Lab I - Direction fields and solution curves*†

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We consider differential equations having the form

$$y' = f(x, y), \qquad \text{that is} \qquad \frac{dy}{dx} = f(x, y). \tag{1}$$

In other words, $\frac{dy}{dx}$ equals some function of x and y.

Example 1.

a. $\frac{dy}{dx} = y$,

b. $\frac{dy}{dx} = y^2 - 2x + 5$.

Usually there is no explicit formula for the solution of such a differential equation, but we will develop a graphical method that at least gives us the *shape* of the solution.

Direction fields.

We want to find a solution $y = y(x)$ of the equation (1), meaning that the function $y(x)$ must satisfy $\frac{dy}{dx} = f(x, y(x))$. Thus at each point (x, y) on the graph of the solution, the slope of the graph must equal the number $f(x, y)$.

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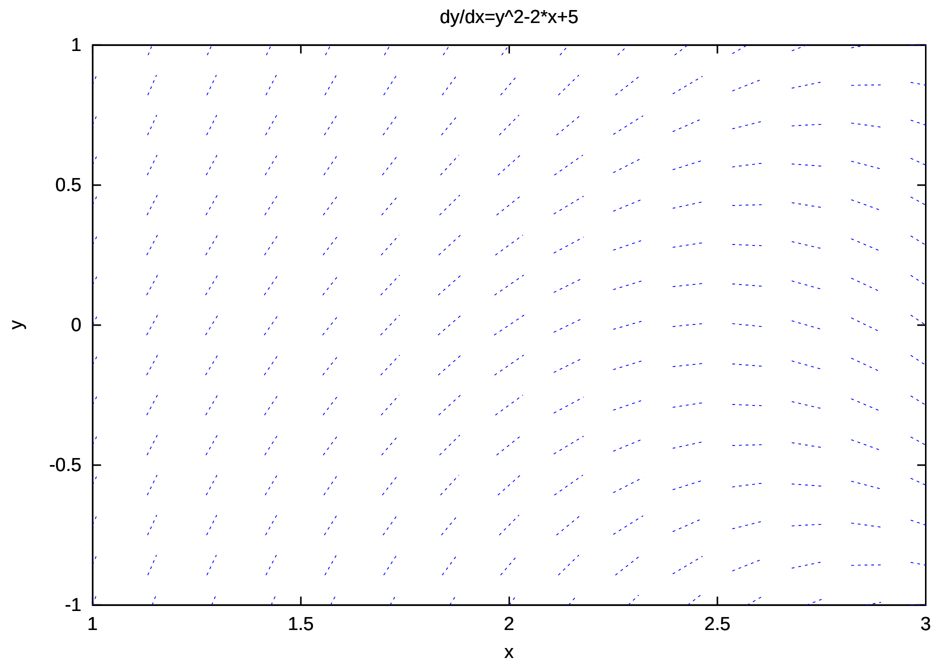
†This document is your guide through the first session with Iode. It is not a homework assignment, and you do not have to turn in any work.

Imagine that at each point (x, y) in the plane, you have drawn a *short* line segment that has slope $f(x, y)$. The collection of all these line segments is called a *direction field* or *slope field* for the differential equation.

For example, here is a portion of the direction field for the equation $\frac{dy}{dx} = y^2 - 2x + 5$, in Example 1(b):



Looking at the point $(2, 0)$ in the middle of this plot, you see the direction field has slope approximately 1 there. And indeed we compute with $x = 2$ and $y = 0$ that $y^2 - 2x + 5 = 0 - 4 + 5 = 1$.

Action 1. (Example 1(a), continued.) Consider $\frac{dy}{dx} = y$. Here the right hand side of the differential equation is $f(x, y) = y$, and so at the point (x, y) we want to draw a *short* line segment with slope y.

1. Try this on some scrap paper right now: start by sketching axes, then find the point $(1, 2)$ and sketch a short segment with slope 2, find $(1, 1)$ and sketch a short segment with slope 1, find $(1, 1/2)$ and sketch a short segment with slope $1/2$, find $(1, 0)$ and sketch a segment with slope 0, find $(1, -1)$ and sketch a segment with slope -1, and find $(1, -2)$ and sketch a segment with slope -2.

2. Now sketch the appropriate segments at some other points in the plane,

say at the points $(2, 3), (2, 2), (2, 1), (2, 0), (2, -1), (2, -2), (2, -3)$, and then at the points $(0, 3), (0, 2), (0, 1), (0, 0), (0, -1), (0, -2), (0, -3)$.

3. Try roughly sketching some curves to smoothly join up some of the line segments, wherever this seems possible. Do the resulting curves have a recognizable shape? If so, what? (*Hint.* Do you already know a solution formula for $\frac{dy}{dx} = y$?)

To reiterate, the point of all this sketching is that a solution $y(x)$ to the differential equation $\frac{dy}{dx} = f(x, y)$ must have the same slope as the corresponding line segments in the direction field, at each point of its graph, because the slope of the graph is $y'(x) = f(x, y)$ and this equals the slope of the line segment, at the point (x, y).

Plotting the direction field.

One can never draw *all* of the line segments in the direction field, but Iode can sketch enough of them to make the picture clear. (It is also possible to sketch the direction field by hand, but this tends to be tedious, like in the example above.)

Action 2.

1. Start up Iode following the instructions in your local guide "First Steps".

2. Pick the first menu item `Direction fields`, and wait for the graphics window to appear. It will show the direction field for the default equation $dy/dx = \sin(y - x)$. (This equation is not particularly important; it is just a nice example.)

