

Figure L6b: Improved vs basic temperature control ($k = 0.25$).

16. Reduce the size of U to see the effect. What does this mean from a practical point of view with regards to the capabilities of the furnace or air conditioning unit?
17. Change the tolerance dT and/or the outside temperature to see to effect on the graph.

Laboratory 7

Implementing Higher-Order Differential Equations

(§4.1 and §4.3 of the Edwards/Penney text)

Numerical solution to IVPs involving higher order ODEs – homogeneous or not, linear or not, can be obtained using the same MATLAB commands as in the first-order by rewriting the ODE in the form of a system of first order ODEs.

Our objectives in this laboratory are as follows:

1. Learn how to reduce a higher-order IVP to a first-order system
2. Learn how to implement a higher-order/system ODE in MATLAB using `ode45.m` and `euler.m`.
3. Get some proficiency in analyzing and comparing solutions for different but similar problems.

Reducing a Higher-Order ODE

(see §4.1, Example 4, of the Edwards/Penney text)

Let's start with an example. Consider the IVP

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = \cos t, \quad \text{with } y(0) = -1, \frac{dy}{dt}(0) = 0. \quad (\text{L7.1})$$

To reduce the order of the ODE we introduce the intermediate unknown function $v = \frac{dy}{dt}$. As a result $\frac{dv}{dt} = \frac{d^2y}{dt^2}$ so that the ODE can be written $\frac{dv}{dt} + 4v + 3y = \cos t$. This equation only involves first-order derivatives, but we now have two unknown functions $y = y(t)$ and $v = v(t)$ with two ODEs. For MATLAB implementations it is necessary to write these ODEs in the form $\frac{d^*}{dt} = \dots$. Thus

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = \cos t \quad \Leftrightarrow \quad \begin{cases} \frac{dy}{dt} = v, \\ \frac{dv}{dt} = \cos t - 4v - 3y. \end{cases} \quad (\text{L7.2})$$

Initial conditions from (L7.1) must also be transformed into initial conditions for y and v . Simply,

$$y(0) = -1, \frac{dy}{dt}(0) = 0 \quad \Leftrightarrow \quad \begin{cases} y(0) = -1, \\ v(0) = 0. \end{cases} \quad (\text{L7.3})$$

We are now ready to implement the IVP in MATLAB.

MATLAB Implementation

The following code shows how to solve the IVP (L7.1) for $0 \leq t \leq 40$ using the MATLAB routine `ode45`.

```
function LAB07ex1
t0 = 0; tf = 40; % initial and final times
y0 = -1; v0 = 0; % initial conditions
[t,Y] = ode45(@f,[t0,tf],[y0,v0]);
y = Y(:,1); v = Y(:,2); % extract y and v from output matrix Y
figure(1); plot(t,y,'b.-',t,v,'r.-'); % time series for y and v
legend('y(t)','v(t)=y''(t)'); grid on % note the use of '' for '
ylim([-1.5,1.5])
figure(2); plot(y,v,'k.-'); % phase plot
xlabel('y'); ylabel('v=y'''); grid on
axis([-1,1,-1.5,1.5])
%-----
function dYdt = f(t,Y)
y = Y(1); v = Y(2);
dYdt = [ v ; cos(t)-4*v-3*y ];
```

Note how the initial conditions are specified in the calling sequence of `ode45`, in the form of an array `[y0,v0]` (`[y0;v0]` is also valid). The routine `ode45` returns a vector `t` containing the times in the interval $[0, 40]$ at which the solution was computed, and a matrix `Y` with two columns of the same length as `t`, one for each of y and v (in that order). Note how the approximations for y and v are retrieved from `Y`.

The time series for both y and v are plotted in the MATLAB figure window 1, and the phase plot showing v vs y is done in the figure window 2.

1. (a) For what (approximate) value(s) of t does y reach a local maximum in the window $0 \leq t \leq 40$? Check by reading the `Y` matrix and the `t` vector. Note that, because the M-file `LAB07ex1.m` is a function file, all the variables are local and thus not available on the Command Window. To read the matrix `Y` and the `t` vector you need to modify the M-file by adding the line `[t Y]`.

