MTH U345 **Ordinary Differential Equations** **Fall 2008**

Lab 1: Slope Fields and Solution Curves

The equation $y' = f(x, y)$ determines a *slope field* (or *direction field*) in the xy-plane: the value $f(x, y)$ is the slope of a tiny line segment at the point (x, y). It has geometrical significance for *solution curves* which are simply graphs of solutions $y(x)$ of the equation: at each point (x_0, y_0) on a solution curve, $f(x_0, y_0)$ is the slope of the tangent line to the curve. If we sketch the slope field, and then try to draw curves which are tangent to this field at each point, we get a good idea how the various solutions behave. In particular, if we pick (x_0, y_0) and try to draw a curve passing through (x_0, y_0) which is everywhere tangent to the slope field, we should get the graph of the solution of the *initial-value problem* $y' = f(x, y),\ y(x_0) = y_0$.

I. SKETCHING BY COMPUTER. The sketching of the slope field is tedious by hand, and is expedited by using a computer; the appropriate software will also enable us to plot solution curves. One such program called "HPGSolver" is available on the CD that comes with the Blanchard-Devaney-Hall textbook; another called "Diffs" is available on most of the computers in the Math Dept Computer Lab, 553 Lake Hall. Both programs are fairly self-explanatory, but one difference between them is that HPGSolver considers ODEs in the form $dy/dt = f(t, y)$, whereas Diffs considers $dy/dx = f(x, y)$. We shall use the former notation, which is consistent with the textbook.

Example 1. $dy/dt = y^2 - yt$.

(1) Enter the equation as $dy/dt = y\,\hat{}\,2 - y * t$.
(2) Change min y to -1 and max y to 4. Change min t to 0 and max t to 4. Click "Show Field".
(3) Select an initial value (t_0, y_0) for your solution in one of two ways: (i) enter the values for t_0 and y_0, and then click "Solution"; or (ii) use the mouse to move the cursor to select an initial value, and then click to sketch the solution curve.
    For example, select $(t_0, y_0) = (0, 1)$ and sketch the solution curve.
(4) Sketch as many solution curves as you need to see the "phase portrait" of your equation. In particular, you will be able to see "rivers," which are places where solution curves concentrate so as to become indistinguishable on the screen.

Exercise 1. $dy/dt = y - t^2$

(a) Plot the slope field for $-3 \le t \le 3$, $-1 \le y \le 4$.
(b) Notice that at some points (t, y) there are horizontal tangents, i.e., $dy/dt = 0$. Approximate some of these points using the cross hairs. Can you find exactly the curve of points where $dy/dt = 0$? (This is the *0-isocline* for the equation.)

(c) Consider the initial condition $y(0) = 0$. From looking at the slope field near $(0, 0)$, what do you expect the solution curve to do as t increases? Check this by selecting an initial condition very close to $(0, 0)$ and plotting the solution curve.

(d) Choose many more initial conditions until you see the phase portrait and the rivers.

(e) Now consider the initial condition $y(0) = 1.5$. From looking at the slope field near $(0, 1.5)$, what do you expect the solution curve to do as t increases? Check this by plotting a solution curve. (You may want to increase the t range to something like $-3 \leq t \leq 5$.)

→**Hand In:** *Written answers to the questions in parts (b), (c), and (e). (In (b), for example, "The curve of points where $dy/dt = 0$ is . . . ".) You may also hand in print-outs of your plots, but your written answers are the most important thing.*

II. ASYMPTOTIC BEHAVIOR. Example 1 and Exercise 1 raise an interesting issue: what happens to a solution $y(t)$ of an initial value problem $dy/dt = f(t, y),\ y(t_0) = y_0$ as t increases? For example, $dy/dt = y,\ y(0) = y_0$ has the solution $y(t) = y_0 e^t$ which is defined for all values t (although $y(t) \to \infty$ as $t \to \infty$). On the other hand, for another equation it may be that a solution $y(t)$ *escapes to infinity* or *blows up* in that $y(t) \to \pm\infty$ as t increases to some finite value t_1.

