526	7935
.52	5945	6133	6342	6575	6835	7131	7475	7883
.53	5886	6076	6286	6519	6781	7078	7425	7840
.54	5827	6019	6230	6465	6727	7026	7374	7792
.55	5769	5962	6175	6410	6674	6973	7323	7744
.56	5712	5906	6120	6356	6620	6921	7272	7695
.57	5655	5851	6065	6303	6563	6869	7221	7646
.58	5599	5795	6011	6249	6513	6817	7170	7597
.59	5543	5741	5958	6196	6461	6765	7119	7548
.60	5488	5687	5904	6144	6415	6714	7068	7498
.61	5434	5633	5852	6092	6359	6672	7017	7449
.62	5379	5580	5799	6040	6307	6611	6966	7399
.63	5326	5528	5148	5988	6256	6560	6915	7348
.64	5273	5476	5696	5937	6205	6509	6864	7298
.65	5220	5424	5645	5887	6154	6458	6813	7247
.66	5169	5373	5595	5836	6104	6407	6762	7197
.67	5117	5323	5544	5786	6054	6357	6711	7145
.68	5066	5272	5495	5736	6004	6206	6660	7094
.69	5016	5223	5445	5687	5954	6256	6609	7043
.70	4966	5174	5396	5638	5905	6206	6559	6991
.71	4916	5125	5348	5590	5856	6156	6508	6939
.72	4868	5076	5300	5541	5807	6107	6457	6887
.73	4819	5029	5252	5493	5759	6057	6406	6835
.74	4771	4981	5205	5446	5710	6008	6355	6783
.75	4724	4934	5158	5399	5662	5959	6305	6730
.76	4677	4888	5111	5352	5615	5910	6254	6677
.77	4630	4841	5065	5305	5567	5861	6203	6624
.78	4584	4796	5020	5259	5520	5813	6153	6571
.79	4538	4751	4974	5213	5473	5764	6102	6518
.80	4493	4706	4929	5168	5427	5716	6052	6464
.81	4449	4661	4885	5122	5380	5668	6002	6411
.82	4404	4617	4840	5078	5334	5620	5952	6357
.83	4360	4574	4797	5033	5289	5573	5901	6303
.84	4317	4530	4753	4989	5243	5525	5851	6250
.85	4274	4488	4710	4945	5198	5478	5801	6195
.86	4232	4445	4667	4901	5153	5431	5751	6141
.87	4190	4403	4625	4858	5108	5384	5701	6086
.88	4148	4362	4583	4815	5064	5338	5651	6032
.89	4107	4320	4541	4773	5020	5291	5602	5977
.90	4066	4280	4500	4730	4976	5245	5552	5923
.91	4025	4239	4459	4688	4932	5199	5503	5868
.92	3985	4199	4418	4647	4889	5153	5453	5813
.93	3946	4159	4378	4605	4846	5108	5404	5758
.94	3906	4120	4338	4564	4803	5063	5355	5703
.95	3867	4081	4299	4524	4761	5017	5306	5648
.96	3829	4042	4259	4483	4718	4973	5257	5593
.97	3791	4004	4220	4443	4676	4928	5208	5538
.98	3753	3966	4182	4403	4635	4883	5160	5482
.99	3716	3929	4144	4364	4593	4839	5111	5427
1.00	3679	3891	4106	4325	4552	4795	5063	5372

A.4. The function $\alpha^{1/\alpha}q(\alpha^{1/\alpha}x;\alpha)$, where $q(x;\alpha)$ is the one-dimensional symmetric stable density (Worsdale, 1976)

x	α										
	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
.00	.3183	.3349	.3485	.3597	.3689	.3765	.3828	.3881	.3924	.3960	.3989
.10	.3152	.3320	.3458	.3571	.3664	.3742	.3806	.3859	.3903	.3939	.3970
.20	.3061	.3234	.3377	.3494	.3591	.3671	.3738	.3794	.3840	.3878	.3910
.30	.2920	.3100	.3249	.3371	.3473	.3558	.3629	.3688	.3737	.3779	.3814
.40	.2744	.2928	.3082	.3210	.3317	.3407	.3482	.3545	.3599	.3644	.3683
.50	.2546	.2731	.3886	.3018	.3129	.3224	.3304	.3371	.3429	.3479	.3521
.60	.2340	.2519	.2674	.2806	.2919	.3017	.3100	.3172	.3233	.3286	.3332
.70	.2136	.2305	.2453	.2582	.2695	.2793	.2879	.2953	.3017	.3074	.3123
.80	.1941	.2095	.2233	.2350	.2465	.2561	.2646	.2721	.2787	.2846	.2897
.90	.1759	.1896	.2021	.2134	.2236	.2328	.2410	.2484	.2550	.2608	.2661
1.00	.1592	.1711	.1821	.1921	.2014	.2098	.2176	.2246	.2310	.2367	.2420
1.05	.1514	.1624	.1726	.1820	.1907	.1987	.2061	.2129	.2191	.2247	.2299
1.10	.1440	.1542	.1635	.1722	.1803	.1878	.1948	.2013	.2073	.2128	.2179
1.15	.1370	.1463	.1548	.1628	.1703	.1773	.1839	.1900	.1957	.2010	.2059
1.20	.1304	.1389	.1466	.1538	.1606	.1671	.1732	.1790	.1844	.1895	.1942
1.25	.1242	.1318	.1388	.1453	.1514	.1573	.1629	.1683	.1734	.1781	.1826
1.30	.1183	.1251	.1313	.1371	.1426	.1479	.1530	.1579	.1626	.1671	.1714
1.35	.1128	.1188	.1243	.1294	.1343	.1390	.1435	.1480	.1523	.1564	.1604
1.40	.1075	.1129	.1177	.1221	.1263	.1304	.1344	.1384	.1424	.1461	.1497
1.45	.1026	.1073	.1114	.1115	.1188	.1223	.1258	.1293	.1327	.1361	.1394
1.50	.0979	.1020	.1056	.1087	.1117	.1146	.1176	.1205	.1235	.1265	.1295
1.55	.0936	.0971	.1000	.1026	.1050	.1074	.1098	.1123	.1148	.1174	.1200
1.60	.0894	.0924	.0948	.0969	.0987	.1006	.1025	.1044	.1065	.1087	.1109
1.65	.0855	.0880	.0899	.0915	.0928	.0942	.0955	.0970	.0987	.1004	.1023
1.70	.0818	.0839	.0853	.0864	.0873	.0881	.0891	.0901	.0913	.0926	.0940
1.75	.0783	.0800	.0810	.0816	.0821	.0825	.0830	.0835	.0843	.0852	.0863
1.80	.0751	.0763	.0769	.0772	.0772	.0772	.0773	.0774	.0777	.0783	.0790
1.85	.0720	.0729	.0731	.0730	.0727	.0723	.0720	.0717	.0716	.0717	.0721
1.90	.0690	.0696	.0696	.0691	.0685	.0677	.0670	.0664	.0659	.0657	.0656
1.95	.0663	.0665	.0662	.0655	.0645	.0634	.0624	.0614	.0606	.0600	.0596
2.00	.0637	.0636	.0630	.0620	.0608	.0595	.0581	.0568	.0557	.0547	.0540
2.05	.0612	.0609	.0601	.0588	.0574	.0557	.0541	.0525	.0511	.0498	.0488
2.10	.0588	.0584	.0573	.0558	.0541	.0523	.0504	.0486	.0469	.0453	.0440
2.15	.0566	.0559	.0547	.0530	.0511	.0491	.0470	.0449	.0430	.0412	.0396
2.20	.0545	.0536	.0522	.0504	.0483	.0461	.0438	.0416	.0394	.0373	.0355
2.25	.0525	.0515	.0499	.0479	.0457	.0433	.0409	.0385	.0361	.0338	.0317
2.30	.0506	.0494	.0477	.0456	.0433	.0408	.0382	.0356	.0331	.0306	.0283
2.35	.0488	.0475	.0456	.0434	.0410	.0384	.0357	.0330	.0303	.0277	.0252
2.40	.0471	.0456	.0437	.0414	.0388	.0361	.0334	.0305	.0277	.0250	.0224
2.45	.0453	.0439	.0418	.0395	.0368	.0341	.0312	.0283	.0254	.0226	.0198
2.50	.0439	.0422	.0401	.0376	.0350	.0322	.0292	.0263	.0233	.0204	.0175
2.55	.0424	.0407	.0385	.0360	.0332	.0304	.0274	.0244	.0214	.0184	.0154
2.60	.0410	.0392	.0369	.0344	.0316	.0287	.0257	.0227	.0196	.0166	.0136
2.65	.0397	.0378	.0354	.0329	.0301	.0272	.0242	.0211	.0180	.0150	.0119
2.70	.0384	.0364	.0341	.0314	.0286	.0257	.0227	.0197	.0166	.0135	.0104
2.75	.0372	.0351	.0327	.0301	.0273	.0244	.0214	.0183	.0152	.0122	.0091
2.80	.0360	.0339	.0315	.0288	.0260	.0231	.0201	.0171	.0140	.0110	.0079
2.85	.0349	.0328	.0303	.0276	.0248	.0219	.0190	.0160	.0130	.0099	.0068
2.90	.0338	.0317	.0292	.0265	.0237	.0205	.0179	.0149	.0120	.0090	.0060
2.95	.0328	.0306	.0281	.0255	.0227	.0198	.0169	.0140	.0111	.0081	.0051
3.00	.0318	.0296	.0271	.0245	.0217	.0189	.0160	.0131	.0102	.0073	.0044