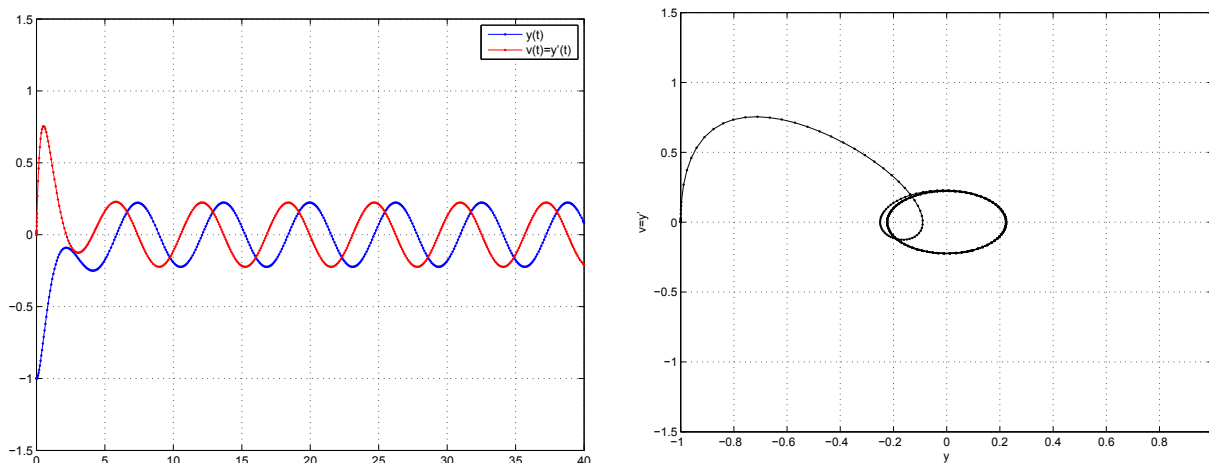


Figure L7a: Time series $y = y(t)$ and $v = v(t) = y'(t)$ (left), and phase plot $v = y'$ vs. y for (L7.1).

- (b) What seems to be the long term behavior of y ?
- (c) Modify the initial conditions and run the file LAB07ex1.m with the modified initial conditions. Does the long term behavior of the solution change? Explain why or why not.
2. Modify the file LAB07ex1.m to solve the IVP $y'' + 2y' + y = \cos t$ with $y(0) = y'(0) = 0$. Plot both time series of y and y' and the phase plot y' vs y . Describe the behavior of the solution y (in particular in the long term).

Additional Considerations

Using euler.m

The routine `euler.m` used in LAB1 can be used without modification to find an approximate solution for (L7.1):

```
function LAB07ex2
% driver for Figure L7b
t0 = 0; tf = 40; % initial and final times
y0 = -1; v0 = 0; % initial conditions
% ode45 solution
[t,Y] = ode45(@f,[t0,tf],[y0,v0]);
y = Y(:,1); v = Y(:,2);
% Euler solution
h = 0.1; N = round((tf-t0)/h);
[te,Ye] = euler(@f,[t0,tf],[y0,v0],N);
ye = Ye(:,1); ve = Ye(:,2);
figure(1); plot(t,y,'b.-',te,ye,'r.-'); % time series for y and v
ylim([-1.5,1.5])
legend('ode45','Euler'); grid on
figure(2); plot(y,v,'b.-',ye,ve,'r.-','Linewidth',2); % phase plot
xlabel('y'); ylabel('v=y'''); grid on
axis([-1,1,-1.5,1.5])
legend('ode45','Euler'); grid on
%-----
function dYdt = f(t,Y)
y = Y(1); v = Y(2);
dYdt = [ v ; cos(t)-4*v-3*y ];
```

For comparison the solution y obtained with `ode45` is included in the plot, see Figure L7b.

