

## 14.7 Do Two-Dimensional Distributions Differ?

We here discuss a useful generalization of the K–S test (§14.3) to *two-dimensional* distributions. This generalization is due to Fasano and Franceschini[1], a variant on an earlier idea due to Peacock [2].

In a two-dimensional distribution, each data point is characterized by an  $(x, y)$  pair of values. An example near to our hearts is that each of the 19 neutrinos that were detected from Supernova 1987A is characterized by a time  $t_i$  and by an energy  $E_i$  (see [3]). We might wish to know whether these measured pairs  $(t_i, E_i)$ ,  $i = 1 \dots 19$  are consistent with a theoretical model that predicts neutrino flux as a function of both time and energy — that is, a two-dimensional probability distribution in the  $(x, y)$  [here,  $(t, E)$ ] plane. That would be a one-sample test. Or, given two sets of neutrino detections, from two comparable detectors, we might want to know whether they are compatible with each other, a two-sample test.

In the spirit of the tried-and-true, one-dimensional K–S test, we want to range over the  $(x, y)$  plane in search of some kind of maximum *cumulative* difference between two two-dimensional distributions. Unfortunately, cumulative probability distribution is not well-defined in more than one dimension! Peacock's insight was that a good surrogate is the *integrated probability in each of four natural quadrants* around a given point  $(x_i, y_i)$ , namely the total probabilities (or fraction of data) in  $(x > x_i, y > y_i)$ ,  $(x < x_i, y > y_i)$ ,  $(x < x_i, y < y_i)$ ,  $(x > x_i, y < y_i)$ . The two-dimensional K–S statistic  $D$  is now taken to be the maximum difference (ranging both over data points and over quadrants) of the corresponding integrated probabilities. When comparing two data sets, the value of  $D$  may depend on which data set is ranged over. In that case, define an effective  $D$  as the average of the two values obtained. If you are confused at this point about the exact definition of  $D$ , don't fret; the accompanying computer routines amount to a precise algorithmic definition.

Figure 14.7.1 gives a feeling for what is going on. The 65 triangles and 35 squares seem to have somewhat different distributions in the plane. The dotted lines are centered on the triangle that maximizes the  $D$  statistic; the maximum occurs in the upper-left quadrant. That quadrant contains only 0.12 of all the triangles, but it contains 0.56 of all the squares. The value of  $D$  is thus 0.44. Is this statistically significant?

Even for fixed sample sizes, it is unfortunately not rigorously true that the distribution of  $D$  in the null hypothesis is independent of the shape of the two-dimensional distribution. In this respect the two-dimensional K–S test is not as natural as its one-dimensional parent. However, extensive Monte Carlo integrations have shown that the distribution of the two-dimensional  $D$  is *very nearly* identical for even quite different distributions, as long as they have the same coefficient of correlation  $r$ , defined in the usual way by equation (14.5.1). In their paper, Fasano and Franceschini tabulate Monte Carlo results for (what amounts to) the distribution of  $D$  as a function of (of course)  $D$ , sample size  $N$ , and coefficient of correlation  $r$ . Analyzing their results, one finds that the significance levels for the two-dimensional K–S test can be summarized by the simple, though approximate, formulas,

$$\text{Probability } (D > \text{observed}) = Q_{KS} \left( \frac{\sqrt{N} D}{1 + \sqrt{1 - r^2} (0.25 - 0.75/\sqrt{N})} \right) \quad (14.7.1)$$

for the one-sample case, and the same for the two-sample case, but with

$$N = \frac{N_1 N_2}{N_1 + N_2}. \quad (14.7.2)$$

The above formulas are accurate enough when  $N \gtrsim 20$ , and when the indicated probability (significance level) is less than (more significant than) 0.20 or so. When the indicated probability is  $> 0.20$ , its value may not be accurate, but the implication that the data and model (or two data sets) are not significantly different is certainly correct. Notice that in the limit of  $r \rightarrow 1$  (perfect correlation), equations (14.7.1) and (14.7.2) reduce to equations (14.3.9) and (14.3.10): The two-dimensional data lie on a perfect straight line, and the two-dimensional K–S test becomes a one-dimensional K–S test.

