Score Evaluation within the Extended Square-root Information Filter

Maria V. Kulikova and Innokenti V. Semoushin

University of the Witwatersrand, Johannesburg, South Africa
Ulyanovsk State University, 42 Leo Tolstoy Str., 432970 Ulyanovsk, Russia

E-mail: mkulikova@cam.wits.ac.za
i.semushin@ulsu.ru http://staff.ulsu.ru/semoushin/

Introduction

The method of maximum likelihood . . .

Introduction

The method of maximum likelihood . . .

- Gradient-search optimization algorithms;

Introduction

The method of maximum likelihood . . .

- Gradient-search optimization algorithms;
- Gradient of the negative Log Likelihood Function;

Introduction

The method of maximum likelihood . . .

- Gradient-search optimization algorithms;
- Gradient of the negative Log Likelihood Function;
- Kalman Filter

Introduction

The method of maximum likelihood ...

- Gradient-search optimization algorithms;
- Gradient of the negative Log Likelihood Function;
- Kalman Filter $\Longrightarrow$ is known to be unstable;

Introduction

The method of maximum likelihood ...

- Gradient-search optimization algorithms;
- Gradient of the negative Log Likelihood Function;
- Kalman Filter $\Longrightarrow$ is known to be unstable;

[P. Park&T. Kailath, 1995] The extended Square Root Information Filter (eSRIF):

Introduction

The method of maximum likelihood . . .

- Gradient-search optimization algorithms;
- Gradient of the negative Log Likelihood Function;
- Kalman Filter $\Longrightarrow$ is known to be unstable;

[P. Park&T. Kailath, 1995] The extended Square Root Information Filter (eSRIF):

- avoids numerical instabilities arising from computational errors;

Introduction

The method of maximum likelihood ...

- Gradient-search optimization algorithms;
- Gradient of the negative Log Likelihood Function;
- Kalman Filter $\Longrightarrow$ is known to be unstable;

[P. Park&T. Kailath, 1995] The extended Square Root Information Filter (eSRIF):

- avoids numerical instabilities arising from computational errors;
- appears to be better suited to parallel implementation and to very large scale integration (VLSI) implementation.

Problem Statement

Consider the discrete-time linear dynamic stochastic system

$$x_{t+1} = F_t x_t + G_t w_t, \qquad t = 0, 1, \ldots, N, \tag{1}$$

$$z_t = H_t x_t + v_t, \qquad t = 1, 2, \ldots, N, \tag{2}$$

with the system state $x_t \in \mathbb{R}^n$, the state disturbance $w_t \in \mathbb{R}^q$, the observed vector $z_t \in \mathbb{R}^m$, and the measurement error $v_t \in \mathbb{R}^m$,

Problem Statement

Consider the discrete-time linear dynamic stochastic system

$$x_{t+1} = F_t x_t + G_t w_t, \qquad t = 0, 1, \ldots, N, \tag{1}$$

$$z_t = H_t x_t + v_t, \qquad t = 1, 2, \ldots, N, \tag{2}$$

with the system state $x_t \in \mathbb{R}^n$, the state disturbance $w_t \in \mathbb{R}^q$, the observed vector $z_t \in \mathbb{R}^m$, and the measurement error $v_t \in \mathbb{R}^m$, such that the initial state x_0 and each w_t, v_t of $\{w_t : t = 0, 1, \ldots\}$, $\{v_t : t = 1, 2, \ldots\}$ are taken from mutually independent Gaussian distributions with the following expectations:

$$\mathbf{E}\left\{\left[\begin{array}{ccc} x_0 & w_t & v_t \end{array}\right]\right\} = \left[\begin{array}{ccc} \bar{x}_0 & 0 & 0 \end{array}\right] \quad \text{and}$$

Problem Statement

$$\mathbf{E}\left\{\begin{bmatrix}(x_0-\bar{x}_0)\\ w_t\\ v_t\end{bmatrix}\begin{bmatrix}(x_0-\bar{x}_0)\\ w_t\\ v_t\end{bmatrix}^T\right\}=\begin{bmatrix}P_0 & 0 & 0\\ 0 & Q_t & 0\\ 0 & 0 & R_t\end{bmatrix}$$

and $\mathbf{E}\left\{w_t w_{t'}^T\right\}=0$, $\mathbf{E}\left\{v_t v_{t'}^T\right\}=0$ if $t\neq t'$.

Problem Statement

$$\mathbf{E}\left\{\begin{bmatrix}(x_0-\bar{x}_0)\\ w_t\\ v_t\end{bmatrix}\begin{bmatrix}(x_0-\bar{x}_0)\\ w_t\\ v_t\end{bmatrix}^T\right\}=\begin{bmatrix}P_0 & 0 & 0\\ 0 & Q_t & 0\\ 0 & 0 & R_t\end{bmatrix}$$

and $\mathbf{E}\left\{w_t w_{t'}^T\right\}=0$, $\mathbf{E}\left\{v_t v_{t'}^T\right\}=0$ if $t \neq t'$. Assume the system is parameterized by a vector $\theta \in \mathbb{R}^p$ of unknown system parameters. This means that all the above characteristics, namely F_t, G_t, H_t, $P_0 \geq 0$, $Q_t \geq 0$ and $R_t > 0$ can depend upon θ (the corresponding notations $F_t(\theta)$, $G_t(\theta)$ and so on, are suppressed for the sake of simplicity).

Problem Statement

For example, (1), (2) may describe a discrete autoregressive (AR) process observed in the presence of additive noise

$$x_{t+1} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \theta_1 & \theta_2 & \theta_3 & \dots & \theta_p \end{bmatrix} x_t + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \gamma \end{bmatrix} w_t ,$$

$$z_t = \begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix} x_t + v_t .$$

In this case θ are the unknown AR parameters which need to be estimated.

Problem Statement

The negative Log Likelihood Function (LLF) for system (1), (2) is given by

$$L_\theta\left(Z_1^N\right) = \frac{1}{2}\sum_{t=1}^{N}\left\{\frac{m}{2}\ln(2\pi) + \ln(\det(R_{e,t})) + e_t^T R_{e,t}^{-1} e_t\right\}$$

with $e_t \stackrel{\text{def}}{=} z_t - H_t \hat{x}_t$ being the zero-mean innovations whose covariance is determined as $R_{e,t} \stackrel{\text{def}}{=} E\{e_t e_t^T\} = H_t P_t H_t^T + R_t$ through matrix P_t, the error covariance of the time updated estimate $\hat{x}_t$ generated by the Kalman Filter.