A.5. Radial functions $\rho_2(r;\alpha)$ of two-dimensional axially symmetric densities

r	α							
	0.2	0.4	0.6	0.8	1.2	1.4	1.6	1.8
0.0	2.888E+5	9.549E+0	7.369E-1	2.645E-1	1.197E-1	1.007E-1	9.016E-2	8.373E-2
.1	1.019E+0	1.166E+0	5.653E-1	2.531E-1	1.188E-1	1.002E-1	8.983E-2	8.348E-2
.2	2.737E-1	4.240E-1	3.623E-1	2.244E-1	1.161E-1	9.875E-2	8.885E-2	8.275E-2
.3	1.253E-1	2.178E-1	2.380E-1	1.891E-1	1.118E-1	9.635E-2	8.724E-2	8.154E-2
.4	7.155E-2	1.320E-1	1.641E-1	1.551E-1	1.062E-1	9.311E-2	8.504E-2	7.988E-2
.5	4.622E-2	8.816E-2	1.183E-1	1.258E-1	9.949E-2	8.913E-2	8.231E-2	7.780E-2
.6	3.228E-2	6.287E-2	8.838E-2	1.020E-1	9.210E-2	8.456E-2	7.911E-2	7.533E-2
.7	2.380E-2	4.698E-2	6.804E-2	8.302E-2	8.434E-2	7.953E-2	7.550E-2	7.252E-2
.8	1.827E-2	3.635E-2	5.367E-2	6.806E-2	7.649E-2	7.418E-2	7.157E-2	6.942E-2
.9	1.445E-2	2.891E-2	4.322E-2	5.627E-2	6.881E-2	6.866E-2	6.739E-2	6.607E-2
1	1.172E-2	2.350E-2	3.541E-2	4.691E-2	6.147E-2	6.309E-2	6.305E-2	6.253E-2
1.1	9.685E-3	1.946E-2	2.944E-2	3.945E-2	5.461E-2	5.759E-2	5.863E-2	5.885E-2
1.2	8.138E-3	1.635E-2	2.478E-2	3.344E-2	4.831E-2	5.224E-2	5.419E-2	5.509E-2
1.3	6.932E-3	1.392E-2	2.110E-2	2.857E-2	4.261E-2	4.714E-2	4.979E-2	5.128E-2
1.4	5.974E-3	1.198E-2	1.814E-2	2.459E-2	3.749E-2	4.233E-2	4.550E-2	4.748E-2
1.5	5.200E-3	1.041E-2	1.573E-2	2.130E-2	3.296E-2	3.785E-2	4.137E-2	4.374E-2
1.6	4.567E-3	9.125E-3	1.375E-2	1.857E-2	2.895E-2	3.372E-2	3.743E-2	4.008E-2
1.7	4.042E-3	8.057E-3	1.210E-2	1.628E-2	2.545E-2	2.996E-2	3.371E-2	3.655E-2
1.8	3.602E-3	7.161E-3	1.071E-2	1.436E-2	2.238E-2	2.655E-2	3.023E-2	3.316E-2
1.9	3.229E-3	6.403E-3	9.538E-3	1.272E-2	1.971E-2	2.349E-2	2.701E-2	2.995E-2
2	2.911E-3	5.755E-3	8.537E-3	1.133E-2	1.739E-2	2.076E-2	2.405E-2	2.692E-2
2.1	2.638E-3	5.199E-3	7.677E-3	1.013E-2	1.537E-2	1.834E-2	2.135E-2	2.409E-2
2.2	2.401E-3	4.717E-3	6.934E-3	9.091E-3	1.361E-2	1.619E-2	1.890E-2	2.147E-2
2.3	2.194E-3	4.298E-3	6.287E-3	8.193E-3	1.209E-2	1.430E-2	1.669E-2	1.906E-2
2.4	2.013E-3	3.930E-3	5.722E-3	7.410E-3	1.076E-2	1.264E-2	1.471E-2	1.685E-2
2.5	1.853E-3	3.606E-3	5.226E-3	6.724E-3	9.599E-3	1.118E-2	1.295E-2	1.484E-2
2.6	1.711E-3	3.320E-3	4.787E-3	6.121E-3	8.586E-3	9.900E-3	1.139E-2	1.303E-2
2.7	1.585E-3	3.065E-3	4.399E-3	5.588E-3	7.700E-3	8.781E-3	1.001E-2	1.141E-2
2.8	1.472E-3	2.838E-3	4.053E-3	5.116E-3	6.922E-3	7.802E-3	8.799E-3	9.956E-3
2.9	1.370E-3	2.634E-3	3.744E-3	4.696E-3	6.239E-3	6.946E-3	7.733E-3	8.668E-3
3	1.279E-3	2.450E-3	3.467E-3	4.321E-3	5.637E-3	6.195E-3	6.799E-3	7.530E-3
3.1	1.196E-3	2.285E-3	3.217E-3	3.986E-3	5.105E-3	5.537E-3	5.983E-3	6.529E-3
3.2	1.121E-3	2.135E-3	2.992E-3	3.684E-3	4.635E-3	4.960E-3	5.270E-3	5.652E-3
3.3	1.053E-3	1.999E-3	2.789E-3	3.413E-3	4.217E-3	4.453E-3	4.649E-3	4.888E-3
3.4	9.911E-4	1.875E-3	2.604E-3	3.168E-3	3.846E-3	4.007E-3	4.107E-3	4.224E-3
3.5	9.342E-4	1.762E-3	2.436E-3	2.946E-3	3.514E-3	3.613E-3	3.635E-3	3.648E-3
3.6	8.820E-4	1.659E-3	2.283E-3	2.744E-3	3.219E-3	3.265E-3	3.224E-3	3.151E-3
3.7	8.340E-4	1.564E-3	2.143E-3	2.561E-3	2.953E-3	2.958E-3	2.865E-3	2.723E-3
3.8	7.898E-4	1.477E-3	2.014E-3	2.394E-3	2.715E-3	2.685E-3	2.552E-3	2.355E-3
3.9	7.489E-4	1.396E-3	1.896E-3	2.241E-3	2.501E-3	2.443E-3	2.279E-3	2.039E-3
4	7.112E-4	1.322E-3	1.788E-3	2.101E-3	2.308E-3	2.227E-3	2.039E-3	1.768E-3