Example 2. $dy/dt = y^2$. If we plot the slope field, we cannot tell if a solution curve escapes to infinity or not. For example, with the initial condition $y(0) = 1$, the solution curve looks somewhat like exponential growth, but it may actually grow so rapidly that the solution reaches $+\infty$ at a finite value of t. To settle the question, notice that the equation is *separable* and so is easily solved:

$$\frac{dy}{dt} = y^2 \quad \Rightarrow \quad \int \frac{dy}{y^2} = \int dt \quad \Rightarrow -\frac{1}{y} = t + C.$$

The initial condition $y(0) = 1$ determines $C = -1$, so the solution is $y(t) = 1/(1 - t)$. We see that indeed the solution blows up at $t = 1$: $y(t) \to +\infty$ as $t \to 1$.

On the other hand, not all solutions of the equation blow up. In fact, with initial condition $y(t_0) = 0$ we encounter the equilibrium solution $y(t) \equiv 0$.

Example 2 has also illustrated a special kind of solution when the equation is *autonomous,* i.e. the function f depends on y but not on t: $dy/dt = f(y)$. Values of y for which $f(y) = 0$ are called *critical points,* and the constant function $y(x) \equiv c$ defines an *equilibrium solution* whose solution curve is just the horizontal straight line $y = c$ in the ty-plane. In Example 2, $y(t) \equiv 0$ is an equilibrium solution. But notice that if we take $y_0 > 0$ very close to zero, then the solution with initial condition $y(t_0) = y_0$ does not remain close to the equilibrium solution, but in fact escapes to infinity at some $t_1 > t_0$. Such equilibrium solutions are called *unstable.* On the other hand, if all solution curves which begin near an equilibrium solution $y(x) \equiv c$ remain near the equilibrium, then the equilibrium is call *stable;* in fact, if all nearby solution curves satisfy $\lim_{x\to\infty} y(x) = c$, then the equilibrium is called a *sink.* (If all solution curves which begin near the equilibrium tend away from it, then the equilibrium is called a *source;* this, however, is not the case for $y = 0$ in Example 2; do you see why?)

Exercise 2. $dy/dt = y(y - 2)$

(a) Plot the slope field for $0 \le t \le 4$, $-2 \le y \le 4$.
(b) Notice that the equation is autonomous; what are the equilibrium solutions of the equation? Determine the stability of each equilibrium solution; is it s sink or a source?
(c) If we take the initial condition $y(0) = 1$, what happens to $y(t)$ as t increases? Does $y(t)$ escape to infinity? Is there a limit of y as $t \to \infty$?
(d) If we take the initial condition $y(0) = 3$, what happens to $y(t)$ as t increases? Can you show that $y(t)$ escapes to infinity? *Hint:* The equation is separable, and leads to an integration by partial fractions.

→**Hand In:** *Written answers to the questions in parts (b), (c), and (d). (In (b), for example, "There is an equilibrium point at ... which is ... a (sink/source). There is also an equilibrium at ... ")*

III. UNIQUENESS. Notice that the solution of the initial value problem

$$\text{(IVP)} \qquad \begin{cases} \dfrac{dy}{dt} & = f(t, y) \\ y(t_0) & = y_0 \end{cases}$$

may escape to infinity, but it should exist at least near the point (t_0, y_0), provided $f(t, y)$ is continuous at (t_0, y_0). Indeed, this is guaranteed by the fundamental Existence Theorem on page 66 of the Blanchard-Devaney-Hall textbook. Notice that uniqueness of the solution (discussed on page 68) requires $\partial f/\partial y$ to be continuous near (t_0, y_0): if $\partial f/\partial y$ is not continuous at (t_0, y_0), then uniqueness may fail.

Exercise 3. $dy/dt = y^{2/3}$

(a) Use Diffs to plot the slope field on $0 \le t \le 4$, $-1 \le y \le 8$.
(b) Sketch the solution curve for the initial condition $y(0) = y_0$ with $y_0 > 0$ as close to zero as you can get. (Note: do *not* use $y_0 < 0$, because the program will have trouble evaluating it.) The result should suggest that the initial value problem $dy/dt = y^{2/3}$, $y(0) = 0$ has a solution with $y(t) \to \infty$ as $t \to t_1$, where t_1 is either a finite value or possibly $t_1 = \infty$. Use separation of variables to find the solution and the value of t_1.
(c) Observe that $y(t) \equiv 0$ is also a solution of $dy/dt = y^{2/3}$, $y(0) = 0$. Thus uniqueness fails. Why does this not violate the Uniqueness Theorem on page 68? Would you expect uniqueness to hold for the problem $dy/dt = y^{2/3}$, $y(0) = 1$?

→**Hand In:** *Written answers to the questions in parts (b) and (c).*