Problem Statement

Let $l_\theta(z_t)$ denote the negative LLF for the t-th measurement z_t in system (1), (2), given measurements $Z_1^{t-1} \stackrel{\text{def}}{=} \{z_1, \ldots, z_{t-1}\}$, then

$$l_\theta(z_t) = \frac{1}{2}\left\{\frac{m}{2}\ln(2\pi) + \ln(\det(R_{e,t})) + e_t^T R_{e,t}^{-1} e_t\right\}. \qquad (3)$$

Problem Statement

Let $l_\theta(z_t)$ denote the negative LLF for the t-th measurement z_t in system (1), (2), given measurements $Z_1^{t-1} \stackrel{\text{def}}{=} \{z_1, \ldots, z_{t-1}\}$, then

$$l_\theta(z_t) = \frac{1}{2}\left\{\frac{m}{2}\ln(2\pi) + \ln(\det(R_{e,t})) + e_t^T R_{e,t}^{-1} e_t\right\}. \qquad (3)$$

By differentiating (3) we obtain

$$\frac{\partial l_\theta(z_t)}{\partial \theta_i} = \frac{1}{2}\frac{\partial}{\partial \theta_i}\left[\ln(\det(R_{e,t}))\right] + \frac{1}{2}\frac{\partial}{\partial \theta_i}\left[e_t^T R_{e,t}^{-1} e_t\right], \; i = \overline{1,p}. \quad (4)$$

As can be seen the computation of (3) and (4) leads to implementation of a Kalman Filter (and its derivative with respect to each parameter) which is known to be unstable.

The eSRIF

[P. Park&T. Kailath, 1995]: assume that $\Pi_0 > 0$, $R_t > 0$ and F_t are invertible. Given $P_0^{-T/2} = \Pi_0^{-T/2}$ and $P_0^{-T/2}\hat{x}_0 = \Pi_0^{-T/2}\bar{x}_0$, then

$$O_t \left[\begin{array}{ccc|c} R_t^{-T/2} & -R_t^{-T/2}H_tF_t^{-1} & R_t^{-T/2}H_tF_t^{-1}G_tQ_t^{T/2} & -R_t^{-T/2}z_t \\ 0 & P_t^{-T/2}F_t^{-1} & -P_t^{-T/2}F_t^{-1}G_tQ_t^{T/2} & P_t^{-T/2}\hat{x}_t \\ 0 & 0 & I_q & 0 \end{array}\right]$$
$$= \left[\begin{array}{ccc|c} R_{e,t}^{-T/2} & 0 & 0 & -\bar{e}_t \\ -P_{t+1}^{-T/2}K_{p,t} & P_{t+1}^{-T/2} & 0 & P_{t+1}^{-T/2}\hat{x}_{t+1} \\ * & * & * & * \end{array}\right]$$

where O_t is any orthogonal transformation such that the matrix on the right-hand side of the formula is block lower triangular.

The eSRIF

The eSRIF is a modification of the conventional one:

$$O_t \begin{bmatrix} R_t^{-T/2} & -R_t^{-T/2} H_t F_t^{-1} & R_t^{-T/2} H_t F_t^{-1} G_t Q_t^{T/2} \\ 0 & P_t^{-T/2} F_t^{-1} & -P_t^{-T/2} F_t^{-1} G_t Q_t^{T/2} \\ 0 & 0 & I_q \end{bmatrix}$$

$$= \begin{bmatrix} R_{e,t}^{-T/2} & 0 & 0 \\ -P_{t+1}^{-T/2} K_{p,t} & P_{t+1}^{-T/2} & 0 \\ * & * & * \end{bmatrix}$$

where O_t is any orthogonal transformation such that the matrix on the right-hand side of the formula is lower triangular.

The eSRIF

The eSRIF is a modification of the conventional one:

$$O_t \left[\begin{array}{ccc|c} R_t^{-T/2} & -R_t^{-T/2}H_tF_t^{-1} & R_t^{-T/2}H_tF_t^{-1}G_tQ_t^{T/2} & -R_t^{-T/2}z_t \\ 0 & P_t^{-T/2}F_t^{-1} & -P_t^{-T/2}F_t^{-1}G_tQ_t^{T/2} & P_t^{-T/2}\hat{x}_t \\ 0 & 0 & I_q & 0 \end{array}\right] = \left[\begin{array}{ccc|c} R_{e,t}^{-T/2} & 0 & 0 & -\bar{e}_t \\ -P_{t+1}^{-T/2}K_{p,t} & P_{t+1}^{-T/2} & 0 & P_{t+1}^{-T/2}\hat{x}_{t+1} \\ * & * & * & * \end{array}\right]$$

where O_t is any orthogonal transformation such that the matrix on the right-hand side of the formula is lower triangular.

The eSRIF

Remark. The predicted estimate now can be found from the entries of the post-array by solving the triangular system

$$\left(\tilde{P}_{t+1}^{-1/2}\right)\left(\hat{x}_{t+1}^{-}\right) = \left(\tilde{P}_{t+1}^{-1/2}\hat{x}_{t+1}^{-}\right). \qquad (5)$$

The LLG in terms of the eSRIF

The Log Likelihood Gradient (LLG) in terms of the eSRIF is given by

$$\frac{\partial l_\theta(z_t)}{\partial \theta_i} = \frac{\partial}{\partial \theta_i}\left[\ln(\det(R_{e,t}^{1/2}))\right] + \bar{e}_t{}^T \frac{\partial \bar{e}_t}{\partial \theta_i}, \quad i = 1, \ldots p,$$

where $R_{e,t}^{1/2}$ is a square-root factor of the matrix $R_{e,t}$, i. e. $R_{e,t} = R_{e,t}^{T/2} R_{e,t}^{1/2}$, and $\bar{e}_t$ are the normalized innovations, i. e. $\bar{e}_t = R_{e,t}^{-T/2} e_t$.