A.6. Radial functions $\rho_3(r;\alpha)$ of three-dimensional spherically symmetric densities (Gusarov, 1998)

r	α							
	0.2	0.4	0.6	0.8	1.2	1.4	1.6	1.8
0.0	2.208E+10	2.370E+2	2.026E+0	2.801E-1	5.612E-2	3.869E-2	3.019E-2	2.541E-2
.1	4.908E+0	4.641E+0	1.244E+0	2.618E-1	5.560E-2	3.847E-2	3.007E-2	2.533E-2
.2	6.682E-1	9.320E-1	5.903E-1	2.179E-1	5.408E-2	3.783E-2	2.972E-2	2.510E-2
.3	2.053E-1	3.331E-1	3.029E-1	1.685E-1	5.167E-2	3.679E-2	2.914E-2	2.472E-2
.4	8.834E-2	1.552E-1	1.708E-1	1.256E-1	4.852E-2	3.539E-2	2.835E-2	2.420E-2
.5	4.580E-2	8.440E-2	1.040E-1	9.252E-2	4.483E-2	3.368E-2	2.737E-2	2.355E-2
.6	2.672E-2	5.079E-2	6.727E-2	6.826E-2	4.081E-2	3.173E-2	2.623E-2	2.278E-2
.7	1.693E-2	3.285E-2	4.565E-2	5.080E-2	3.665E-2	2.959E-2	2.494E-2	2.190E-2
.8	1.138E-2	2.242E-2	3.220E-2	3.826E-2	3.252E-2	2.734E-2	2.355E-2	2.094E-2
.9	8.019E-3	1.595E-2	2.345E-2	2.920E-2	2.855E-2	2.503E-2	2.207E-2	1.989E-2
1	5.858E-3	1.174E-2	1.753E-2	2.259E-2	2.485E-2	2.272E-2	2.054E-2	1.879E-2
1.1	4.408E-3	8.875E-3	1.341E-2	1.769E-2	2.148E-2	2.046E-2	1.898E-2	1.765E-2
1.2	3.398E-3	6.865E-3	1.045E-2	1.403E-2	1.845E-2	1.829E-2	1.742E-2	1.648E-2
1.3	2.675E-3	5.415E-3	8.283E-3	1.126E-2	1.578E-2	1.624E-2	1.589E-2	1.530E-2
1.4	2.142E-3	4.342E-3	6.662E-3	9.125E-3	1.346E-2	1.433E-2	1.441E-2	1.412E-2
1.5	1.742E-3	3.532E-3	5.428E-3	7.470E-3	1.145E-2	1.259E-2	1.299E-2	1.297E-2
1.6	1.435E-3	2.909E-3	4.472E-3	6.171E-3	9.736E-3	1.100E-2	1.164E-2	1.184E-2
1.7	1.196E-3	2.424E-3	3.723E-3	5.141E-3	8.276E-3	9.574E-3	1.037E-2	1.076E-2
1.8	1.007E-3	2.039E-3	3.128E-3	4.316E-3	7.040E-3	8.306E-3	9.200E-3	9.719E-3
1.9	8.562E-4	1.731E-3	2.650E-3	3.650E-3	5.995E-3	7.187E-3	8.123E-3	8.737E-3
2	7.337E-4	1.481E-3	2.262E-3	3.107E-3	5.114E-3	6.206E-3	7.141E-3	7.815E-3
2.1	6.334E-4	1.276E-3	1.944E-3	2.661E-3	4.372E-3	5.351E-3	6.253E-3	6.956E-3
2.2	5.506E-4	1.107E-3	1.681E-3	2.293E-3	3.747E-3	4.611E-3	5.456E-3	6.162E-3
2.3	4.815E-4	9.662E-4	1.463E-3	1.986E-3	3.220E-3	3.971E-3	4.746E-3	5.435E-3
2.4	4.235E-4	8.480E-4	1.279E-3	1.729E-3	2.774E-3	3.420E-3	4.118E-3	4.772E-3
2.5	3.744E-4	7.480E-4	1.124E-3	1.512E-3	2.398E-3	2.948E-3	3.564E-3	4.172E-3
2.6	3.326E-4	6.629E-4	9.926E-4	1.328E-3	2.078E-3	2.543E-3	3.079E-3	3.634E-3
2.7	2.967E-4	5.901E-4	8.802E-4	1.172E-3	1.807E-3	2.197E-3	2.656E-3	3.153E-3
2.8	2.658E-4	5.274E-4	7.837E-4	1.037E-3	1.576E-3	1.900E-3	2.289E-3	2.726E-3
2.9	2.391E-4	4.731E-4	7.003E-4	9.221E-4	1.379E-3	1.647E-3	1.971E-3	2.349E-3
3	2.158E-4	4.260E-4	6.280E-4	8.224E-4	1.210E-3	1.430E-3	1.697E-3	2.018E-3
3.1	1.954E-4	3.848E-4	5.650E-4	7.359E-4	1.065E-3	1.245E-3	1.462E-3	1.729E-3
3.2	1.775E-4	3.486E-4	5.099E-4	6.605E-4	9.398E-4	1.086E-3	1.259E-3	1.478E-3
3.3	1.617E-4	3.168E-4	4.616E-4	5.945E-4	8.319E-4	9.492E-4	1.086E-3	1.261E-3
3.4	1.477E-4	2.887E-4	4.189E-4	5.367E-4	7.384E-4	8.320E-4	9.373E-4	1.074E-3
3.5	1.353E-4	2.638E-4	3.812E-4	4.857E-4	6.572E-4	7.310E-4	8.103E-4	9.128E-4
3.6	1.242E-4	2.416E-4	3.478E-4	4.407E-4	5.865E-4	6.440E-4	7.016E-4	7.754E-4
3.7	1.143E-4	2.218E-4	3.180E-4	4.008E-4	5.247E-4	5.687E-4	6.086E-4	6.583E-4
3.8	1.054E-4	2.040E-4	2.914E-4	3.654E-4	4.705E-4	5.035E-4	5.290E-4	5.587E-4
3.9	9.742E-5	1.881E-4	2.676E-4	3.338E-4	4.230E-4	4.469E-4	4.609E-4	4.742E-4
4	9.022E-5	1.737E-4	2.463E-4	3.056E-4	3.811E-4	3.977E-4	4.025E-4	4.027E-4
4.1	8.370E-5	1.608E-4		2.803E-4		3.547E-4	3.523E-4	3.423E-4
4.2	7.779E-5	1.491E-4		2.576E-4		3.171E-4	3.092E-4	2.914E-4
4.3	7.243E-5	1.385E-4		2.371E-4		2.842E-4	2.720E-4	2.484E-4
4.4	6.754E-5	1.288E-4		2.187E-4		2.552E-4	2.400E-4	2.122E-4
4.5	6.308E-5	1.200E-4		2.020E-4		2.298E-4	2.122E-4	1.817E-4
4.6	5.900E-5	1.120E-4		1.869E-4		2.073E-4	1.882E-4	1.560E-4
4.7	5.527E-5	1.047E-4		1.732E-4		1.874E-4	1.673E-4	1.343E-4
4.8	5.184E-5	9.794E-5		1.608E-4		1.697E-4	1.491E-4	1.159E-4
4.9	4.869E-5	9.178E-5		1.494E-4		1.540E-4	1.333E-4	1.004E-4
5	4.579E-5	8.611E-5		1.391E-4		1.401E-4	1.194E-4	8.731E-5

A.7. Strictly stable densities expressed via elementary functions, special functions and quadratures

ONE-DIMENSIONAL DISTRIBUTIONS $q^C(x;\alpha,\delta)$

$$q(x;1/4,1/4) = \frac{1}{2\pi}x^{-4/3}\int_0^\infty \exp\left\{-\frac{1}{4}x^{-1/3}\left(y^4+y^{-2}\right)\right\}dy; \tag{A.7.1}$$

$$q(x;1/3,1/3) = \frac{1}{2\pi}x^{-3/2}\int_0^\infty \exp\left\{-\frac{1}{3\sqrt{3x}}\left(y^3+y^{-3}\right)\right\}dy$$

$$= \frac{1}{3\pi}x^{-3/2}K_{1/3}\left(\frac{2}{3\sqrt{3}}x^{-1/2}\right), \tag{A.7.2}$$

where $x > 0$, and $K_{1/3}(z)$ is the Macdonald function (modified Bessel function of the third kind);

$$q(x;1/2,1/2) = \frac{1}{2\sqrt{\pi}}x^{-3/2}\exp\{-1/(4x)\}, \quad x>0; \tag{A.7.3}$$

$$q(x;1/2,0) = \frac{1}{2\sqrt{2\pi}}x^{-3/2}\left\{\cos\left[\frac{1}{4x}\left(\frac{1}{2}-C\left(\sqrt{2/(\pi x)}\right)\right)\right]\right.$$