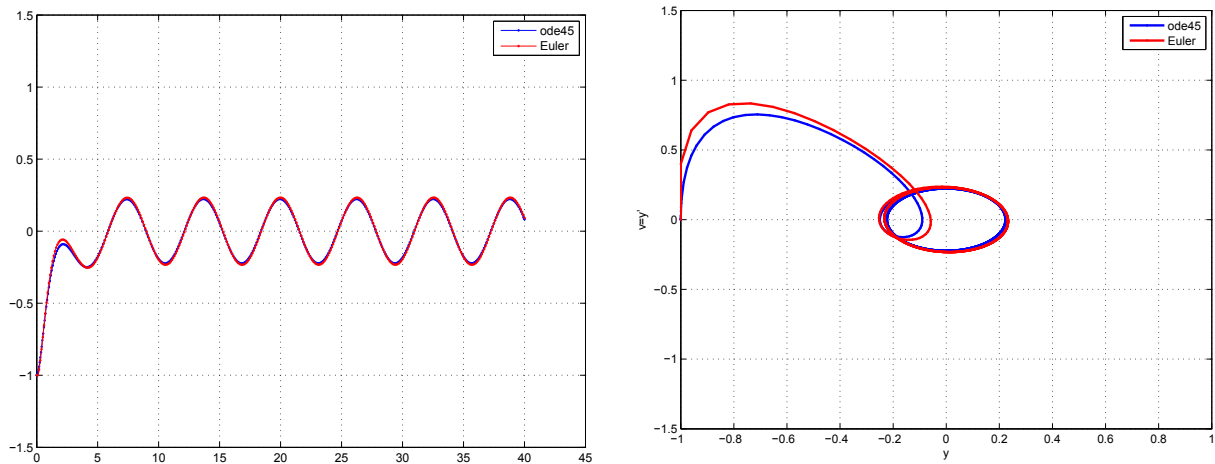


Figure L7b: Time series $y = y(t)$ (left) and phase plane $v = y'$ vs. y (right) obtained from `ode45.m` and `euler.m` for the problem (L7.1).

3. Modify the file `LAB07ex2.m` in order to solve the problem (L7.1) using the improved Euler method from Laboratory 2. Compare the results with those obtained from `ode45` and `euler` by plotting pictures similar to L7b but including the Improved Euler approximation as well.

Nonlinear Problems

Nonlinear problems do not present any additional difficulty from an implementation point of view (they may present new numerical challenges for integration routines like `ode45`). As an example consider the modified problem

$$\frac{d^2y}{dt^2} + 4y^2 \frac{dy}{dt} + 3y = \cos t, \quad \text{with } y(0) = -1, \frac{dy}{dt}(0) = 0. \tag{L7.4}$$

The ODE (L7.4) is very similar to (L7.1) except for the y^2 term in the left-hand side. Because of the factor y^2 the ODE (L7.4) is nonlinear, while (L7.1) is linear. There is however very little to change in the implementation of (L7.1) to solve (L7.4). The driver part in `LAB07ex1` is not modified (same initial conditions) while the ODE definition becomes

```
function dYdt = f(t,Y)
y = Y(1); v = Y(2);
dYdt = [ v ; cos(t)-4*y^2*v-3*y ];
```

The new output is shown in Fig L7c.

4. (a) Compare the output of Figs L7a and L7c. Describe the changes in the behavior of the solution in the short term.
 - (b) Increase the time interval of computation. Compare the long time behavior of both problems (L7.1) and (L7.4), in particular the amplitude of oscillations.
5. Solve numerically the IVP

$$\frac{d^2y}{dt^2} + 4|y| \frac{dy}{dt} + 3y = \cos t, \quad \text{with } y(0) = -1, \frac{dy}{dt}(0) = 0$$

in the interval $0 \leq t \leq 40$. Is the behavior of the solution significantly different from that of the solution of (L7.4)? Comment. In particular, compare the size of the long term oscillations in y with that of the problems (L7.1) and (L7.4).