The LLG in terms of the eSRIF

Taking into account that matrix $R_{e,t}^{1/2}$ is upper triangular, we can show

The LLG in terms of the eSRIF

Taking into account that matrix $R_{e,t}^{1/2}$ is upper triangular, we can show

$$\frac{\partial}{\partial\theta_i}\left[\ln\left(\det(R_{e,t}^{1/2})\right)\right] = \mathbf{tr}\left[R_{e,t}^{-1/2}\,\frac{\partial\left(R_{e,t}^{1/2}\right)}{\partial\theta_i}\right], \quad i = 1,\ldots,p,$$

where $\mathbf{tr}[\,\cdot\,]$ is a trace of matrix.

The LLG in terms of the eSRIF

Taking into account that matrix $R_{e,t}^{1/2}$ is upper triangular, we can show

$$\frac{\partial}{\partial\theta_i}\left[\ln\left(\det(R_{e,t}^{1/2})\right)\right] = \mathbf{tr}\left[R_{e,t}^{-1/2}\,\frac{\partial\left(R_{e,t}^{1/2}\right)}{\partial\theta_i}\right], \quad i = 1,\ldots,p,$$

where $\mathbf{tr}[\,\cdot\,]$ is a trace of matrix.
Finally, we obtain the expression for the LLG in terms of the eSRIF:

$$\frac{\partial l_\theta(z_t)}{\partial\theta_i} = -\mathbf{tr}\left[R_{e,t}^{1/2}\,\frac{\partial\left(R_{e,t}^{-1/2}\right)}{\partial\theta_i}\right] + \bar{e}_t^T\frac{\partial\bar{e}_t}{\partial\theta_i}, \; i = 1,\ldots,p. \quad (6)$$

LLG Evaluation

$$\frac{\partial l(z_t)}{\partial \theta_i} = -tr\left[R_{e,t}^{1/2} \frac{\partial \left(R_{e,t}^{-1/2} \right)}{\partial \theta_i} \right] + \bar{e}_t^T \frac{\partial \bar{e}_t}{\partial \theta_i}, \; i = 1, \ldots, p. \qquad (6)$$

LLG Evaluation

$$\frac{\partial l(z_t)}{\partial \theta_i} = -tr\left[R_{e,t}^{1/2} \frac{\partial \left(R_{e,t}^{-1/2} \right)}{\partial \theta_i} \right] + \bar{e}_t^T \frac{\partial \bar{e}_t}{\partial \theta_i}, \; i = 1, \ldots, p. \quad (6)$$

LLG Evaluation

$$\frac{\partial l(z_t)}{\partial \theta_i} = -tr\left[R_{e,t}^{1/2} \frac{\partial \left(R_{e,t}^{-1/2} \right)}{\partial \theta_i} \right] + \bar{e}_t^T \frac{\partial \bar{e}_t}{\partial \theta_i}, \quad i = 1, \ldots, p. \qquad (6)$$

The eSRIF:

$$O_t \left[\begin{array}{ccc|c} R_t^{-T/2} & -R_t^{-T/2} H_t F_t^{-1} & R_t^{-T/2} H_t F_t^{-1} G_t Q_t^{T/2} & -R_t^{-T/2} z_t \\ 0 & P_t^{-T/2} F_t^{-1} & -P_t^{-T/2} F_t^{-1} G_t Q_t^{T/2} & P_t^{-T/2} \hat{x}_t \\ 0 & 0 & I_q & 0 \end{array} \right]$$

$$= \left[\begin{array}{ccc|c} R_{e,t}^{-T/2} & 0 & 0 & -\bar{e}_t \\ -P_{t+1}^{-T/2} K_{p,t} & P_{t+1}^{-T/2} & 0 & P_{t+1}^{-T/2} \hat{x}_{t+1} \\ * & * & * & * \end{array} \right]$$

LLG Evaluation

$$\frac{\partial l(z_t)}{\partial \theta_i} = -tr\left[R_{e,t}^{1/2} \frac{\partial \left(R_{e,t}^{-1/2} \right)}{\partial \theta_i} \right] + \bar{e}_t^T \frac{\partial \bar{e}_t}{\partial \theta_i}, \quad i = 1, \ldots, p. \qquad (6)$$

The eSRIF:

$$O_t \left[\begin{array}{ccc|c} R_t^{-T/2} & -R_t^{-T/2} H_t F_t^{-1} & R_t^{-T/2} H_t F_t^{-1} G_t Q_t^{T/2} & -R_t^{-T/2} z_t \\ 0 & P_t^{-T/2} F_t^{-1} & -P_t^{-T/2} F_t^{-1} G_t Q_t^{T/2} & P_t^{-T/2} \hat{x}_t \\ 0 & 0 & I_q & 0 \end{array} \right]$$

$$= \left[\begin{array}{ccc|c} R_{e,t}^{-T/2} & 0 & 0 & -\bar{e}_t \\ -P_{t+1}^{-T/2} K_{p,t} & P_{t+1}^{-T/2} & 0 & P_{t+1}^{-T/2} \hat{x}_{t+1} \\ * & * & * & * \end{array} \right]$$

LLG Evaluation

$$\frac{\partial l(z_t)}{\partial \theta_i} = -tr\left[R_{e,t}^{1/2} \frac{\partial \left(R_{e,t}^{-1/2} \right)}{\partial \theta_i} \right] + \bar{e}_t^T \frac{\partial \bar{e}_t}{\partial \theta_i}, \; i = 1, \ldots, p. \qquad (6)$$

The eSRIF:

$$O_t \left[\begin{array}{ccc|c} R_t^{-T/2} & -R_t^{-T/2} H_t F_t^{-1} & R_t^{-T/2} H_t F_t^{-1} G_t Q_t^{T/2} & -R_t^{-T/2} z_t \\ 0 & P_t^{-T/2} F_t^{-1} & -P_t^{-T/2} F_t^{-1} G_t Q_t^{T/2} & P_t^{-T/2} \hat{x}_t \\ 0 & 0 & I_q & 0 \end{array} \right]$$

$$= \left[\begin{array}{ccc|c} R_{e,t}^{-T/2} & 0 & 0 & -\bar{e}_t \\ -P_{t+1}^{-T/2} K_{p,t} & P_{t+1}^{-T/2} & 0 & P_{t+1}^{-T/2} \hat{x}_{t+1} \\ * & * & * & * \end{array} \right]$$

LLG Evaluation

Lemma 1. Let

$$QA = L \qquad (7)$$

where Q is any orthogonal transformation such that the matrix on the right-hand side of formula (7) is lower triangular and A is a nonsingular matrix. If the elements of A are differentiable functions of a parameter θ then the upper triangular matrix U in

$$Q'_\theta Q^T = \bar{U}^T - \bar{U} \qquad (8)$$

is, in fact, the strictly upper triangular part of the matrix $QA'_\theta L^{-1}$:

$$QA'_\theta L^{-1} = \bar{L} + D + \bar{U} \qquad (9)$$

where $\bar{L}$, D and $\bar{U}$ are, respectively, strictly lower triangular, diagonal and strictly upper triangular.