$$\left.+\sin\left[\frac{1}{4x}\left(\frac{1}{2}-S\left(\sqrt{2/(\pi x)}\right)\right)\right]\right\}, \tag{A.7.4}$$

where

$$C(z) = \int_0^z \cos\left(\pi t^2/2\right)\,dt$$

and

$$S(z) = \int_0^z \sin\left(\pi t^2/2\right)\,dt$$

are the Fresnel integrals;

$$q(x;1/2,\delta) = \frac{1}{\pi}x^{-3/2}\Re\left\{\sqrt{\pi}\zeta e^{-\zeta^2}-2i\zeta w(\zeta)\right\}, \tag{A.7.5}$$

where

$$\zeta = -iz/2 = -i\frac{1}{2\sqrt{x}}\exp\{i(\delta+1/2)\pi/2\}$$

and

$$w(\zeta) = e^{-\zeta^2}\int_0^\zeta e^{t^2}dt$$

is the function tabulated for complex-valued ζ in (Karpov, 1965);

$$q(x;2/3,2/3) = \frac{1}{\sqrt{3\pi}}x^{-1}\exp\left(-\frac{2}{27}x^{-2}\right)W_{1/2,1/6}\left(\frac{4}{27}x^{-2}\right), \quad x>0; \tag{A.7.6}$$

$$q(x; 2/3, 0) = \frac{1}{2\sqrt{3\pi}} |x|^{-1} \exp\left(\frac{2}{27} x^{-2}\right) W_{-1/2,1/6}\left(\frac{4}{27} x^{-2}\right); \tag{A.7.7}$$

$$q(x; 1, 0) = \frac{1}{\pi(1 + x^2)}; \tag{A.7.8}$$

$$q(x; 3/2, -1/2) = \frac{1}{2\sqrt{3\pi}} x^{-1} \exp\left(\frac{2}{27} x^3\right) W_{-1/2,1/6}\left(\frac{4}{27} x^3\right), \quad x > 0; \tag{A.7.9}$$

$$q(x; 3/2, 1/2) = \frac{1}{\sqrt{3\pi}} x^{-1} \exp\left(-\frac{2}{27} x^3\right) W_{1/2,1/6}\left(\frac{4}{27} x^3\right), \quad x > 0; \tag{A.7.10}$$

$$q(x; 2, \delta) = \frac{1}{2\sqrt{\pi}} \exp\{-x^2/4\}; \tag{A.7.11}$$

$$q(x; \alpha, \alpha) = (1/\alpha) x^{-2} H_{11}^{10}\left(\frac{1}{x} \middle| \begin{matrix} (-1, 1) \\ (-1/\alpha, 1/\alpha) \end{matrix}\right), \quad \alpha < 1, \quad x > 0; \tag{A.7.12}$$

$$q(x; \alpha, 2 - \alpha) = (1/\alpha) H_{11}^{10}\left(x \middle| \begin{matrix} (1 - 1/\alpha, 1/\alpha) \\ (0, 1) \end{matrix}\right), \quad \alpha > 1, \quad x > 0; \tag{A.7.13}$$

$$q(x; \alpha, \delta) = (1/\alpha) x^{-2} H_{22}^{11}\left(\frac{1}{x} \middle| \begin{matrix} (-1, 1), (-\gamma, \gamma) \\ (-1/\alpha, 1/\alpha), (-\gamma, \gamma) \end{matrix}\right), \tag{A.7.14}$$

$$\gamma = (1 + \delta/\alpha)/2, \quad \alpha < 1, \quad |\delta| < \alpha, \quad x > 0;$$

$$q(x; \alpha, \delta) = (1/\alpha) H_{22}^{11}\left(x \middle| \begin{matrix} (1 - 1/\alpha, 1/\alpha), (1 - \gamma, \gamma) \\ (0, 1), (1 - \gamma, \gamma) \end{matrix}\right); \tag{A.7.15}$$

$$\gamma = (1 + \delta/\alpha)/2, \quad \alpha > 1, \quad \delta < 2 - \alpha, \quad x > 0.$$

RADIAL FUNCTIONS $\rho_N(r; \alpha)$

$$\rho_N(r; 2/3) = \frac{\Gamma(N/2 + 1/3)\Gamma(N/2 + 2/3)}{2\sqrt{3\pi^N}\Gamma(5/6)\Gamma(7/6)} \times r^{-N} \exp\left(\frac{2}{27} r^{-2}\right) W_{-N/2,1/6}\left(\frac{4}{27} r^{-2}\right); \tag{A.7.16}$$

$$q_N(r; 1) = \frac{\Gamma((N + 1)/2)}{[\pi(1 + r^2)]^{(N+1)/2}}; \tag{A.7.17}$$

$$q_N(r; 2) = (4\pi)^{-N/2} \exp\{-r^2/4\}; \tag{A.7.18}$$

$$q_N(r; \alpha) = (1/2)(r\sqrt{\pi})^{-N} H_{21}^{11}\left(\frac{2}{r} \middle| \begin{matrix} (1 - N/2, 1/2), (1, 1/2) \\ (1, 1/\alpha) \end{matrix}\right), \quad \alpha < 1; \tag{A.7.19}$$

$$q_N(r; \alpha) = (2/\alpha)(2\sqrt{\pi})^{-N} H_{12}^{11}\left(\frac{r^2}{4} \middle| \begin{matrix} (1 - N/\alpha, 2/\alpha) \\ (0, 1)\,(1 - N/2, 1) \end{matrix}\right), \quad \alpha \geq 1. \tag{A.7.20}$$

A.8. Fractional integro-differential operators

RIEMANN–LIOUVILLE FRACTIONAL INTEGRALS ($\alpha > 0$):

$$\left(I_{+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{f(\xi)\, d\xi}{(x-\xi)^{1-\alpha}}, \tag{A.8.1}$$

$$\left(I_{0+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} \frac{f(\xi)\, d\xi}{(x-\xi)^{1-\alpha}}, \tag{A.8.2}$$

$$\left(I_{-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} \frac{f(\xi)\, d\xi}{(\xi-x)^{1-\alpha}},$$

$$(\mathbb{I}^{\alpha} f)(x) = (e^{-i\pi\alpha} I_{-}^{\alpha})(x). \tag{A.8.3}$$

RIEMANN–LIOUVILLE FRACTIONAL DERIVATIVES ($0 < \alpha < 1$):

$$\left(D_{+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{-\infty}^{x} \frac{f(\xi)\, d\xi}{(x-\xi)^{\alpha}}, \tag{A.8.4}$$

$$\left(D_{0+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{0}^{x} \frac{f(\xi) d\xi}{(x-\xi)^{\alpha}}, \tag{A.8.5}$$

$$\left(D_{-}^{\alpha} f\right)(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{x}^{\infty} \frac{f(\xi)\, d\xi}{(\xi-x)^{\alpha}}.$$

MARCHAUD FRACTIONAL DERIVATIVES ($0 < \alpha < 1$):

$$\left(\mathbb{D}_{+}^{\alpha} f\right)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_{0}^{\infty} \frac{f(x) - f(x-\xi)}{\xi^{1+\alpha}} d\xi$$

$$= \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{x} \frac{f(x) - f(\xi)}{(x-\xi)^{1+\alpha}} d\xi, \tag{A.8.6}$$

$$\left(\mathbb{D}_{-}^{\alpha} f\right)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_{0}^{\infty} \frac{f(x) - f(x+\xi)}{\xi^{1+\alpha}} d\xi. \tag{A.8.7}$$

RIESZ POTENTIAL ($\alpha > 0$, $\alpha \neq 1, 3, 5, \ldots$):

$$\left(I^{\alpha} f\right)(x) = \frac{1}{2\cos(\alpha\pi/2)} \left(\left(I_{+}^{\alpha} f\right)(x) + \left(I_{-}^{\alpha} f\right)(x)\right)$$

$$= \frac{1}{2\Gamma(\alpha)\cos(\alpha\pi/2)} \int_{-\infty}^{\infty} \frac{f(\xi)\, d\xi}{|x-\xi|^{1-\alpha}}. \tag{A.8.8}$$

where I_{+}^{α} and I_{-}^{α} are defined by (A.8.1) and (A.8.3) respectively.

