Course 216: Ordinary Differential Equations

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These notes cover the ODEs course given in 2007-2008 by Dr. John Stalker.

Contents

Terminology

Scalar equation A single ODE.

System of equations Several ODEs.

Order The order of an ODE is the order of the highest derivative appearing in it.

Linear / Non-linear A linear ODE is an ODE that is linear, etc.

Homogeneous / Inhomogeneous Homogeneous means no constant terms present. Inhomogeneous means constant terms are present.

Invariants An invariant of a system of ODEs is a function of the dependent and independent variables and their derivatives which is constant for any solution of the equation. They can be used to place bounds on solutions.

Part I
Solving Linear ODEs

1 Reduction of Order

- Any higher order ODE or system of ODEs can be reduced to a system of first order ODEs by introducing new variables to replace the derivatives in the original equation/system.
- For example, the third order equation

$$c_1 x'''(t) + c_2 x''(t) + c_3 x'(t) + c_4 x(t) = 0$$

can be reduced to a first order system using the following set of substitutions:

$$x_1 = x,\ x_2 = x',\ x_3 = x''$$

giving:

$$x_1' = x_2,\ x_2' = x_3,\ x_3' = -\frac{c_4}{c_1}x_1 - \frac{c_3}{c_1}x_2 - \frac{c_2}{c_1}x_3$$

We can write this in matrix form:

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{c_4}{c_1} & -\frac{c_3}{c_1} & -\frac{c_2}{c_1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

- Hence, any ODE or system of ODEs can be written in the following matrix form:

$$\vec{x}'(t) = A(t)\vec{x}(t)$$

which has solution:

$$\vec{x}(t) = \exp(tA)\,\vec{x}(0)$$

2 Computing Matrix Exponentials

- The exponential of the matrix tA is given by:

$$\exp(tA) = \sum_{n=0}^{\infty} \frac{1}{n!} t^n A^n$$

- For a diagonal matrix,

$$\exp \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & b & \dots & 0 \\ \vdots & & \ddots & \\ 0 & \dots & 0 & n \end{pmatrix} = \begin{pmatrix} \exp(a) & 0 & \dots & 0 \\ 0 & \exp(b) & \dots & 0 \\ \vdots & & \ddots & \\ 0 & \dots & 0 & \exp(n) \end{pmatrix}$$

- Given two matrices A and B then

$$\exp(A+B) = \exp(A)\exp(B)$$

if $AB = BA$. Note that any scalar multiple of the identity commutes with all matrices.

- **2 by 2 Matrices**

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{a+d}{2} & 0 \\ 0 & \frac{a+d}{2} \end{pmatrix} + \begin{pmatrix} \frac{a-d}{2} & b \\ c & \frac{d-a}{2} \end{pmatrix} = B + C$$

and we have $BC = CB$ so that $\exp(B+C) = \exp B \, \exp C$. Letting $\mu = \frac{a+d}{2}$, we then have

$$\exp(tA) = \exp(tB)\exp(tC)$$

$$\Rightarrow \exp(tA) = \begin{pmatrix} \exp(\mu t) & 0 \\ 0 & \exp(\mu t) \end{pmatrix} \exp(tC)$$

Now, the discriminant Δ of A is

$$\Delta = \Big(\mathrm{tr} A\Big)^2 - 4\,\mathrm{det} A$$

and $C^2 = \frac{\Delta}{4} I$. This leads to three cases:
i) $\Delta = 0$, then

$$\exp(tC) = I + tC$$

ii) $\Delta < 0$, then

$$\exp(tC) = \cos\Big(\frac{t\sqrt{-\Delta}}{2}\Big) I + \frac{\sin(\frac{t\sqrt{-\Delta}}{2})}{\frac{\sqrt{-\Delta}}{2}} C$$

iii) $\Delta > 0$, then

$$\exp(tC) = \cosh\Big(\frac{t\sqrt{\Delta}}{2}\Big) I + \frac{\sinh(\frac{t\sqrt{\Delta}}{2})}{\frac{\sqrt{\Delta}}{2}} C$$

- $n \times n$ **Matrices**

Every n by n matrix A is similar to its Jordan form J, which can be written as the sum of a diagonal and a nilpotent matrix, $J = D + N$. We have

$$A = PJP^{-1}$$

$$\Rightarrow \exp(tA) = P\,\exp(tJ)\,P^{-1}$$

$$\Rightarrow \exp(tA) = P\,\exp(tD)\,\exp(tN)\,P^{-1}$$

The Jordan form J has the eigenvalues of A on the diagonal, and some ones below the diagonal, depending on whether the eigenvalues are distinct. The columns of the matrix P are the eigenvectors of A. The entries of P can also be found once you know J, using $AP = PJ$.