Algorithm LLG-eSRIF

I. For each θ_i, $i = 1, \ldots, p$, apply the eSRIF

$$O_t \left[\begin{array}{ccc|c} R_t^{-T/2} & -R_t^{-T/2} H_t F_t^{-1} & R_t^{-T/2} H_t F_t^{-1} G_t Q_t^{T/2} & -R_t^{-T/2} z_t \\ 0 & P_t^{-T/2} F_t^{-1} & -P_t^{-T/2} F_t^{-1} G_t Q_t^{T/2} & P_t^{-T/2} \hat{x}_t \\ 0 & 0 & I_q & 0 \end{array} \right] = \left[\begin{array}{ccc|c} R_{e,t}^{-T/2} & 0 & 0 & -\bar{e}_t \\ -P_{t+1}^{-T/2} K_{p,t} & P_{t+1}^{-T/2} & 0 & P_{t+1}^{-T/2} \hat{x}_{t+1} \\ * & * & * & * \end{array} \right]$$

where O_t is any orthogonal transformation such that the matrix on the right-hand side of the formula is block lower triangular.

Algorithm LLG-eSRIF

II. For each θ_i, $i = 1, \ldots, p$, calculate

$$O_t \left[\begin{array}{ccc|c} \frac{\partial}{\partial \theta_i}\left(R_t^{-T/2}\right) & \frac{\partial}{\partial \theta_i}\left(S_t^{(1)}\right) & \frac{\partial}{\partial \theta_i}\left(S_t^{(2)}\right) & \frac{\partial}{\partial \theta_i}\left(S_t^{(3)}\right) \\ 0 & \frac{\partial}{\partial \theta_i}\left(S_t^{(4)}\right) & \frac{\partial}{\partial \theta_i}\left(S_t^{(5)}\right) & \frac{\partial}{\partial \theta_i}\left(S_t^{(6)}\right) \\ 0 & 0 & 0 & 0 \end{array} \right] = \left[\begin{array}{ccc|c} X_i & Y_i & M_i & L_i \\ N_i & V_i & W_i & K_i \\ * & * & * & * \end{array} \right],$$

O_t is the same orthogonal transformation as in the eSRIF and

$$S_t^{(1)} = -R_t^{-T/2} H_t F_t^{-1}, \quad S_t^{(2)} = R_t^{-T/2} H_t F_t^{-1} G_t Q_t^{T/2}, \quad S_t^{(3)} = -R_t^{-T/2} z_t,$$
$$S_t^{(4)} = P_t^{-T/2} F_t^{-1}, \quad S_t^{(5)} = -P_t^{-T/2} F_t^{-1} G_t Q_t^{T/2}, \quad S_t^{(6)} = P_t^{-T/2} \hat{x}_t.$$

Algorithm LLG-eSRIF

III. For each θ_i, $i = 1, \ldots, p$, compute

$$J_i = \begin{bmatrix} X_i & Y_i & M_i \\ N_i & V_i & W_i \end{bmatrix} \begin{bmatrix} R_{e,t}^{-T/2} & 0 & 0 \\ -P_{t+1}^{-T/2} K_{p,t} & P_{t+1}^{-T/2} & 0 \\ * & * & * \end{bmatrix}^{-1}.$$

Algorithm LLG-eSRIF

IV. For each θ_i, $i = 1, \dots, p$, split the matrices

$$J_i = \overbrace{\Big[\ \underbrace{L_i + D_i + U_i}_{m+n}\ \Big|\ \ * * *\ \ \Big]}^{m+n+q} \Big\}{m+n}$$

where L_i, D_i and U_i are the strictly lower triangular, diagonal and strictly upper triangular parts of J_i, respectively.

Algorithm LLG-eSRIF

V. For each θ_i, $i = 1, \ldots, p$, compute the following quantities:

$$\begin{bmatrix} \dfrac{\partial R_{e,t}^{-T/2}}{\partial \theta_i} & 0 \\ -\dfrac{\partial \left(\tilde{P}_{t+1}^{-T/2} K_{p,t} \right)}{\partial \theta_i} & \dfrac{\partial \tilde{P}_{t+1}^{-T/2}}{\partial \theta_i} \end{bmatrix} = \left[L_i + D_i + U_i^T \right] \times \begin{bmatrix} R_{e,t}^{-T/2} & 0 \\ -\tilde{P}_{t+1}^{-T/2} K_{p,t} & \tilde{P}_{t+1}^{-T/2} \end{bmatrix}, \qquad (12)$$

Algorithm LLG-eSRIF

$$\frac{\partial \overline{e_t}}{\partial \theta_i} = \left[\frac{\partial R_{e,t}^{-T/2}}{\partial \theta_i} - X_i \right] R_{e,t}^{T/2} \overline{e_t} + Y_i F_t \hat{x}_t - L_i, \qquad (13)$$

Algorithm LLG-eSRIF

$$\frac{\partial \bar{e}_t}{\partial \theta_i} = \left[\frac{\partial R_{e,t}^{-T/2}}{\partial \theta_i} - X_i \right] R_{e,t}^{T/2} \bar{e}_t + Y_i F_t \hat{x}_t - L_i, \tag{13}$$

$$\begin{aligned} \frac{\partial S_{t+1}^{(6)}}{\partial \theta_i} = & \left[\frac{\partial \left(\tilde{P}_{t+1}^{-T/2} K_{p,t} \right)}{\partial \theta_i} + N_i \right] R_{e,t}^{T/2} \bar{e}_t \\ & + \left[\frac{\partial \tilde{P}_{t+1}^{-T/2}}{\partial \theta_i} - V_i \right] F_t \hat{x}_t + K_i. \end{aligned} \tag{14}$$