RIESZ DERIVATIVE ($0 < \alpha < 1$):

$$D^{\alpha} f \equiv \left(I^{\alpha}\right)^{-1} f = \frac{\alpha}{2\Gamma(1-\alpha)\cos(\alpha\pi/2)} \int_{-\infty}^{\infty} \frac{f(x) - f(x-\xi)}{|\xi|^{1+\alpha}} d\xi$$

$$= \frac{\alpha}{2\Gamma(1-\alpha)\cos(\alpha\pi/2)} \int_0^\infty \frac{2f(x) - f(x-\xi) - f(x+\xi)}{\xi^{1+\alpha}} d\xi$$
$$= [2\cos(\alpha\pi/2)]^{-1} \left(\mathbb{D}_+^\alpha f + \mathbb{D}_-^\alpha f\right), \tag{A.8.9}$$

where $\mathbb{D}_+^\alpha$ and $\mathbb{D}_-^\alpha$ are given by (A.8.6) and (A.8.7).

FELLER POTENTIAL ($0 < \alpha < 1$):

$$\left(M_{u,v}^\alpha f\right)(x) = u\left(I_+^\alpha f\right)(x) + v\left(I_-^\alpha f\right)(x)$$
$$= \int_{-\infty}^\infty \frac{u + v + (u-v) sign(x-\xi)}{|x-\xi|^{1-\alpha}} f(\xi)\, d\xi, \tag{A.8.10}$$

where $u^2 + v^2 \neq 0$. In particular,

$$M_{u,v}^\alpha = 2u\cos(\alpha\pi/2) I^\alpha,$$

where I^α is given by (A.8.8).

INVERSE FELLER POTENTIAL ($0 < \alpha < 1$):

$$\left(M_{u,v}^\alpha\right)^{-1} f = \frac{\alpha}{2A\Gamma(1-\alpha)} \int_{-\infty}^\infty \frac{u + v + (u-v)\,\mathrm{sign}(x-\xi)}{|x-\xi|^{1+\alpha}} [f(x) - f(\xi)]\, d\xi$$
$$= \frac{\alpha}{2A\Gamma(1-\alpha)} \int_0^\infty \left[(u+v)f(x) - uf(x-\xi) - vf(x+\xi)\right] \xi^{-1-\alpha}\, d\xi, \tag{A.8.11}$$

where

$$A = [(u+v)\cos(\alpha\pi/2)]^2 + [(u-v)\sin(\alpha\pi/2)]^2.$$

In particular,

$$\left(M_{1,0}^\alpha\right)^{-1} = \mathbb{D}_+^\alpha,$$
$$\left(M_{0,1}^\alpha\right)^{-1} = \mathbb{D}_-^\alpha,$$
$$\left(M_{u,u}^\alpha\right)^{-1} f = [2u\cos(\alpha\pi/2)]^{-1} D^\alpha f,$$

where D^α is defined by (A.8.9).

n-DIMENSIONAL RIESZ INTEGRO-DIFFERENTIAL OPERATOR

$$(-\Delta_n)^{-\alpha/2} f = \frac{1}{\gamma_n(\alpha)} \int_{\mathbb{R}^n} \frac{f(\xi)\, d\xi}{|x-\xi|^{n-\alpha}}, \tag{A.8.12}$$

where

$$\alpha > 0, \quad \alpha \neq n, n+2, n+4, \ldots,$$
$$\gamma_n(\alpha) = 2^\alpha \pi^{n/2} \Gamma(\alpha/2)/\Gamma((n-\alpha)/2),$$

and

$$(-\Delta_n)^{\alpha/2} f = \frac{1}{d_{n,l}(\alpha)} \int_{\mathbb{R}^n} \sum_{k=0}^{l} (-1)^k \binom{l}{k} f(x - k\xi) |\xi|^{-n-\alpha} d\xi \quad \text{(A.8.13)}$$

where

$$\alpha > 0, \quad l = [\alpha] + 1,$$

$$d_{n,l}(\alpha) = \frac{\pi^{1+n/2}}{2^\alpha \Gamma(1 + \alpha/2) \Gamma((n + \alpha)/2) \sin(\alpha\pi/2)} \sum_{k=0}^{l} (-1)^k \binom{l}{k} k^\alpha.$$

In particular, if $n = 1$, then

$$\gamma_1(\alpha) = 2\Gamma(\alpha) \cos(\alpha\pi/2),$$

$$d_{1,1}(\alpha) = -2\Gamma(-\alpha) \cos(\alpha\pi/2), \qquad \alpha < 1,$$

and operators (A.8.12), (A.8.13) coincide with (A.8.8), (A.8.9) respectively.

FOURIER TRANSFORMS $\hat{F}_n f \equiv \int_{\mathbb{R}^n} e^{ik\cdot x} f(x)\, dx, \ \hat{F}_1 \equiv \hat{F}$:

$$\hat{F}\left(I_\pm^\alpha f\right) = |k|^{-\alpha} \exp\{\pm i\alpha(\pi/2) \operatorname{sign} k\} \hat{F} f, \quad 0 < \alpha < 1; \quad \text{(A.8.14)}$$

$$\hat{F}\left(D_\pm^\alpha f\right) = |k|^{\alpha} \exp\{\mp i\alpha(\pi/2) \operatorname{sign} k\} \hat{F} f, \quad \alpha \geq 0; \quad \text{(A.8.15)}$$

$$\hat{F}\left(M_{u,v}^\alpha f\right) = [(u + v)\cos(\alpha\pi/2) + i(u - v)\sin(\alpha\pi/2) \operatorname{sign} k] \times |k|^{-\alpha} \hat{F} f, \quad 0 < \alpha < 1; \quad \text{(A.8.16)}$$

$$\hat{F}_n\left((-\Delta_n)^{\alpha/2} f\right) = |k|^\alpha \hat{F}_n f. \quad \text{(A.8.17)}$$

In particular,

$$\hat{F}_1((-\Delta_1)^{-\alpha/2} f) \equiv \hat{F}_1(I^\alpha f) = |k|^{-\alpha} \hat{F}_1 f. \quad \text{(A.8.18)}$$

LAPLACE TRANSFORM $\hat{L} f \equiv \int_0^\infty e^{-\lambda x} f(x)\, dx$:

$$\hat{L}\left(I_{0+}^\alpha f\right) = \lambda^{-\alpha} \left(\hat{L} f\right), \quad \text{(A.8.19)}$$

$$\hat{L}\left(D_{0+}^\alpha f\right) = \lambda^{\alpha} \left(\hat{L} f\right). \quad \text{(A.8.20)}$$

For any $f(x)$, $x \in \mathbb{R}^N$, satisfying the condition

$$\|f\|_p = \left\{ \int_{\mathbb{R}^N} |f(x)|^p dx \right\}^{1/p}, \qquad 1 < p < N/\alpha,$$

the relation

$$(I^\alpha f)(x) = \frac{1}{\Gamma(\alpha/\mu)} \int_0^\infty dt t^{\alpha/\mu - 1} \int_{\mathbb{R}^N} f(x - x')p(x', t; \mu)\, dx'$$

is true, where

$$p(x, t; \mu) = t^{-N/\mu} q_N(xt^{-1/\mu}; \mu)$$

is the distribution density of the spherically symmetric N-dimensional stable process with the characteristic parameter μ:

$$\int_{\mathbb{R}^N} p(x, t; \mu) e^{i(k,x)} dx = e^{-t|k|^\mu}.$$

The special cases of this relation with $\mu = 1$ and $\mu = 2$ were considered in (Stein & Weiss, 1960; Johnson, 1973) respectively.