3. In the graphics window, look along the diagonal where the line $y = x$ would be. The line segments should all be horizontal, since $dy/dx = \sin(y - x) = \sin(0) = 0$ when $y = x$.

4. Look where the x-axis would be (across the middle of the plot). The line segments have slopes $dy/dx = \sin(y - x) = \sin(-x)$, since $y = 0$ along the x-axis. You should find as you look left to right along the axis (from $x = -\pi$ to $x = 0$ to $x = \pi$) that the slopes of the segments change from positive to zero to negative, because $\sin(-x)$ is positive for $-\pi < x < 0$ and is negative for $0 < x < \pi$.

5. Look along the y-axis. Argue like in Part 4 to describe in one or two sentences how the slopes of the segments change, as you go up the y-axis. Explain *why* the slopes change this way:......

Plotting a solution.

Once you have plotted the direction field, you can just "sketch along the line segments" to get solution curves. (You can look at the direction field on screen and imagine sketching smooth curves that join up nearby line segments.)

But usually we want to find the solution curve that goes through a given "initial point" (x_0, y_0) in the plane. That is, we want the solution to satisfy the *initial condition* $y(x_0) = y_0$, for some given numbers x_0 and y_0. To graphically find this solution, you just locate the point (x_0, y_0) on the plot of the direction field and then "sketch along the line segments" from that point.

Action 3.

1. Now go back to the `Direction fields` window of Iode. Type some initial condition coordinates into the boxes on the right of the window. (You can enter any coordinates you like, but today it is best to choose coordinates that lie within the viewing window.) Then click on `Plot solution`.

2. A solution graph should appear in your window. Does it follow more-or-less along some of the line segments in the direction field? (The solution curve usually won't exactly hit the line segments shown, because only some of the line segments in the direction field are plotted, and these are probably not the ones we need for this particular solution curve.)

   The graph is a numerical approximation to the true solution of the differential equation. Later we will study the numerical methods used.

3. Another way to plot solutions is to just click in the graph area of the window. This tells Iode to plot a solution curve through the click point. In other words, it tells Iode to take the click coordinates (x_0, y_0) as the initial condition: $y(x_0) = y_0$.

4. Now spend 10 minutes playing around with the direction fields module of Iode, trying out the different control buttons and menu items, figuring out what they do. (*Note.* Ignore any "`Other`" options.) Time spent now learning the tricks of Iode will make you a "power user" later in the semester. Things you can try:

   • Plot more solutions, then right-click in the graph and undo them.

   • Left-drag the mouse to create a zoom box over part of the direction field. Then undo your zoom with a right-click.

   • Enter a new differential equation, using the `Equation` menu. See the end of this Lab for functions you can use, and for hints on valid Matlab syntax, e.g., type `x*exp(-(x^2)/5)+y/2` for $xe^{-x^2/5} + y/2$.

   • Use the `Equation` menu to change the display parameters, in particular to increase the number of line segments. Is the direction field easier to understand with 30 segments in each direction? 50? How many seems the "right" number?

   • Try relabelling the variables from x, y to t, x. (Many of the equations we will study are in terms of t and x.)

   • Try out the `Options` features, especially regarding zooming out. Then add a caption to your plot.

   • Increase the step size to 0.5 or 1, and plot some solutions. Do these graphs help explain why this parameter is called "step size"?

   • Print your graph, using the `File` menu.

Mathematical expressions in Matlab, Octave

[See also the Iode Manual, at `www.math.uiuc.edu/iode/documentation.html`.]

For simple expressions, we use the usual keyboard characters:

`2*x`	means $2x$,
`(x^3-1)/6`	means $(x^3 - 1)/6$,
`pi`	means π.

Built-in functions

`exp(x)`	exponential, e^x
`log(x)`	natural logarithm, $\ln x$
`log10(x)`	base 10 logarithm, $\log_{10} x$
`abs(x)`	absolute value, $\|x\|$
`sqrt(x)`	square root, $\sqrt{x}$
`sign(x)`	signum function, which equals $\begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$

`sin(x)`		`sinh(x)`	
`cos(x)`	trigonometric	`cosh(x)`	hyperbolic
`tan(x)`	functions	`tanh(x)`	trigonometric
`cot(x)`	(x in radians)	`coth(x)`	functions
`sec(x)`		`sech(x)`	
`csc(x)`		`csch(x)`	

`asin(x)`		`asinh(x)`	
`acos(x)`	inverse	`acosh(x)`	inverse
`atan(x)`	trigonometric	`atanh(x)`	hyperbolic
`acot(x)`	functions	`acoth(x)`	trigonometric
`asec(x)`		`asech(x)`	functions
`acsc(x)`		`acsch(x)`	

Example 2.

`sin(exp(y))^4`	means $\sin^4(e^y)$,
`acos(exp(1)^(-1))`	means $\arccos(e^{-1})$.

Logical expressions in Matlab, Octave and Iode

Expressions like `x>=2` are treated as logical functions, and return a value of either 1 (true) or 0 (false). So `x>=2` is the "step" function that equals $\begin{cases} 1 & \text{if } x \geq 2 \\ 0 & \text{otherwise} \end{cases}$.

Example 3. Logical functions help us create functions defined in pieces:

`(t^2)*(t<4)` means $\begin{cases} t^2 & \text{if } t < 4 \\ 0 & \text{otherwise} \end{cases}$.