Algorithm LLG-eSRIF

VI. Finally, we have all values to compute the LLG according to (6):

$$\frac{\partial l_\theta(z_t)}{\partial \theta_i} = -\,\mathbf{tr}\left[R_{e,t}^{1/2} \frac{\partial \left(R_{e,t}^{-1/2} \right)}{\partial \theta_i} \right] + \bar{e}_t^T \frac{\partial \bar{e}_t}{\partial \theta_i}, \quad i = \overline{1,p}.$$

Algorithm LLG-eSRIF

VI. Finally, we have all values to compute the LLG according to (6):

$$\frac{\partial l_\theta(z_t)}{\partial \theta_i} = -\,\mathbf{tr}\left[R_{e,t}^{1/2} \frac{\partial \left(R_{e,t}^{-1/2} \right)}{\partial \theta_i} \right] + \bar{e}_t^T \frac{\partial \bar{e}_t}{\partial \theta_i}, \quad i = \overline{1,p}.$$

The LLG-eSRIF: Stage I

The LLG-eSRIF: Stages II —V

Algorithm LLG-eSRIF

VI. Finally, we have all values to compute the LLG according to (6):

$$\frac{\partial l_\theta(z_t)}{\partial \theta_i} = -\,\mathbf{tr}\left[R_{e,t}^{1/2} \frac{\partial\left(R_{e,t}^{-1/2}\right)}{\partial \theta_i} \right] + \bar{e}_t^T \frac{\partial \bar{e}_t}{\partial \theta_i}, \quad i = \overline{1,p}.$$

The LLG-eSRIF: Stage I

The LLG-eSRIF: Stages II —V

Algorithm LLG-eSRIF

VI. Finally, we have all values to compute the LLG according to (6):

$$\frac{\partial l_\theta(z_t)}{\partial \theta_i} = -\,\mathbf{tr}\left[R_{e,t}^{1/2} \frac{\partial \left(R_{e,t}^{-1/2} \right)}{\partial \theta_i} \right] + \bar{e}_t^T \frac{\partial \bar{e}_t}{\partial \theta_i}, \quad i = \overline{1,p}.$$

The LLG-eSRIF:

Stage I

The LLG-eSRIF:

Stages II —V

Algorithm LLG-eSRIF

The LLG-eSRIF consists of two parts:

Algorithm LLG-eSRIF

The LLG-eSRIF consists of two parts:

- The source filtering algorithm, i.e. eSRIF.

Algorithm LLG-eSRIF

The LLG-eSRIF consists of two parts:

- The source filtering algorithm, i.e. eSRIF.
- The ”differentiated” part: Stages II — V.

Algorithm LLG-eSRIF

The LLG-eSRIF consists of two parts:

- The source filtering algorithm, i.e. eSRIF.
- The ”differentiated” part: Stages II — V.

Thus, the Algorithm LLG-eSRIF is ideal for simultaneous

Algorithm LLG-eSRIF

The LLG-eSRIF consists of two parts:

- The source filtering algorithm, i.e. eSRIF.
- The ”differentiated” part: Stages II — V.

Thus, the Algorithm LLG-eSRIF is ideal for simultaneous

- state estimation

Algorithm LLG-eSRIF

The LLG-eSRIF consists of two parts:

- The source filtering algorithm, i.e. eSRIF.
- The ”differentiated” part: Stages II — V.

Thus, the Algorithm LLG-eSRIF is ideal for simultaneous

- state estimation
- parameter identification.

Algorithm LLG-eSRIF

Remark. Since the matrices in LLG (6) are triangular, only the diagonal elements of $R_{e,t}^{1/2}$ and $\frac{\partial\left(R_{e,t}^{-1/2}\right)}{\partial\theta_i}$ need to be computed. Hence, the Algorithm LLG-eSRIF allows the $m \times m$-matrix inversion of $R_{e,t}$ to be avoided in the evaluation of LLG.

Ill-Conditioned Example Problems

Problem 1 Given:

$$P_0 = \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix}, H = \begin{bmatrix} 1, & 0 \end{bmatrix}, R = e^2\theta,\ F = I_2,\ Q = 0,$$

$$G = \begin{bmatrix} 0, & 0 \end{bmatrix}^T,$$ 0¡ e ¡¡ 1

where θ is an unknown parameter, I_2 is an identity 2×2 matrix; to simulate roundoff we assume

Ill-Conditioned Example Problems

Problem 1 Given:

$$P_0 = \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix}, H = \begin{bmatrix} 1, & 0 \end{bmatrix}, R = e^2\theta,\ F = I_2,\ Q = 0,$$

$$G = \begin{bmatrix} 0, & 0 \end{bmatrix}^T,$$ 0¡ e ¡¡ 1

where θ is an unknown parameter, I_2 is an identity 2×2 matrix; to simulate roundoff we assume $e + 1 \neq 1$ but $e^2 + 1 \overset{r}{=} 1$.

Ill-Conditioned Example Problems

Problem 1 Given:

$$P_0 = \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix}, H = \begin{bmatrix} 1, & 0 \end{bmatrix}, R = e^2\theta,\ F = I_2,\ Q = 0,$$

$$G = \begin{bmatrix} 0, & 0 \end{bmatrix}^T,$$ 0¡ e ¡¡ 1

where θ is an unknown parameter, I_2 is an identity 2×2 matrix; to simulate roundoff we assume $e + 1 \neq 1$ but $e^2 + 1 \overset{r}{=} 1$.

Calculate: $(P_1)'_\theta$ at the point $\theta = 1$.

Ill-Conditioned Example Problems

Problem 1 Given:

$$P_0 = \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix}, H = \begin{bmatrix} 1, & 0 \end{bmatrix}, R = e^2\theta,\ F = I_2,\ Q = 0,$$

$$G = \begin{bmatrix} 0, & 0 \end{bmatrix}^T,$$ 0¡ e ¡¡ 1

where θ is an unknown parameter, I_2 is an identity 2×2 matrix; to simulate roundoff we assume $e + 1 \neq 1$ but $e^2 + 1 \stackrel{r}{=} 1$.