A.9. Approximation of inverse distribution function $r(x) = F^{-1}(x)$ for simulation of three-dimensional random vectors with density $q_3(\mathbf{r};\alpha)$ (Uchaikin & Gusarov, 1998)

$$r(x) = \begin{cases} (x/A)^{1/3} + x[B/(1-x)]^{1/\alpha} P_n^{(\alpha)}(x), & \alpha \le 1, \\ (x/A)^{1/3} + (5/9)(C/A)^2 (x/A)^{5/3} + g(x) - xP_n^{(\alpha)}(x), & \alpha > 1, \end{cases}$$

where

$$A = 2\Gamma(3/\alpha)/(3\alpha\pi), \qquad B = \Gamma(\alpha+2)\sin(\alpha\pi/2)/(\alpha\pi/2),$$
$$C = 8\Gamma(5/\alpha)/(5!\alpha\pi), \qquad D = \Gamma(2\alpha+2)|\sin(\alpha\pi)|/(\alpha\pi),$$

$$g(x) = \left[\sqrt{(B/D)^2 + 2(1-x)/D} - B/D\right]^{-1/\alpha} - \left[\sqrt{(B/D)^2 + 2/D} - B/D\right]^{-1/\alpha},$$

$$xP_n^{(\alpha)}(x) = c_0 x + c_1 x^2 + \ldots + c_n x^{n+1}.$$
$$P_3^{(0.2)}(x) = 0.04310 + 0.47961x + 0.43761x^2 + 0.03968x^3,$$
$$P_3^{(0.3)}(x) = 0.36633 + 0.68591x - 0.09155x^2 + 0.03931x^3,$$
$$P_2^{(0.4)}(x) = 0.77792 + 0.27989x - 0.05781x^2,$$
$$P_2^{(0.5)}(x) = 1.01155 + 0.09253x - 0.10407x^2,$$
$$P_2^{(0.6)}(x) = 1.06878 + 0.14079x - 0.20957x^2,$$
$$P_2^{(0.7)}(x) = 1.02330 + 0.28358x - 0.30687x^2,$$
$$P_2^{(0.8)}(x) = 0.94223 + 0.39823x - 0.34046x^2,$$
$$P_3^{(0.9)}(x) = 0.86897 + 0.43915x - 0.36207x^2 + 0.05395x^3,$$
$$P_3^{(1.0)}(x) = 0.80193 + 0.44032x - 0.34080x^2 + 0.09855x^3,$$
$$P_3^{(1.1)}(x) = 0.39456 + 3.54231x - 3.06908x^2 + 2.55379x^3,$$
$$P_3^{(1.2)}(x) = 0.34583 + 2.37756x - 1.18707x^2 + 0.47064x^3,$$
$$P_3^{(1.3)}(x) = 0.26378 + 1.95839x - 1.08096x^2 + 0.53840x^3,$$
$$P_3^{(1.4)}(x) = 0.16540 + 1.84557x - 1.42423x^2 + 0.89226x^3,$$
$$P_3^{(1.5)}(x) = 0.05993 + 1.96872x - 2.24189x^2 + 1.57409x^3,$$
$$P_5^{(1.6)}(x) = 0.04606 + 1.59258x - 3.18405x^2 + 7.19358x^3 - 8.69865x^4 + 4.37779x^5,$$
$$P_6^{(1.7)}(x) = 0.00210 + 0.84060x + 2.58118x^2 - 16.69826x^3 + 39.86698x^4 - 43.13309x^5 + 17.89178x^6,$$

$$P_9^{(1.8)}(x) = -0.34685 + 6.66547x - 41.66960x^2 + 133.46730x^3 - 189.59325x^4$$
$$+ 22.42745x^5 + 241.46185x^6 - 222.98818x^7$$
$$+ 18.19747x^8 + 33.83475x^9.$$

A.10. Some statistical terms

CONFIDENCE p-PERCENT INTERVAL for parameter α is the interval (c_1, c_2) such that $\mathsf{P}\{\alpha \in (c_1, c_2)\} = (1-p)/100$.

CONVERGENCE IN PROBABILITY: a sequence $X_1, X_2, \ldots$ converges in probability to a constant c, $X_n \xrightarrow{p} c$, if $\mathsf{P}\{|x_n - c| > 0\} \to 0$ as $n \to \infty$ for any $\varepsilon > 0$.

It is equivalent to the weak convergence of the distribution function $F_{X_n}(x)$ to the degenerate distribution function $e(x-c)$: $F_{X_n}(x) \Rightarrow e(x-c)$, $n \to \infty$.

ASYMPTOTIC EFFICIENCY of the estimator $\hat{\alpha} = \hat{\alpha}(X_1, \ldots, X_n)$ is the limit $e_{as}(\bar{\alpha}) = \lim_{n\to\infty} e(\hat{\alpha}(X_1, \ldots, X_n))$.

EFFICIENCY OF AN ESTIMATOR $\hat{\alpha}$ is the ratio $e(\hat{\alpha}) = \min \operatorname{Var} \hat{\alpha} / \operatorname{Var} \hat{\alpha}$ where $\min \operatorname{Var} \hat{\alpha}$ is the lowest variance of the estimator and $\operatorname{Var} \hat{\alpha}$ is its true variance.

ASYMPTOTICALLY EFFICIENT ESTIMATOR is an estimator $\hat{\alpha}$ with asymptotic efficiency $e_{as}(\bar{\alpha}) = 1$.

CONSISTENT ESTIMATOR of a parameter α is an estimator $\hat{\alpha} = \hat{\alpha}(X_1, \ldots, X_n)$ converging in probability to α as $n \to \infty$: $\hat{\alpha} \xrightarrow{p} \alpha$.

$1/\sqrt{n}$-CONSISTENT ESTIMATOR of a parameter α is a consistent estimator $\hat{\alpha}$ for which $\mathsf{P}\{|\hat{\alpha}(X_1, \ldots, X_n) - \alpha| > 0\} \propto 1/\sqrt{n}$ as $n \to \infty$.

EFFICIENT ESTIMATOR is an estimator having the lowest variance for a sample with a given finite size, i.e., an estimator with efficiency $e(\hat{\alpha}) = 1$.

MAXIMUM LIKELIHOOD ESTIMATOR (MLE) for parameter α is the solution of the equation $\partial \ln L(\alpha)/\partial\alpha = 0$ depending on the sample $X_1, \ldots, X_n$. Here $L(\alpha)$ is the likelihood function.

UNBIASED ESTIMATOR of parameter α of distribution function $F_X(x;\alpha)$ is an estimator $\hat{\alpha} = \hat{\alpha}(X_1, \ldots, X_n)$ satisfying the condition $\mathsf{E}\hat{\alpha} = \alpha$.

FRACTILE (QUANTILE) x_p is the solution of equation $F_X(x_p) = p$, $0 < p < 1$.

LIKELIHOOD FUNCTION for a sample $X_1, \ldots, X_n$ of random variables with a common probability density $P_X(x;\alpha)$ is $L(x_1, \ldots, x_n; \alpha) = p_X(x_1;\alpha)\ldots p_X(x_n;\alpha)$.

A p PERCENT TRUNCATED SAMPLE MEAN is the arithmetical mean of the middle p percent of the ranked observations.

A.11. Some auxiliary formulae for statistical estimators

We introduce the abbreviated notation

$$U = \operatorname{sign} Y_E(\nu, \theta, \tau), \qquad U_0 = U - \mathsf{E}U, \tag{A.11.1}$$

$$V = \ln |Y_E(\nu, \theta, \tau)|, \qquad V_0 = V - \mathsf{E}V, \tag{A.11.2}$$

and present formulae obtained in (Zolotarev, 1986) by means of characteristic transformation applied to form E:

$$\mathsf{E}U^0 = \mathsf{E}V^0 = \mathsf{E}U^2 = 1, \quad \mathsf{E}U = \theta, \quad \mathsf{E}V = \tau, \tag{A.11.3}$$

$$\mathsf{E}UV = \theta\tau = \mathsf{E}U\mathsf{E}V, \tag{A.11.4}$$

$$\mathsf{E}V^2 = \tau^2 + \pi^2[2\nu - 3\theta^2 + 1]/12, \tag{A.11.5}$$

$$\mathsf{E}U_0^r = (-1)^r \left[\sum\nolimits_0 \binom{r}{j} \theta^{r-j} - \sum\nolimits_1 \binom{r}{j} \theta^{r-j+1}\right], \tag{A.11.6}$$

$$\mathsf{E}U_0^r V_0^2 = \left[\pi^2(1-\theta^2)/6\right] \sum\nolimits_0 \binom{r}{j} \theta^{r-j} + [\pi^2(2\nu - \theta^2 - 1)/12](-1)^r \mathsf{E}U_0^r, \tag{A.11.7}$$

$$\mathsf{E}U_0^r V_0^4 = [\pi^4(1-\theta^4)/20] \sum\nolimits_0 \binom{r}{j} \theta^{r-j} + [\pi^4(8\nu^2 - \theta^4 - 7)/120](-1)^r \mathsf{E}U_0^r, \tag{A.11.8}$$

where $\sum_0$ and $\sum_1$ denote summation over even and odd values of j, respectively, not exceeding r.

