2. Higher-order Linear ODE's

2A. Second-order Linear ODE's: General Properties

2A-1. On the right below is an abbreviated form of the ODE on the left:

$$(*) \qquad y'' + p(x)y' + q(x)y = r(x) \qquad\qquad Ly = r(x) \ ;$$

where L is the *differential operator*:

$$L = D^2 + p(x)D + q(x) \ .$$

a) If u_1 and u_2 are any two twice-differentiable functions, and c is a constant, then

$$L(u_1 + u_2) = L(u_1) + L(u_2) \qquad \text{and} \qquad L(c\,u) = c\,L(u).$$

Operators which have these two properties are called **linear** . Verify that L is linear, i.e., that the two equations are satisfied.

b) Show that if y_p is a solution to (*), then all other solutions to (*) can be written in the form

$$y = y_c + y_p \ ,$$

where y_c is a solution to the *associated homogeneous equation* $Ly = 0$.

2A-2.

a) By eliminating the constants, find a second-order linear homogeneous ODE whose general solution is $y = c_1e^x + c_2e^{2x}$.

b) Verify for this ODE that the IVP consisting of the ODE together with the initial conditions

$$y(x_0) = y_0, \qquad y'(x_0) = y'_0 \qquad y_0,\ y'_0 \ \text{ constants}$$

is always solvable.

2A-3.

a) By eliminating the constants, find a second-order linear homogeneous ODE whose general solution is $y = c_1x + c_2x^2$.

b) Show that there is no solution to the ODE you found in part (a) which satisfies the initial conditions $y(0) = 1, \quad y'(0) = 1$.

c) Why doesn't part (b) contradict the existence theorem for solutions to second-order linear homogeneous ODE's? (Book: Theorem 2, p. 110.)

2A-4. Consider the ODE $y'' + p(x)y' + q(x)y = 0$.

a) Show that if p and q are continuous for all x, a solution whose graph is tangent to the x-axis at some point must be identically zero, i.e., zero for all x.

b) Find an equation of the above form having x^2 as a solution, by calculating its derivatives and finding a linear equation connecting them. Why isn't part (a) contradicted, since the function x^2 has a graph tangent to the x axis at 0?

2A-5. Show that the following pairs of functions are linearly independent, by calculating their Wronskian.

a) $e^{m_1 x}$, $e^{m_2 x}$, $m_1 \neq m_2$ b) e^{mx}, xe^{mx} (can $m = 0$?)

2A-6. Consider $y_1 = x^2$ and $y_2 = x|x|$. (Sketch the graph of y_2.)

a) Show that $W(y_1, y_2) \equiv 0$ (i.e., is identically zero).

b) Show that y_1 and y_2 are not linearly dependent on any interval (a, b) containing 0. Why doesn't this contradict theorem 3b, p. 116 in your book?

2A-7. Let y_1 and y_2 be two solutions of $y'' + p(x)y' + q(x)y = 0$.

a) Prove that $\dfrac{dW}{dx} = -p(x)W$, where $W = W(y_1, y_2)$, the Wronskian.

b) Prove that if $p(x) = 0$, then $W(y_1, y_2)$ is always a constant.

c) Verify (b) by direct calculation for $y'' + k^2 y = 0$, $k \neq 0$, whose general solution is $y_1 = c_1 \sin kx + c_2 \cos kx$.

2B. Reduction of Order

2B-1. Find a second solution y_2 to $y'' - 2y' + y = 0$, given that one solution is $y_1 = e^x$, by three methods:

a) putting $y_2 = ue^x$ and determining $u(x)$ by substituting into the ODE;

b) determining $W(y_1, y_2)$ using Exercise 2A-7a, and from this getting y_2;

c) by using the general formula $y_2 = y_1 \displaystyle\int \frac{1}{y_1^2} e^{-\int p\,dx} dx$.

d) If you don't get the same answer in each case, account for the differences. (What is the most general form for y_2?)

2B-2. In Exercise 2B-1, prove that the general formula in part (c) for a second solution gives a function y_2 such that y_1 and y_2 are linearly independent. (Calculate their Wronskian.)

2B-3. Use the method of reduction of order (as in 2B-1a) to find a second solution to

$$x^2 y'' + 2xy' - 2y = 0 \ ,$$

given that one solution is $y_1 = x$.