Calculate: $(P_1)'_\theta$ at the point $\theta = 1$.

For the $\theta = 1$, this example illustrates the initialization problems [Kaminski P.G., Bryson A.E., Schmidt S.F., 1971] that result when $H_1 \Pi_0 H_1^T + R_1$ is rounded to $H_1 \Pi_0 H_1^T$.

Comparison (Problem 1)

Filter	Exact Answer	Rounded Answer

Comparison (Problem 1)

Filter	Exact Answer	Rounded Answer
'diff' KF	$(P_1)'_\theta = \begin{bmatrix} \frac{e^2}{1+e^2} & 0 \\ 0 & 1 \end{bmatrix}$	$\stackrel{r}{=} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Comparison (Problem 1)

Filter	Exact Answer	Rounded Answer
'diff' KF	$(P_1)'_\theta = \begin{bmatrix} \frac{e^2}{1+e^2} & 0 \\ 0 & 1 \end{bmatrix}$	$\stackrel{r}{=} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
'diff' IF	$\left(P_1^{-1}\right)'_\theta = -\begin{bmatrix} \frac{1+e^2}{e^2} & 0 \\ 0 & 1 \end{bmatrix}$	$\stackrel{r}{=} -\begin{bmatrix} \frac{1}{e^2} & 0 \\ 0 & 1 \end{bmatrix}$

Comparison (Problem 1)

Filter	Exact Answer	Rounded Answer
'diff' KF	$(P_1)'_\theta = \begin{bmatrix} \frac{e^2}{1+e^2} & 0 \\ 0 & 1 \end{bmatrix}$	$\stackrel{r}{=} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
'diff' IF	$\left(P_1^{-1}\right)'_\theta = -\begin{bmatrix} \frac{1+e^2}{e^2} & 0 \\ 0 & 1 \end{bmatrix}$	$\stackrel{r}{=} -\begin{bmatrix} \frac{1}{e^2} & 0 \\ 0 & 1 \end{bmatrix}$
LLG-eSRIF	$\left(P_1^{-1/2}\right)'_\theta - \frac{1}{2}\begin{bmatrix} \frac{\sqrt{1+e^2}}{e} & 0 \\ 0 & 1 \end{bmatrix}$	$\stackrel{r}{=} -\frac{1}{2}\begin{bmatrix} \frac{1}{e} & 0 \\ 0 & 1 \end{bmatrix}$

Ill-Conditioned Example Problems

Problem 2 Given:

$$P_0 = \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix}, H = \begin{bmatrix} 1, & 1 \end{bmatrix}, R = e^2\theta,\ F = I_2,\ Q = 0,$$

$$G = \begin{bmatrix} 0, & 0 \end{bmatrix}^T,$$ 0¡ e ¡¡ 1

where θ is an unknown parameter, I_2 is an identity 2×2 matrix; to simulate roundoff we assume

Ill-Conditioned Example Problems

Problem 2 Given:

$$P_0 = \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix}, H = \begin{bmatrix} 1, & 1 \end{bmatrix}, R = e^2\theta,\ F = I_2,\ Q = 0,$$

$$G = \begin{bmatrix} 0, & 0 \end{bmatrix}^T,$$ 0¡ e ¡¡ 1

where θ is an unknown parameter, I_2 is an identity 2×2 matrix; to simulate roundoff we assume $e + 1 \neq 1$ but $e^2 + 1 \stackrel{r}{=} 1$.

Ill-Conditioned Example Problems

Problem 2 Given:

$$P_0 = \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix}, H = \begin{bmatrix} 1, & 1 \end{bmatrix}, R = e^2\theta,\ F = I_2,\ Q = 0,$$

$$G = \begin{bmatrix} 0, & 0 \end{bmatrix}^T,$$ 0¡ e ¡¡ 1

where θ is an unknown parameter, I_2 is an identity 2×2 matrix; to simulate roundoff we assume $e + 1 \neq 1$ but $e^2 + 1 \stackrel{r}{=} 1$.

Calculate: $(P_1)'_\theta$ at the point $\theta = 1$.

Comparison (Problem 2)

Filter	Exact Answer	Rounded Answer

Comparison (Problem 2)

Filter	Exact Answer	Rounded Answer
'diff' KF	$\frac{1}{2+e^2}\begin{bmatrix} 1+e^2 & -1 \\ -1 & 1+e^2 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Comparison (Problem 2)

Filter	Exact Answer	Rounded Answer
'diff' KF	$\frac{1}{2+e^2}\begin{bmatrix} 1+e^2 & -1 \\ -1 & 1+e^2 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$
'diff' IF	$-\frac{1}{e^2}\begin{bmatrix} 1+e^2 & 1 \\ 1 & 1+e^2 \end{bmatrix}$	$-\frac{1}{e^2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Comparison (Problem 2)

Filter	Exact Answer	Rounded Answer
'diff' KF	$\frac{1}{2+e^2}\begin{bmatrix} 1+e^2 & -1 \\ -1 & 1+e^2 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$
'diff' IF	$-\frac{1}{e^2}\begin{bmatrix} 1+e^2 & 1 \\ 1 & 1+e^2 \end{bmatrix}$	$-\frac{1}{e^2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
LLG-eSRIF	$-\frac{1}{2}\begin{bmatrix} \sqrt{\frac{2+e^2}{1+e^2}} & \frac{1}{e\sqrt{1+e^2}} \\ 0 & \frac{\sqrt{1+e^2}}{e} \end{bmatrix}$	$-\frac{1}{2}\begin{bmatrix} \sqrt{2} & \frac{1}{e} \\ 0 & \frac{1}{e} \end{bmatrix}$

Numerical Results

Example. Let the test system (1), (2) be defined as follows:

$$x_{t+1} = \begin{bmatrix} 1 & \Delta t \\ 0 & e^{-\Delta t/\tau} \end{bmatrix} x_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_t,$$

$$z_t = \begin{bmatrix} 1 & 0 \end{bmatrix} x_t + v_t$$

where τ is an unknown parameter which needs to be estimated.