The exponential of the nilpotent matrix N is computed directly using the exponential formula.

Note that in the case of a higher order scalar equation, we only need the first row of P, as we are just looking for $x(t)$.

3 Higher Order Scalar ODEs

- Consider a higher order scalar ODE,

$$c_n \frac{d^n x}{dt^n} + \ldots + c_2 \frac{d^2 x}{dt^2} + c_1 \frac{dx}{dt} + c_0 x = 0$$

which we can write as

$$p\left(\frac{d}{dt}\right) x = 0$$

where p is the polynomial

$$p(s) = c_n s^n + \ldots + c_2 s^2 + c_1 s + c_0 = 0$$

which has roots λ_i.

- A basis for the solution space is then

$$\Big\{ \exp(\lambda_1 t), t \; \exp(\lambda_1 t), \ldots, t^{r_1 - 1} \exp(\lambda_1 t), \ldots, \exp(\lambda_k t), \ldots, t^{r_k - 1} \exp(\lambda_k t) \Big\}$$

where the λ_i are the individual roots of the equation and r_i is the multiplicity of the i^{th} root.

- In the inhomogeneous case, we have $p(\frac{d}{dt})x = f$, and have the special case where f itself satisfies some differential equation $q(\frac{d}{dt})f = 0$. Hence

$$q\left(\frac{d}{dt}\right) p\left(\frac{d}{dt}\right) x = 0$$

and we can form a basis for the solution space using the roots of $r(s) = q(s)p(s)$. It is then possible to evaluate the coefficients of the particular solution to the inhomogeneous equation by evaluating $p(\frac{d}{dt})x = f$

4 Non-constant Coefficients

- **Homogeneous Scalar Equations**

The homogeneous equation

$$x'(t) = a(t)x(t)$$

has unique solution:

$$x(t) = x(0) \; \exp\left(\int_0^t a(s) ds \right)$$

- **Inhomogeneous Scalar Equations**

  The inhomogeneous equation
  $$x'(t) = a(t)x(t) + f(t)$$
  has unique solution:
  $$x(t) = x(0)\,\exp\left(\int_0^t a(s)ds\right) + \int_0^t \exp\left(\int_s^t a(r)dr\right) f(s)ds$$

- **Systems**

  The equation
  $$\vec{x}'(t) = A(t)\vec{x}(t) + \vec{f}(t)$$
  has unique solution:
  $$\vec{x}(t) = W(t)\vec{x}(0) + \int_0^t W(t)W^{-1}(s)\vec{f}(s)ds$$
  where $W(t)$ satisfies the matrix initial value problem
  $$W'(t) = A(t)W(t),\ \ W(0) = I$$

5 Method of Wronski

- Consider a second order scalar linear homogeneous ODE:
  $$p(t)x''(t) + q(t)x'(t) + r(t)x(t) = 0 \qquad (1)$$
  which has a two-dimensional solution space.

- We define
  $$w(t) = x_1(t)x_2'(t) - x_1'(t)x_2(t)$$
  giving
  $$p(t)w'(t)+q(t)w(t) = x_1(t)\Big[p(t)x_2''(t)+q(t)x_2'(t)+r(t)x_2(t)\Big]-x_2(t)\Big[p(t)x_1''(t)+q(t)x_1'(t)+r(t)x_1(t)\Big]$$
  so if x_1, x_2 solve (1) then $w(t)$ solves
  $$p(t)w'(t) + q(t)w(t) = 0 \qquad (2)$$

- Hence, if we have x_1 a solution to (1) and w a solution to (2), we can then find x_2 such that x_2 is a solution to (1), and is linearly independent to x_1.