Let $X_1, \dots, X_n$ be a collection of independent and identically distributed random variables with finite fourth moment, and set

$$a = \mathsf{E}X_1, \quad b^2 = \operatorname{Var} X_1, \quad c^4 = \mathsf{E}(X_1 - a)^4,$$

$$\overline{X} = \frac{1}{n} \sum_{j=1}^{n} X_j, \quad S_X^2 = \frac{1}{n-1} \sum_{j=1}^{n} \sum_{j=1}^{n} (X_j - \overline{X})^2.$$

Then

$$\mathsf{E}\overline{X} = a, \quad \operatorname{Var} \overline{X} = b^2/n, \quad \mathsf{E}S_X^2 = b^2, \tag{A.11.9}$$

$$\operatorname{Var} S_X^2 = (c^4 - b^4)/n + 2b^4/[n(n-1)]. \tag{A.11.10}$$

The proofs can be found in (Kendall & Stuart, 1967).

Let (L, M) be a pair of uncorrelated random variables with zero means and finite fourth moments and $(L_1, M_1), \dots, (L_n, M_n)$ be a collection of mutually independent pairs of random variables, each distributed as (L, M). Constructing from the n-tuples $(L_1, \dots, L_n)$ and $(M_1, \dots, M_n)$ the sample variances S_L^2 and S_M^2 one can prove (see (Zolotarev, 1986)) that for any $n \geq 2$

$$\operatorname{cov}(S_L^2, S_M^2) = \mathsf{E}(S_L^2 S_M^2) - (\mathsf{E}S_L^2)(\mathsf{E}S_M^2) = (1/n) \operatorname{cov}(L^2, M^2). \tag{A.11.11}$$

A.12. Functional derivatives

Let

$$F = F(u(\cdot))$$

be the value of a functional of argument $u(x)$, and let $\delta u(x)$ be the variation of the function $u(x)$ in some domain Δx_0 about x_0; outside this domain, $\delta u = 0$. We say that the functional (or variational) derivative of the functional $F(u(\cdot))$ at a point x_0 is the limit

$$\frac{\delta F(u(\cdot))}{\delta u(x_0)} = \lim_{\substack{|\Delta x| \to 0 \\ \max|\delta u| \to 0}} \frac{F(u(\cdot) + \delta u(\cdot)) - F(u(\cdot))}{\int_{\Delta x_0} \delta u(x)\, dx}$$

under the condition that this limit exists and depends on neither the form of $\delta u(x)$, nor how the domain Δx_0 shrinks to a point, nor how $|\delta u|$ tends to zero.

The rules of functional differentiation given below immediately follow from the definition.

(1) If A_1 and A_2 are constants, and $F_1(u(\cdot))$ and $F_2(u(\cdot))$ are functionals, then

$$\frac{\delta}{\delta u(x)} \left[A_1 F_1(u(\cdot)) + A_2 F_2(u(\cdot))\right] = A_1 \frac{\delta F_1(u(\cdot))}{\delta u(x)} + A_2 \frac{\delta F_2(u(\cdot))}{\delta u(x)}$$

and

$$\frac{\delta}{\delta u(x)} \left[F_1(u(\cdot)) F_2(u(\cdot))\right] = \frac{\delta F_1(u(\cdot))}{\delta u(x)} F_2(u(\cdot)) + F_1(u(\cdot)) \frac{\delta F_2(u(\cdot))}{\delta u(x)}.$$

(2) If $z = F(u(\cdot))$, and $f(z)$ is an ordinary function, then

$$\frac{\delta f F(u(\cdot))}{\delta u(x)} = f'(F(u(\cdot))) \frac{\delta F(u(\cdot))}{\delta u(x)},$$

where $f'(z)$ stands for the ordinary derivative.

(3) If $\Phi(u(\cdot))$ and $G(x; u(\cdot))$ are functionals, and the latter depends on a parameter x, then

$$\frac{\delta \Phi(G(\cdot; u(\cdot)))}{\delta u(x)} = \int \frac{\delta \Phi(G(\cdot; u(\cdot)))}{\delta G(x'; u(\cdot))} \frac{\delta G(x'; u(\cdot))}{\delta u(x)} dx'$$

We will present here a derivation of the formula for the functional derivative of arbitrary order n of a product of an arbitrary number of functionals $Q_i(u(\cdot))$, whose particular case was used in Section 11.4:

$$D_u^{(n)}(1,\dots,n)\prod_1^k Q_k(u(\cdot))$$
$$= S_n(1,\dots,n)\sum_{N_k=n}\binom{n}{n_1\dots n_k}\prod_{i=1}^k D_u^{(n_i)}(N_{i-1}+1,\dots,N_i)Q_i(u(\cdot)), \quad \text{(A.12.1)}$$

where

$$D_u^{(n)}(1,\dots,n)Q(u(\cdot)) \equiv \frac{\delta^n Q(u(\cdot))}{\delta u(x_1)\dots\delta u(x_n)}, \quad \text{(A.12.2)}$$
$$N_k = n_1+\dots+n_k, \quad n_i = 0,1,2,\dots, \quad N_0 = 0, \quad \text{(A.12.3)}$$

and

$$S_n(1,\dots,n) = \frac{1}{n!}\sum_{i_1\dots i_n}^{n}{}'$$

is the symmetrization operator:

$$S_n(1,\dots,n)1 = 1,$$
$$S_n(1,\dots,n)f_1(1) = \frac{1}{n}\sum_{i=1}^n f_1(i),$$
$$S_n(1,\dots,n)f_2(1,2) = \frac{1}{n(n-1)}\sum_{i\neq j}^n f_2(i,j),$$

and so on. Sometimes it is convenient to take

$$g_n(1,\dots,n) \stackrel{s}{=} f_n(1,\dots,n)$$

instead of

$$g_n(1,\dots,n) = S_n(1,\dots,n)f_n(1,\dots,n).$$

Because (A.12.1) is true for $k = 1$, we consider the case $k = 2$. The validity of (A.12.1) is obvious for $n = 1$ and $n = 2$. Let us assume that (A.12.1) holds true for some $n > 2$:

$$D_u^{(n)}(1,\dots,n)[Q_1Q_2] = S_n(1,\dots,n)\sum_{n_1+n_2=2}\binom{n}{n_1 n_2}$$
$$\times Q_1^{(n_1)}(1,\dots,n_1)Q_2^{(n_2)}(n_1+1,\dots,n_1+n_2). \quad \text{(A.12.4)}$$

Here we use the notation $Q^{(n)}(1,\dots,n)$ instead of (A.12.2). Applying $D_u^{(1)}(n+1)$ to both sides of (A.12.4), we obtain

$$D_u^{(n+1)}(1,\dots,n+1)\,[Q_1Q_2] = S_n(1,\dots,n)\left\{\frac{n!}{0!n!}\left[Q_1^{(1)}(n+1)Q_2^{(n)}(1,\dots,n)\right.\right.$$
$$\left.+\,Q_1Q_2^{(n+1)}(1,\dots,n,n+1)\right] + \frac{n!}{1!(n-1)!}\left[Q_1^{(2)}(1,n+1)Q_2^{(n-1)}(2,\dots,n)\right.$$
$$\left.+\,Q_1^{(1)}(1)Q_2^{(n)}(2,\dots,n+1)\right] + \frac{n!}{2!(n-2)!}\left[Q_1^{(3)}(1,2,n+1)Q_2^{(n-2)}(3,\dots,n)\right.$$
$$\left.+\,Q_1^{(2)}(1,2)Q_2^{(n-1)}(3,\dots,n+1)\right] + \dots$$
$$\left.+\,\frac{n!}{n!0!}\left[Q_1^{(n+1)}(1,\dots,n+1)Q_2 + Q_1^{(n)}(1,\dots,n)Q_2^{(1)}(n+1)\right]\right\}$$

