Practical Tests with Global Error Evaluation Techniques in Multistep Methods

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ODE's & Multistep Methods

- Consider the system of ODE

$$x'(t) = f\big(x(t)\big),\ t \in [0, T],\ x(t_0) = x_0, \qquad (1)$$

and the grid

$$w = \{t_0 = 0, t_1, t_2, \ldots, t_N = T\}$$

- Multistep methods

$$\sum_{j=0}^{l} \alpha_j x_{k+1-j} = h \sum_{j=0}^{l} \beta_j f(x_{k+1-j}) \qquad (2)$$

Local & Global Errors

- Local error

$$e(t_k) = x(t_k) - \tilde{x}_k$$

where $\tilde{x}_k$ *is computed using a multistep method with exact input.*

- Global error

$$E(t_k) = x(t_k) - x_k$$

Global error evaluation

Methods:

- Solving for the correction

$$E_h'(t) = f(P_h - E_h(t)) - P_h'(t),\ E_h(0) = 0$$

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- Zadunaisky's technique

$$\tilde{x}' = f(\tilde{x}) + d(h), \; \tilde{x}(0) = x(0)$$

where the perturbation $d(h) = P_h' - f(P_h)$

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- Solving the linearised discrete variational equation

Global error evaluation (Cont)

- Richardson extrapolation

$$E_k = \frac{x_k^h - x_k^{\frac{h}{2}}}{1 - 2^{-s}}$$

where s is the order of the multistep method.

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- Using two different methods

$$E_k = x_k^1 - x_k^2$$

Test Problems

- An oscillatory problem described by

$$x'(t) = x(t)cos(t),$$

with the initial condition $x(0) = 1$ for $t \in [0, 1]$.

The exact solution to this problem is

$$x(t) = e^{\sin(x(t))}.$$

Test Problems (Cont)

- A stiff ODE: Butcher test problem

$$x'(t) = \lambda(x(t) - \sin(\mu t)) + \mu\cos(\mu t)$$

for $t \in [0, 1]$ with the initial condition $x(0) = 1$.

It is known that the solution to this problem is:

$$x(t) = \sin(\mu t) + e^{\lambda t}.$$

Graphs

We present in the following figures the graphs of the errors in the estimation provided by different methods.

Conclusion