2B-4. Find the general solution on the interval $(-1, 1)$ to the ODE

$$(1 - x^2)y'' - 2xy' + 2y = 0,$$

given that $y_1 = x$ is a solution.

2C. Second-order Linear ODE's with Constant Coefficients

2C-1. Find the general solution, or the solution satisfying the given initial conditions, to each of the following:

a) $y'' - 3y' + 2y = 0$

b) $y'' + 2y' - 3y = 0; \quad y(0) = 1,\ y'(0) = -1$

c) $y'' + 2y' + 2y = 0$

d) $y'' - 2y' + 5y = 0; \quad y(0) = 1,\ y'(0) = -1$

e) $y'' - 4y' + 4y = 0; \quad y(0) = 1,\ y'(0) = 1$

2C-2. Show by using the Wronskian criterion that $e^{ax}\cos bx$ and $e^{ax}\sin bx$ are linearly independent. Are there any restrictions on the constants a and b ?

2C-3. Consider $y'' + cy' + 4y = 0$, c constant. For each statement below, tell for what value(s) of c it holds (indicate reasoning):

a) the equation has oscillatory solutions

b) all solutions are damped oscillations

2C-4. **Euler's equidimensional equation** is the ODE

$$x^2y'' + pxy' + qy = 0; \qquad p,\ q \text{ constants.}$$

a) Show that setting $x = e^t$ changes it into an equation with constant coefficients.

b) Use this to find the general solution to $x^2y'' + xy' + y = 0$.

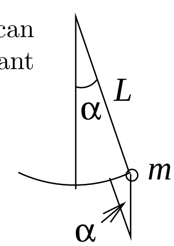
2C-5. The equation $mx'' + cx' + kx = 0$ represents the motion of a damped spring-mass system. (The independent variable is the time t.)

How are the constants m, c, k related if the system is critically damped (i.e., just on the edge of being oscillatory)?

2C-6. Show that the angle α of the pendulum swinging with small amplitude (so you can use the approximation $\sin\alpha \approx \alpha$) approximately obeys a second-order ODE with constant coefficients. Use

$$L = \text{length}, \qquad m = \text{ mass}, \qquad \text{damping } = mc\frac{d\alpha}{dt}, \quad \text{for some constant } c$$

If the motion is undamped, i.e., $c = 0$, express the period in terms of L, m, and the gravitational constant g.



2C-7. For each of the following, tell what you would use as the trial solution in determining a particular solution by the method of undetermined coefficients

a) $y'' + 2y' + 2y = x + e^x$

b) $y'' - 4y' = \cos 2x$

c) $y'' + 4y = 3\cos 2x$

d) $y'' - 2y' + y = 3e^x$

e) $y'' - 3y' + 2y = e^{-x} + 3e^{2x}$

f) $y'' - 6y' + 9y = 2xe^{3x}$

2C-8. Find the general solution, or the solution satisfying the given initial conditions:

a) $y'' - 6y' + 5y = e^x$

b) $y'' + 4y = 2\cos x, \quad y(0) = 0,\ y'(0) = 1$

c) $y'' + y' + y = 2x\,e^x$

d) $y'' - y = x^2, \quad y(0) = 0,\ y'(0) = -1$

2C-9. Consider the ODE $y'' + p(x)y' + q(x)y = r(x)$.

a) Prove that if y_i is a particular solution when $r = r_i(x)$, $(i = 1, 2)$, then $y_1 + y_2$ is a particular solution when $r = r_1 + r_2$. (Use the ideas of Exercise 2A-1.)

b) Use part (a) to find a particular solution to $y'' + 2y' + 2y = 2x + \cos x$.

2C-10. A series RLC-circuit is modeled by either of the ODE's (the second equation is just the derivative of the first)

$$Lq'' + Rq' + \frac{q}{C} = \mathcal{E},$$

$$Li'' + Ri' + \frac{i}{C} = \mathcal{E}',$$

where $q(t)$ is the charge on the capacitor, and $i(t)$ is the current in the circuit; $\mathcal{E}(t)$ is the applied electromotive force (from a battery or generator), and the constants L, R, C are respectively the inductance of the coil, the resistance, and the capacitance, measured in some compatible system of units.

a) Show that if $R = 0$ and $\mathcal{E} = 0$, then $q(t)$ varies periodically, and find the period. (Assume $L \neq 0$.)

b) Assume $\mathcal{E} = 0$; how must R, L, C be related if the current oscillates?

c) If $R = 0$ and $\mathcal{E} = E_0 \sin \omega t$, then for a certain ω_0, the current will have large amplitude whenever $\omega \approx \omega_0$. What is the value of ω_0. (Indicate reason.)