The left-hand side of this equation is a symmetric function of $n+1$ arguments, and we can make the change $S_n(1,\dots,n) \to S_{n+1}(1,\dots,n+1)$. Since

$$S_n(1,\dots,n+1)\left[Q_1^{(1)}(n+1)Q_2^{(n)}(1,\dots,n)\right]$$
$$= S_n(1,\dots,n+1)\left[Q_1^{(1)}(1)Q_2^{(n)}(2,\dots,n+1)\right],$$

$$S_n(1,\dots,n+1)\left[Q_1^{(2)}(1,n+1)Q_2^{(n-1)}(2,\dots,n)\right]$$
$$= S_n(1,\dots,n+1)\left[Q_1^{(2)}(1,2)Q_2^{(n-1)}(3,\dots,n+1)\right]$$

etc., we rewrite the expression as

$$D_u^{(n+1)}(1,\dots,n+1)\,[Q_1Q_2] = S_{n+1}(1,\dots,n+1)$$
$$\times\left\{\frac{(n+1)!}{0!(n+1)!}Q_1Q_2^{(n+1)}(1,\dots,n+1) + \frac{(n+1)!}{1!n!}Q_1^{(1)}(1)Q_2^{(n+1)}(2,\dots,n+1) + \dots\right.$$
$$\left.+\,\frac{(n+1)!}{(n+1)!0!}Q_1^{(n+1)}(1,\dots,n+1)Q_2\right\}. \quad \text{(A.12.5)}$$

It is easy to see that equation (A.12.5) coincides with (A.12.4) after changing n to $n+1$. Therefore, (A.12.4) holds true for any integer n.

Let (A.12.1) be true for some integer $k > 2$:

$$D_u^{(n)}(1,\dots,n)[Q_1,\dots,Q_k] = S_n(1,\dots,n)\sum_{n_1+\dots+n_k=n}\binom{n}{n_1\dots n_k}Q_1^{(n_1)}(1,\dots,n_1)$$
$$\times\, Q_2^{(n_2)}(n_1+1,\dots,n_1+n_2)\dots Q_k^{(n_k)}(n_1+\dots+n_{k-1}+1,\dots,n). \quad \text{(A.12.6)}$$

Introducing $P_k \equiv Q_1\dots Q_k$, we consider the expression

$$D_u^{(n)}(1,\dots,n)[Q_1,\dots,Q_kQ_{k+1}] = D_u^{(n)}(1,\dots,n)[P_kQ_{k+1}]$$

which can be rewritten as

$$D_u^{(n)}(1,\dots,n)[P_kQ_{k+1}] = S_n(1,\dots,n)\sum_{n_1+n_2=n}\binom{n}{n_1n_2}P_k^{(n)}(1,\dots,n_1) \times Q_{k+1}^{(n_1)}(n_1+1,\dots,n_1+n_2) \quad \text{(A.12.7)}$$

which immediately follows from (A.12.4). Applying formula (A.12.6) to the expression

$$P_k^{(n_1)}(1,\dots,n_1) \equiv D_u^{(n)}(1,\dots,n_1)[Q_1\dots Q_k],$$

from (A.12.7) we obtain

$$D_u^{(n)}(1,\dots,n)[Q_1\dots Q_{k+1}] = S_n(1,\dots,n)\sum_{n_1+n_2=n}\binom{n}{n_1n_2} \times \left\{S_{n_1}(1,\dots,n_1)\sum_{n_1'+\dots+n_k'=n_1}\binom{n_1}{n_1'\dots n_k'}Q_1^{(n_1')}(1,\dots,n_1')\dots \dots Q_k^{(n_k')}(n_1'+\dots+n_{k-1}',\dots,n_1)\right\}Q_{k+1}^{(n_2)}(n_1'+1,\dots,n_1+n_2).$$

Replacing n_2 by n_{k+1}', removing the unnecessary symmetrizing operator from the braces, recalling that

$$\sum_{n_1+n_2=n}\ \sum_{n_1'+\dots+n_k'=n_1} \equiv \sum_{n_1'+\dots+n_{k+1}'=n},$$

$$\binom{n}{n_1n_2}\binom{n_1}{n_1'\dots n_k'} = \binom{n}{n_1'\dots n_{k+1}'},$$

and dropping the primes about the sum indices $n_1',\dots,n_k'$, we finally arrive to the formula which differs from (A.12.6) only in k replaced by $k+1$, which we have to prove.

Conclusion

Thus, we have had an opportunity to admire the elegance of the theory of stable laws, which also turns out to be helpful while solving various actual problems. The latter witnesses that it is worthwhile to consider the densities of stable laws as a somewhat useful class of special functions. This assertion becomes even better grounded if we observe that a great body of well-known special functions appear to be closely related to stable laws.

The former, theoretical part of this book contains a large pile of information concerning stable laws, which, surely, are interesting enough to attract attention to them and to searching for new, yet unknown to us, properties of these objects.

We tried to make the latter part of the book cover a wide spectrum of examples of occurrence of stable laws in many problems of physics, technics, astronomy, and economics, which should inspire searching for new applications of these interesting probability laws. Of course, the success of such a search depends, to a great extent, on the choice of a mathematical model, which, on the one hand, should be assumed to be adequate to the phenomenon under investigation, and, on the other hand, should be convenient to investigate, because we are able to make use of the vast toolbox of known mathematical methods.

The wealth of features of stable laws inspire specialists to search for various analogues and extensions, which, they hope, possess as intriguing and useful properties as stable laws do. In the last 2–3 decades, new classes of distributions appeared both in the theory of limit theorems and beyond it, in the frames of the general theory of special functions. In this connection, it is worthwhile to mention the so-called semi-stable laws introduced by V.M. Kruglov (1979) as a subclass of infinitely divisible laws in a certain scheme of summation of independent random variables, and the pseudo-stable laws introduced by A.R. Zinger (1965) in the course of solving the well-known problem due to B.V. Gnedenko. Later, Kruglov established that semi-stable laws can, in some special cases, be treated as a particular case of pseudo-stable ones.

Moreover, the analytic extension of the series representing the densities of stable laws beyond the usual domains leaded V.M. Zolotarev (1986) to the consideration of, in essence, a new subclass of special functions, which were referred to as trans-stable. These functions, as concerns their mathematical properties and manifestations, behave as the densities of stable laws, but in addition, can be valuable tools in those cases where stable laws themselves, due to usual restrictions on their characterizing parameters, cannot be applied. An interesting generalization of one-sided stable densities was suggested by Schneider in 1987.

Speaking about the construction of mathematical models, we naturally dwell upon mathematical modelling as a whole. It is generally agreed that mathematics recently has become a universal tool to carry out theoretical investigations in various fields of human practice. It is worthwhile to notice that mathematics has passed a way from a science about formulas, as profanes thought, to a science about models of various actual phenomena. We are witnesses of somewhat peculiar counterflow of tendencies: on the one hand, more and more facts accumulated by mathematics find applications; and on the other hand, they exaggerate towards formalization of those fields of our practice where no or very little mathematics was used before.

Mathematical models, as an adequate reflection of actual reality, phenomena and processes, can evolve in two ways.

First, a mathematical model can appear to be a formalization of some qualitative ('soft') model of a phenomenon where causal relationships exist of the following kind: something grows at the sacrifice of decrease of some other factor, but no quantitative expressions exist. If some factors can be partially expressed in quantitative terms, then this model, as passing from 'soft' to 'hard' kind, is naturally referred to as 'semi-soft', and in the case where all quantitative characteristics entering into the model are 'computable', we say that such a model is 'hard'.

Second, having a bulk of experimental observation at our disposal, we are able to make an attempt to describe mathematically this flow of experimental data by means of some known model. For example, flows of observations of logarithms of stock exchange rates resemble realizations of stable stochastic processes with independent increments. Parameters of these processes should be chosen appropriately to give a good fit to observed data. The advantage of such an approach to constructing a model consists of the fact that, being a 'hard' model, it allows us to judge the mechanism of formation of the model and therefore, of the qualitative effects in the corresponding 'soft' model.

As an example of a 'hard' model created with the use of logical reasoning, we can consider the model (Section 18.1) of diffusion of cosmic rays; the whole Chapter 17 serves as an example of making a model fit the observed data by an appropriate choice of parameters. This approach was first used by Mandelbrot (1960).

The concept of a 'soft' model is illustrated by the model of asymmetry phenomenon suggested by V. Geodakyan (1993).

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