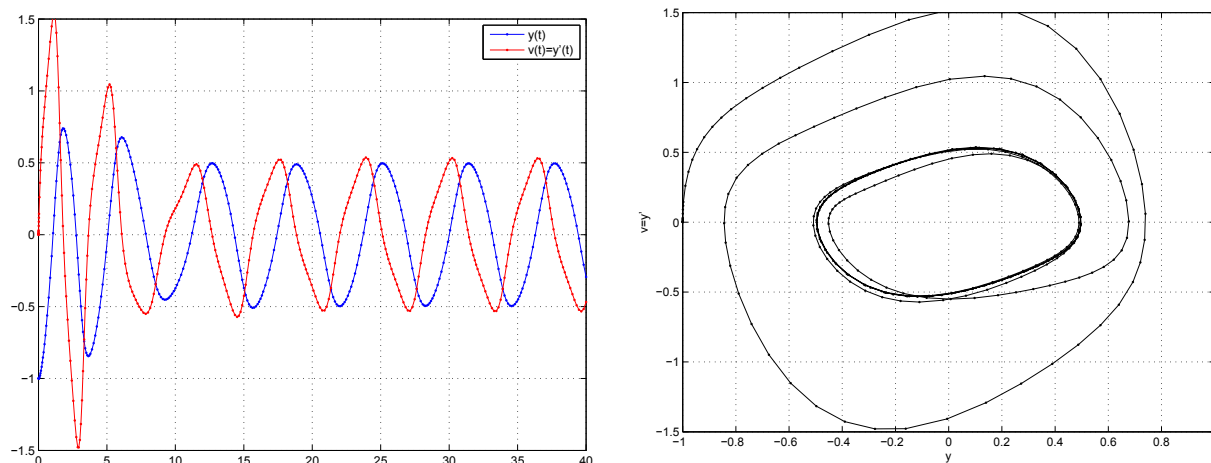


Figure L7c: Time series $y = y(t)$ and $v = v(t) = y'(t)$ (left), and phase plot $v = y'$ vs. y for (L7.4).

6. Solve numerically the IVP

$$\frac{d^2y}{dt^2} + 4y \frac{dy}{dt} + 3y = \cos t, \quad \text{with } y(0) = -1, \frac{dy}{dt}(0) = 0$$

in the interval $0 \leq t \leq 40$. Is the behavior of the solution significantly different from that of the solution of (L7.4)? Comment.

A Third-Order Problem

Consider the third-order IVP

$$\frac{d^3y}{dt^3} + 4y^2 \frac{d^2y}{dt^2} + 8y \left(\frac{dy}{dt} \right)^2 + 3 \frac{dy}{dt} = -\sin t, \quad \text{with } y(0) = -1, \frac{dy}{dt}(0) = 0, \frac{d^2y}{dt^2}(0) = 4. \quad (\text{L7.5})$$

Introducing $v = \frac{dy}{dt}$ and $w = \frac{d^2y}{dt^2}$ we obtain $\frac{dv}{dt} = w$ and $\frac{dw}{dt} = \frac{d^3y}{dt^3} = -\sin t - 4y^2w - 8yv^2 - 3v$. Moreover, $v(0) = \frac{dy}{dt}(0) = 0$ and $w(0) = \frac{d^2y}{dt^2}(0) = 4$. Thus (L7.5) is equivalent to

$$\begin{cases} \frac{dy}{dt} = v, \\ \frac{dv}{dt} = w, \\ \frac{dw}{dt} = -\sin t - 4y^2w - 8yv^2 - 3v \end{cases} \quad \text{with } \begin{cases} y(0) = -1, \\ v(0) = 0, \\ w(0) = 4. \end{cases} \quad (\text{L7.6})$$

This problem is implemented as follows.

```
function LAB07ex3
t0 = 0; tf = 40; % initial and final times
y0 = -1; v0 = 0; w0 = 4; % initial conditions
[t,Y] = ode45(@f,[t0,tf],[y0,v0,w0]);
y = Y(:,1); v = Y(:,2); w = Y(:,3); % extract y, v, w from Y
figure(1);
plot(t,y,'b.-',t,v,'r.-',t,w,'m.-'); % time series
legend('y(t)','v(t)=y''(t)','w(t)=y''''); % note the use of ''
grid on; ylim([-1.5,1.5]);
figure(2); plot3(y,v,w,'k.-'); % 3D phase plot
xlabel('y'); ylabel('v=y'''); zlabel('w=y'''');
view(-40,60); grid on
%-----
```

```
function dYdt = f(t,Y)
y = Y(1); v = Y(2); w = Y(3);
dYdt = [ v ; w; -sin(t)-4*y^2*w-8*y*v^2-3*v ];
```

The output is shown in Fig. L7d. The limits in the vertical axis of the plot on the left were deliberately set to the same ones as in Fig. L7c for comparison purposes using the MATLAB command `ylim([-1.5,1.5])`.