2D. Variation of Parameters

2D-1. Find a particular solution by variation of parameters:

a) $y'' + y = \tan x$

b) $y'' + 2y' - 3y = e^{-x}$

c) $y'' + 4y = \sec^2 2x$

2D-2. Bessel's equation of order p is $x^2y'' + xy' + (x^2 - p^2)y = 0$.
For $p = \frac{1}{2}$, two independent solutions for $x > 0$ are

$$y_1 = \frac{\sin x}{\sqrt{x}} \quad \text{and} \quad y_2 = \frac{\cos x}{\sqrt{x}}, \quad x > 0.$$

Find the general solution to

$$x^2y'' + xy' + (x^2 - \tfrac{1}{4})\, y = x^{3/2} \cos x\ .$$

2D-3. Consider the ODE $y'' + p(x)y' + q(x)y = r(x)$.

a) Show that the particular solution obtained by variation of parameters can be written as the definite integral

$$y \;=\; \int_a^x \frac{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1(x) & y_2(x) \end{vmatrix}}{W(y_1(t), y_2(t))}\, r(t)\, dt\ .$$

(Write the functions v_1 and v_2 (in the Variation of Parameters formula) as definite integrals.)

b) If instead the particular solution is written as an indefinite integral, there are arbitrary constants of integration, so the particular solution is not precisely defined. Explain why this doesn't matter.

2D-4. When *must* you use variation of parameters to find a particular solution, rather than the method of undetermined coefficients?

2E. Complex Numbers

All references are to Notes C: Complex Numbers

2E-1. Change to polar form: a) $-1+i$ b) $\sqrt{3}-i$.

2E-2. Express $\dfrac{1-i}{1+i}$ in the form $a+bi$ by two methods: one using the Cartesian form throughout, and one changing numerator and denominator to polar form. Show the two answers agree.

2E-3.* Show the distance between any two complex points z_1 and z_2 is given by $|z_2 - z_1|$.

2E-4. Prove two laws of complex conjugation:

for any complex numbers z and w, a) $\overline{z+w} = \overline{z} + \overline{w}$ b) $\overline{zw} = \overline{z}\,\overline{w}$.

2E-5.* Suppose $f(x)$ is a polynomial with *real* coefficients. Using the results of 2E-4, show that if $a+ib$ is a zero, then the complex conjugate $a-ib$ is also a zero. (Thus, complex roots of a real polynomial occur in conjugate pairs.)

2E-6.* Prove the formula $e^{i\theta}e^{i\theta'} = e^{i(\theta+\theta')}$ by using the definition (Euler's formula (9)), and the trigonometric addition formulas.

2E-7. Calculate each of the following two ways: by changing to polar form and using DeMoivre's formula (13), and also by using the binomial theorem.

a) $(1-i)^4$ b) $(1+i\sqrt{3})^3$

2E-8.* By using DeMoivre's formula (13) and the binomial theorem, express $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

2E-9. Express in the form $a+bi$ the six sixth roots of 1.

2E-10. Solve the equation $x^4 + 16 = 0$.

2E-11.* Solve the equation $x^4 + 2x^2 + 4 = 0$, expressing the four roots in both the polar form and the Cartesian form $a+bi$.

2E-12.* Calculate A and B explicitly in the form $a+bi$ for the cubic equation on the first page of Notes C, and then show that $A+B$ is indeed real, and a root of the equation.

2E-13.* Prove the law of exponentials (16), as suggested there.

2E-14. Express $\sin^4 x$ in terms of $\cos 4x$ and $\cos 2x$, using (18) and the binomial theorem. Why would you not expect $\sin 4x$ or $\sin 2x$ in the answer?

2E-15. Find $\displaystyle\int e^{2x}\sin x\,dx$ by using complex exponentials.

2E-16. Prove (18): a) $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$, b) $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$.