Numerical Results

Example. Let the test system (1), (2) be defined as follows:

$$x_{t+1} = \begin{bmatrix} 1 & \Delta t \\ 0 & e^{-\Delta t/\tau} \end{bmatrix} x_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_t ,$$

$$z_t = \begin{bmatrix} 1 & 0 \end{bmatrix} x_t + v_t$$

where τ is an unknown parameter which needs to be estimated. For the test problem, $\tau^*=15$ was chosen as the true value of parameter τ.

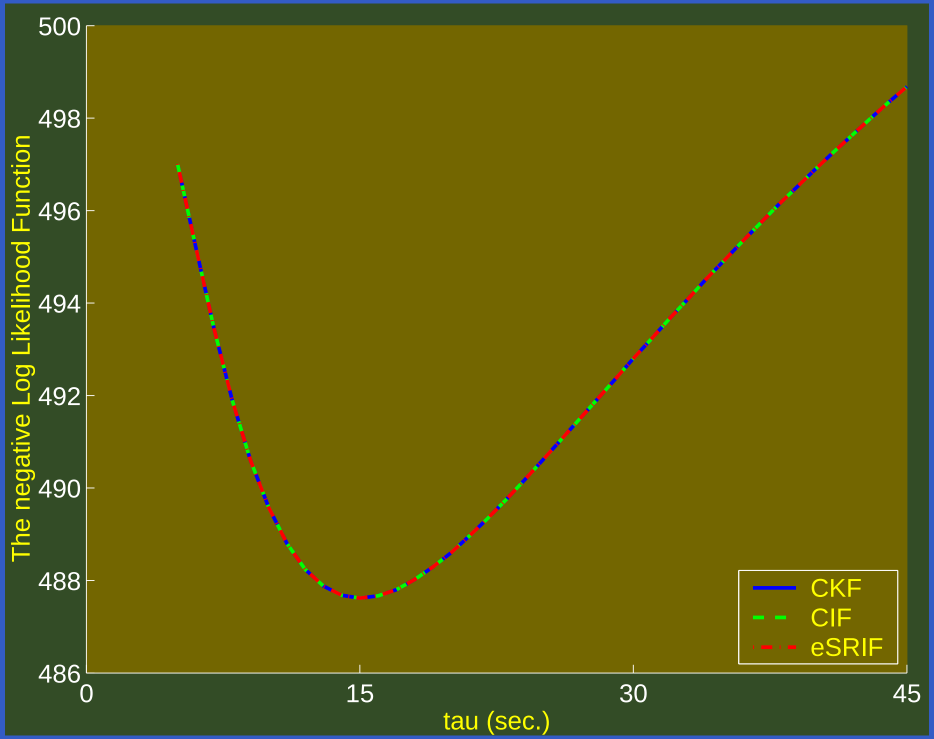
Numerical Results

The negative Log Likelihood Function

The Log Likelihood Gradient

Numerical Results

The negative Log Likelihood Function



The Log Likelihood Gradient

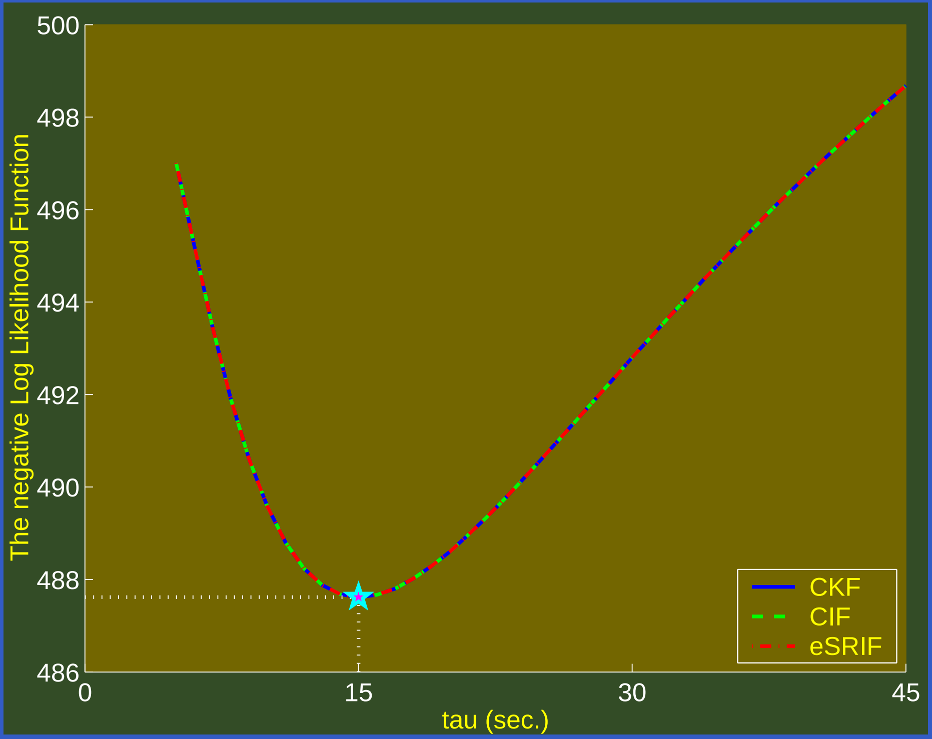
Numerical Results

The negative Log Likelihood Function

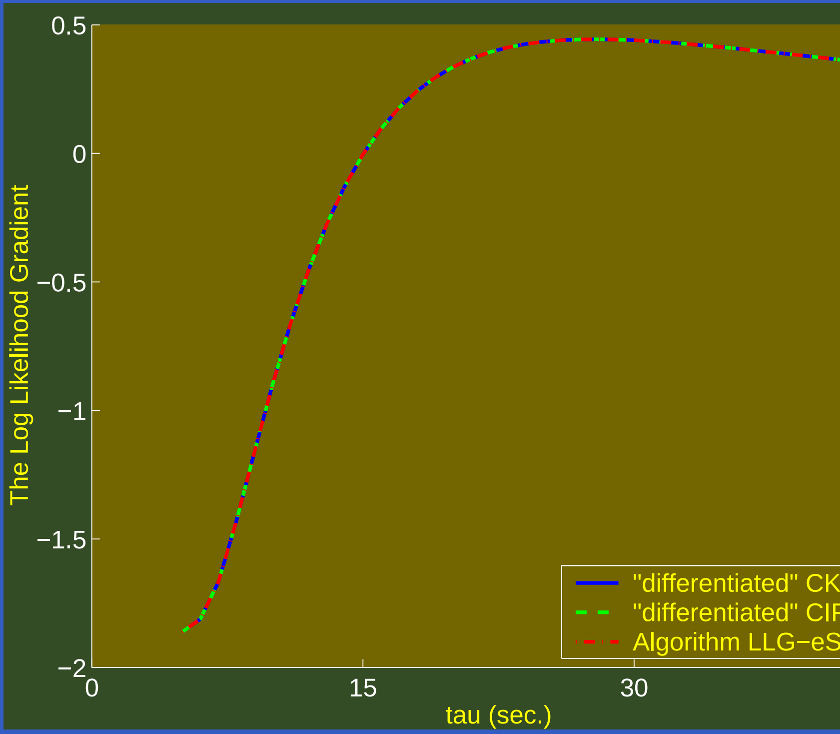
The Log Likelihood Gradient