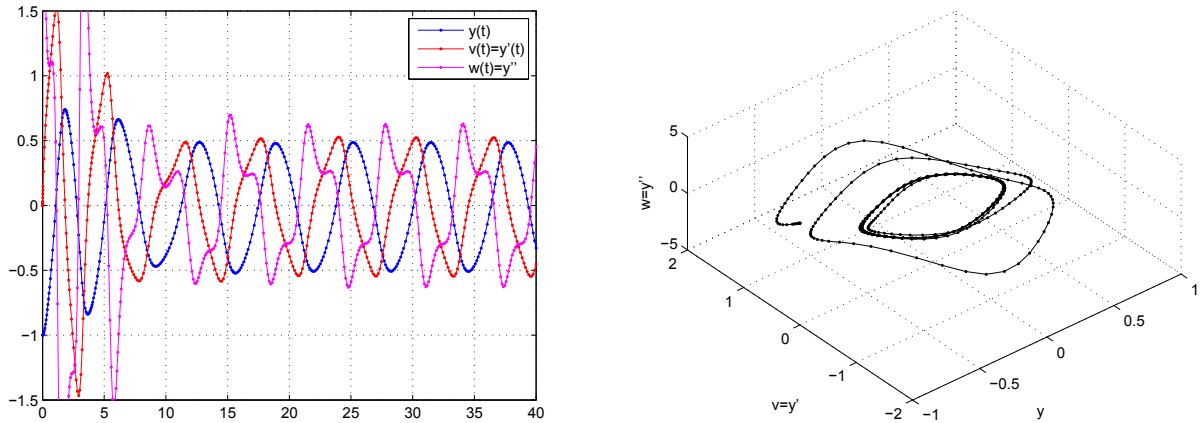


Figure L7d: Time series $y = y(t)$, $v = v(t) = y'(t)$, and $w = w(t) = y''(t)$ (left), and 3D phase plot $v = y'$ vs. y vs $w = y''$ for (L7.5) (rotated with `view = [-40,60]`).

7. (a) Compare the output of Figs L7c and L7d. What do you notice?
- (b) Differentiate the ODE in (L7.4) and compare to the ODE in (L7.5).
- (c) Explain why the solution of (L7.4) also satisfies the initial conditions in (L7.5).
- (d) Solve the problems (L7.4) and (L7.5) using Euler's method with step-size $h = .1$. Are the solutions identical? Comment.

Alternate Reduction

The choice of intermediate function v in the reduction of (L7.1) to first-order ODEs is not limited to $v = \frac{dy}{dt}$, although it is the simplest from a practical point of view. Other choices may be used, which may be more revealing from a purely mathematical point of view or lead to more accurate numerical results.

To reduce (L7.1) to a set of two first-order ODEs consider for example the choice $v = \frac{dy}{dt} + y$, i.e., $\frac{dy}{dt} = v - y$. Then

$$\frac{dv}{dt} = \frac{d^2y}{dt^2} + \frac{dy}{dt} = \left(\cos t - 4\frac{dy}{dt} - 3y \right) + \frac{dy}{dt} = \cos t - 3(v - y) - 3y = \cos t - 3v. \tag{L7.7}$$

To get an initial condition for v substitute $t = 0$ in the definition of v :

$$v(0) = \frac{dy}{dt}(0) + y(0) = 0 + (-1) = -1. \tag{L7.8}$$

Thus

$$(L7.1) \Leftrightarrow \begin{cases} \frac{dy}{dt} = v - y, \\ \frac{dv}{dt} = \cos t - 3v \end{cases} \quad \text{with} \quad \begin{cases} y(0) = -1, \\ v(0) = -1. \end{cases} \tag{L7.9}$$

Note that (L7.7) is independent of y and can be integrated directly, either analytically here, or numerically, for v . The solution y can then be obtained from v by direct integration or numerical quadrature.

8. Solve (L7.9) numerically using `ode45`:

(a) by solving the problem as a system;

(b) by first solving numerically for v (use only (L7.8) and (L7.9)), then use the following MATLAB commands to obtain y :

```
% assume v was obtained as [tv,v] = ode45(...)
[ty,y] = ode45(@odey,[0,12],??,[],tv,v); % fill-in IC
...
%-----
function dydt = odey(t,y,tv,ty)
v = interp1(tv,ty,t);           % interpolate v
dydt = v-y;
```

(c) Compare the solutions from part (a) and part (b).

9. Is there any other value of r such that the choice $v = \frac{dy}{dt} - ry$ leads to an ODE in v which is independent of y ?