2E-17.* Derive formula (20): $D(e^{(a+ib)x}) = (a+ib)e^{(a+ib)x}$ from the definition of complex exponential and the derivative formula (19): $D(u+iv) = Du + iDv$.

2E-18.* Find the three cube roots of unity in the $a+bi$ form by locating them on the unit circle and using elementary geometry.

2F. Linear Operators and Higher-order ODE's

2F-1. Find the general solution to each of the following ODE's:

a) $(D-2)^3(D^2+2D+2)\,y=0$ b) $(D^8+2D^4+1)\,y=0$

c) $y^{(4)}+y=0$ d) $y^{(4)}-8y''+16y=0$

e) $y^{(6)}-y=0$ (use 2E-9) f) $y^{(4)}+16y=0$ (use 2E-10)

2F-2. Find the solution to $y^{(4)}-16y=0$, which in addition satisfies the four side conditions $y(0)=0$, $y'(0)=0$, $y(\pi)=1$, and $|y(x)|<K$ for some constant K and all $x>0$.

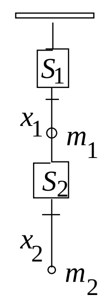
2F-3. Find the general solution to

a) $(D^3-D^2+2D-2)\,y=0$ b) $(D^3+D^2-2)\,y=0$

c) $y^{(3)}-2y'-4=0$ d) $y^{(4)}+2y''+4y=0$

(By high-school algebra, if m is a zero of a polynomial $p(D)$, then $(D-m)$ is a factor of $p(D)$. If the polynomial has integer coefficients and leading coefficient 1, then any integer zeros of $p(D)$ must divide the constant term.)

2F-4. A system consisting of two coupled springs is modeled by the pair of ODE's (we take the masses and spring constants to be 1; in the picture the S_i are springs, the m_i are the masses, and x_i represents the distance of mass m_i from its equilibrium position (represented here by a short horisontal line)):



$$x_1''+2x_1-x_2=0, \qquad x_2''+x_2-x_1=0\ .$$

a) Eliminate x_1 to get a 4th order ODE for x_2.

b) Solve it to find the general solution.

2F-5. Let $y=e^{2x}\cos x$. Find y'' by using operator formulas.

2F-6. Find a particular solution to

a) $(D^2+1)y=4e^x$ b) $y^{(3)}+y''-y'+2y=2\cos x$

c) $y''-2y'+4y=e^x\cos x$ d) $y''-6y'+9y=e^{3x}$

(Use the methods in Notes O; use complex exponentials where possible.)

2F-7. Find a particular solution to the general first-order linear equation with constant coefficients, $y'+ay=f(x)$, by assuming it is of the form $y_p=e^{-ax}u$, and applying the exponential-shift formula.

2G. Stability of Linear ODE's with Constant Coefficients

2G-1. For the equation $y'' + 2y' + cy = 0$, c constant,

(i) tell which values of c correspond to each of the three cases in Notes S, p.1;

(ii) for the case of two real roots, tell for which values of c both roots are negative, both roots are positive, or the roots have different signs.

(iii) Summarize the above information by drawing a c-axis, and marking the intervals on it corresponding to the different possibilities for the roots of the characteristic equation.

(iv) Finally, use this information to mark the interval on the c-axis for which the corresponding ODE is stable. (The stability criterion using roots is what you will need.)

2G-2. Prove the stability criterion (coefficient form) (Notes S,(8)), in the direction $\Longrightarrow$.

(You can assume that $a_0 > 0$, after multiplying the characteristic equation through by -1 if necessary. Use the high-school algebra relations which express the coefficients in terms of the roots.)

2G-3. Prove the stability criterion in the coefficient form (Notes S,(8)) in the direction $\Longleftarrow$. Use the quadratic formula, paying particular attention to the case of two real roots.

2G-4.* *Note: in what follows, formula references (11), (12), etc. are to Notes S.*

(a) Prove the higher-order stability criterion in the coefficient form (12).

(You can use the fact that a real polynomial factors into linear and quadratic factors, corresponding respectively to its real roots and its pairs of complex conjugate roots. You will need (11) and the stability criterion in the coefficient form for second-order equations.)

(b) Prove that the converse to (12) is true for those equations all of whose characteristic roots are real.

(Use an indirect proof — assume it is false and derive a contradiction.)