- Then, given (1) and x_1:

$$w(t) = w(0)\ \exp\left(-\int_0^t \frac{q(s)}{p(s)} ds\right)$$

and as $\frac{d}{dt}\left(\frac{x_2(t)}{x_1(t)}\right) = \frac{w(t)}{x_1^2(t)}$,

$$\frac{x_2(t)}{x_1(t)} = \frac{x_2(0)}{x_1(0)} + \int_0^t \frac{w(s)}{x_1(s)^2} ds$$

- The general solution is then any linear combination of x_1 and x_2:

$$x(t) = c_1 x(t) + c_2 x_2(t)$$

Part II
Stability

6 Non-linear ODEs

- **Non-linear ODEs**
  A non-linear ODE is of the form

$$\vec{x}'(t) = \vec{F}\Big(\vec{x}(t), t\Big)$$

- **Autonomous Systems**
  An autonomous system is of the form

$$\vec{x}'(t) = \vec{F}\Big(\vec{x}(t)\Big)$$

7 Equilibria and Stability

- **Equilibria**
  An equilibrium of an autonomous system $\vec{x}'(t) = \vec{F}\Big(\vec{x}(t)\Big)$ is a $\vec{c}$ such that

$$\vec{F}(\vec{c}) = 0$$

  i.e. the equilibria of a system are the zeros of $\vec{F}$.

- **Stability**
  An equilibrium $\vec{c}$ is said to be stable if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that if

$$||\, \vec{x}(0) - \vec{c}\, || \leq \delta$$

  then

$$||\, \vec{x}(t) - \vec{c}\, || \leq \varepsilon$$

  for all positive t.

- **Asymptotic Stability**

  An equilibrium $\vec{c}$ is said to be asymptotically stable if $\exists\,\delta > 0$ such that

$$||\,\vec{x}(0) - \vec{c}\,|| \leq \delta \Rightarrow \lim_{t\to\infty} \vec{x}(t) = \vec{c}$$

- **Strict Stability**

  An equilibrium $\vec{c}$ is said to be strictly stable if it is both stable and asymptotically stable.

- **Stability and Invariants**

  If $\vec{c}$ is an equilibrium of an autonomous system and E is a continuously differentiable invariant of the system which has a strict local minimum at $\vec{c}$, then $\vec{c}$ is stable but not asymptotically stable.

- **Stability of Linear Constant Coefficient First Order Systems**

  These are systems

$$\vec{x}'(t) = A\vec{x}(t)$$

  with solution

$$\vec{x}(t) = \exp(tA)\vec{x}(0) = P\exp(tJ)P^{-1}\vec{x}(0)$$

  $\vec{0}$ is always an equilibrium, and each equilibrium is stable/asymptotically stable if and only if $\vec{0}$ is stable/asymptotically stable.

  We can determine the stability of the system by considering the real parts of the eigenvalues of A:

Real Parts	**Stable**	**Asymptotically Stable**
all < 0	Yes	Yes
all ≤ 0, geometric multiplicity = algebraic multiplicity for all imaginary eigenvalues	Yes	No
all ≤ 0, geometric multiplicity < algebraic multiplicity for some imaginary eigenvalue	No	No
some > 0	No	No

  In the 2 by 2 case, then if trace $A < 0$ and det $A \geq 0$, then $\vec{0}$ is strictly stable. If trace $A \leq 0$ and det $A \geq 0$ then $\vec{0}$ is stable. Otherwise it is not stable or asymptotically stable.

  In the scalar high order case where $p(\frac{d}{dt})x = 0$, $p(s)$ a polynomial, if all roots of $p(s) = 0$ have negative real parts, then we have strict stability. If all roots have non-positive real parts, and all imaginary roots have multiplicity one, then we have stability but not strict stability. Otherwise, neither stability nor asymptotic stability.

8 Linearisation

- The linearisation of an autonomous system $\vec{x}'(t) = \vec{F}\Big(\vec{x}(t)\Big)$ about an equilibrium $\vec{c}$ is the matrix A defined by

$$a_{jk} = \frac{\partial F_j}{\partial x_k}(\vec{c})$$

- If all eigenvalues of A have negative real parts, then $\vec{c}$ is strictly stable.
- If some eigenvalue of A has positive real part, then $\vec{c}$ is neither stable nor asymptotically stable.
- Otherwise, we learn nothing.

9 Method of Lyapunov

- **Lyapunov Function**

  A Lyapunov function for the equilibrium $\vec{c}$ of an autonomous system is a continuously differentiable function V with a strict local minimum at $\vec{c}$ such that

$$\sum_j \frac{\partial V}{\partial x_j} F_j \le 0$$

- **Strict Lyapunov Function**

  A strict Lyapunov function is a Lyapunov function satisfying

$$\sum_j \frac{\partial V}{\partial x_j} F_j \le -r\Big[V(\vec{x}) - V(\vec{c})\Big]$$

  for some positive r.

- An equilibrium $\vec{c}$ is stable if it admits a Lyapunov function, and strictly stable if it admits a strict Lyapunov function.