Numerical Results

The negative Log Likelihood Function

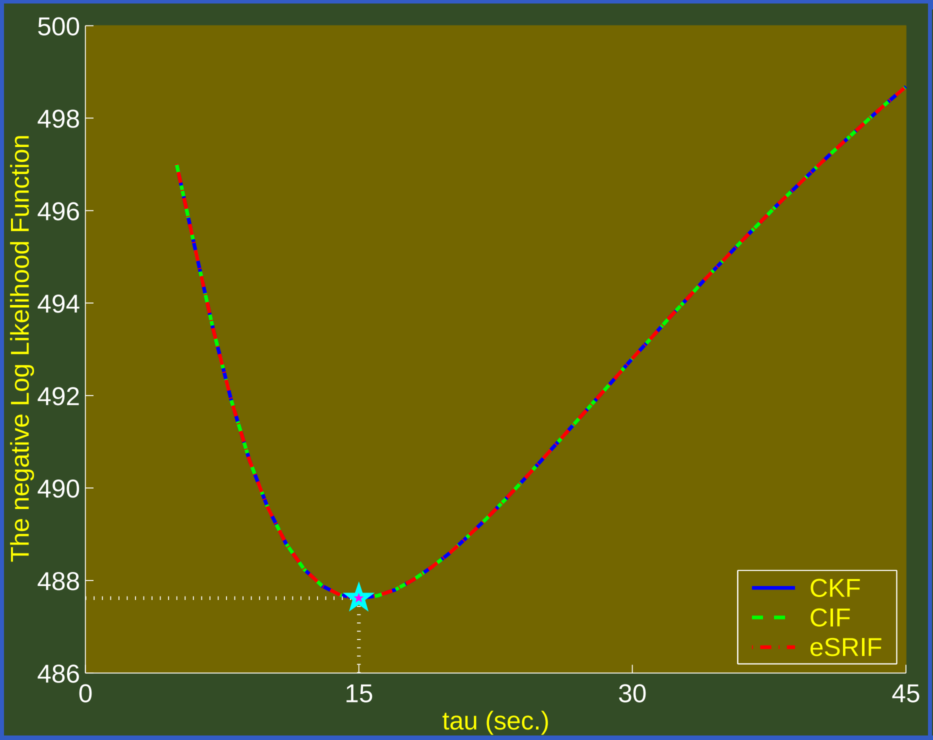


The Log Likelihood Gradient

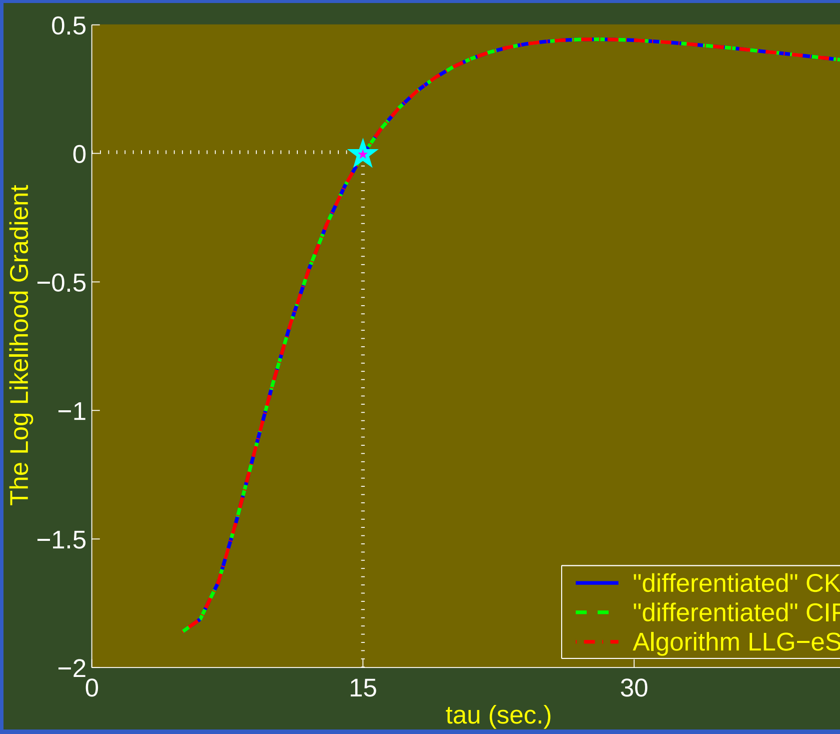


Numerical Results

The negative Log Likelihood Function



The Log Likelihood Gradient



Conclusion

In this paper, the new algorithm for evaluating the Log Likelihood Gradient (score) of linear discrete-time dynamic systems has been developed. The necessary theory has been given and substantiated by the computational experiments. Two ill-conditioned example problems have been constructed to show the superior perfomance of the Algorithm LLG-eSRIF over the conventional approach. All of these are good reasons to use the presented algorithm in practice.

Q & A

Maria Kulikova

University of the Witwatersrand, Johannesburg
South Africa