(c) To illustrate that the converse to (12) is in general false, show by using the criterion (11) that the equation $y''' + y'' + y' + 6y = 0$ is not stable. (Find a root of the characteristic equation by inspection, then use this to factor the characteristic polynomial.)

2G-5.* (a) Show when n = 2, the Routh-Hurwitz conditions (Notes S, (13)) are the same as the conditions given for second-order ODE's in (8).

(b) For the ODE $y''' + y'' + y' + cy = 0$, use the Routh-Hurwitz conditions to find all values of c for which the ODE is stable.

2G-6.* Take as the input $r(t) = At$, where A is a constant, in the ODE

$$ay'' + by' + cy = r(t), \qquad a,\ b,\ c \text{ constants}, \quad t = \text{time}. \tag{1}$$

a) Assume $a, b, c > 0$ and find by undetermined coefficients the steady-state solution. Express it in the form $K(t - d)$, where K and d are constants depending on the parameter A and on the coefficients of the equation.

b) We may think of d as the "time-delay". Going back to the two physical interpretations of (1) (i.e., springs and circuits), for each interpretation, express d in terms of the usual constants of the system (m-b-k, or R-L-C, depending on the interpretation).

2H. Impulse Response and Convolution

2H-1. Find the unit impulse response $w(t)$ to $y'' - k^2 y = f(t)$.

2H-2.* a) Find the unit impulse response $w(t)$ to $y'' - (a+b)y' + aby = f(t)$.

b) As $b \to a$, the associated homogeneous system turns into one having the repeated characteristic root a, and te^{at} as its weight function, according to Example 2 in the Notes. So the weight function $w(t)$ you found in part (a) should turn into $t\,e^{at}$, even though the two functions look rather different.

Show that indeed, $\lim\limits_{b \to a} w(t) = t\,e^{at}$. (Hint: write $b = a + h$ and find $\lim_{h \to 0}$.)

2H-3. a) Use (10) in Notes I to solve $y'' + 4y' + 4y = f(x)$, $y(0) = y'(0) = 0$, $x \ge 0$, where $f(x) = e^{-2x}$.
Check your answer by using the method of undetermined coefficients.

b)* Build on part (a) by using (10) to solve the IVP if $f(x) = \begin{cases} e^{-2x}, & 0 \le x \le 1; \\ 0, & x > 1\ . \end{cases}$

2H-4. Let $\phi(x) = \displaystyle\int_0^x (2x + 3t)^2 dt$. Calculate $\phi'(x)$ two ways:

a) by using Leibniz' formula
b) directly, by calculating $\phi(x)$ explicitly, and differentiating it.

2H-5.* Using Leibniz' formula, verify directly that these IVP's have the solution given:

a) $y'' + k^2 y = f(x)$, $y(0) = y'(0) = 0$; $\qquad y_p = \dfrac{1}{k}\displaystyle\int_0^x \sin k(x-t)\, f(t)\, dt.$

b) $y'' - 2ky' + k^2 y = f(x)$, $y(0) = y'(0) = 0$; $\qquad y_p = \displaystyle\int_0^x (x-t)e^{k(x-t)}\, f(t)\, dt.$

2H-6.* Find the following convolutions, as explicit functions $f(x)$:

a) $e^{ax} * e^{ax} = xe^{ax}$ (cf. (15)) b) $1 * x$ c) $x * x^2$

2H-7.* Give, with reasoning, the solution to Example 7.

2H-8.* Show $y' + ay = r(x)$, $y(0) = 0$ has the solution $y_p = e^{-ax} * r(x)$ by
a) Leibniz' formula
b) solving the IVP by the first-order method, using a definite integral (cf. Notes D).

2H-9.* There is an analogue of (10) for the IVP with non-constant coefficients:

$$(*) \qquad y'' + p(x)y' + q(x)y = f(x), \quad y(0) = y'(0) = 0.$$

It assumes you know the complementary function: $y_c = c_1 u(x) + c_2 v(x)$. It says

$$y(x) = \int_0^x g(x,t)\, f(t)\, dt, \qquad \text{where } g(x,t) = \frac{\begin{vmatrix} u(t) & v(t) \\ u(x) & v(x) \end{vmatrix}}{\begin{vmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{vmatrix}}.$$

By using Leibniz' formula, prove this solves the IVP (*).

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18.03 Notes and Exercises