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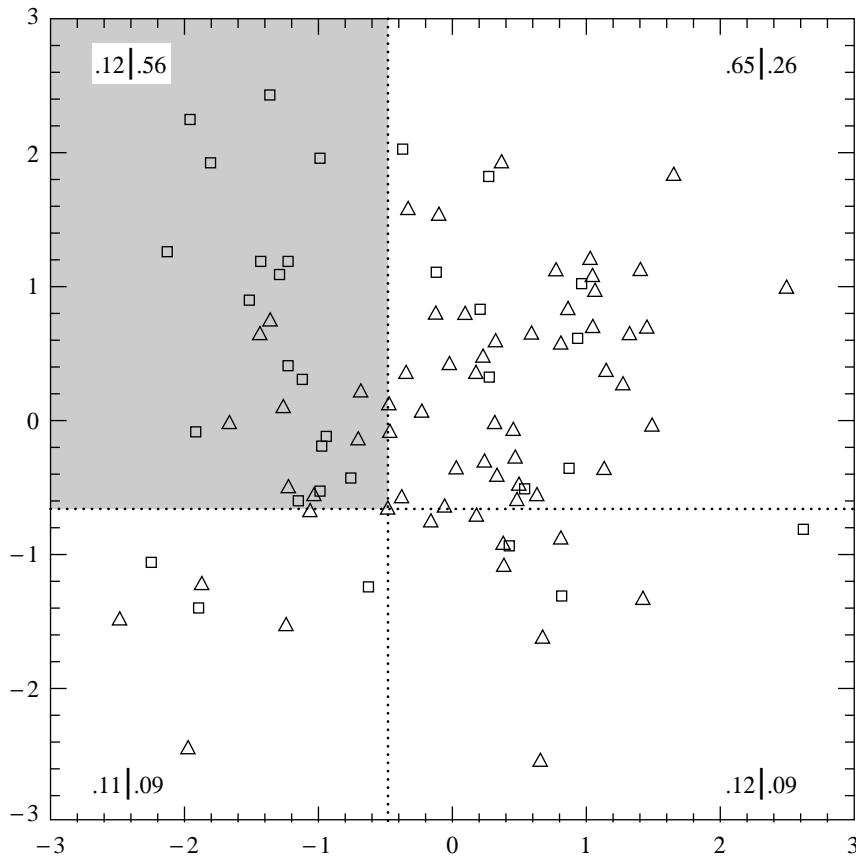


Figure 14.7.1. Two-dimensional distributions of 65 triangles and 35 squares. The two-dimensional K-S test finds that point one of whose quadrants (shown by dotted lines) maximizes the difference between fraction of triangles and fraction of squares. Then, equation (14.7.1) indicates whether the difference is statistically significant, i.e., whether the triangles and squares must have different underlying distributions.

The significance level for the data in Figure 14.7.1, by the way, is about 0.001. This establishes to a near-certainty that the triangles and squares were drawn from different distributions. (As in fact they were.)

Of course, if you do not want to rely on the Monte Carlo experiments embodied in equation (14.7.1), you can do your own: Generate a lot of synthetic data sets from your model, each one with the same number of points as the real data set. Compute  $D$  for each synthetic data set, using the accompanying computer routines (but ignoring their calculated probabilities), and count what fraction of the time these synthetic  $D$ 's exceed the  $D$  from the real data. That fraction is your significance.

One disadvantage of the two-dimensional tests, by comparison with their one-dimensional progenitors, is that the two-dimensional tests require of order  $N^2$  operations: Two nested loops of order  $N$  take the place of an  $N \log N$  sort. For small computers, this restricts the usefulness of the tests to  $N$  less than several thousand.

We now give computer implementations. The one-sample case is embodied in the routine `ks2d1s` (that is, 2-dimensions, 1-sample). This routine calls a straightforward utility routine `quadct` to count points in the four quadrants, and it calls a user-supplied routine `quadv1` that must be capable of returning the integrated probability of an analytic model in each of four quadrants around an arbitrary  $(x, y)$  point. A trivial sample `quadv1` is shown; realistic `quadv1`s can be quite complicated, often incorporating numerical quadratures over

analytic two-dimensional distributions.

```
SUBROUTINE ks2d1s(x1,y1,n1,quadv1,d1,prob)
  INTEGER n1
  REAL d1,prob,x1(n1),y1(n1)
  EXTERNAL quadv1
```

C *USES* *pearsn, probks, quadct, quadv1*

Two-dimensional Kolmogorov-Smirnov test of one sample against a model. Given the  $x$  and  $y$  coordinates of  $n1$  data points in arrays  $x1(1:n1)$  and  $y1(1:n1)$ , and given a user-supplied function  $quadv1$  that exemplifies the model, this routine returns the two-dimensional K-S statistic as  $d1$ , and its significance level as  $prob$ . Small values of  $prob$  show that the sample is significantly different from the model. Note that the test is slightly distribution-dependent, so  $prob$  is only an estimate.

```
  INTEGER j
  REAL dum,dumm,fa,fb,fc,fd,ga,gb,gc,gd,r1,rr,sqen,probks
  d1=0.0
  do 11 j=1,n1                               Loop over the data points.
    call quadct(x1(j),y1(j),x1,y1,n1,fa,fb,fc,fd)
    call quadv1(x1(j),y1(j),ga,gb,gc,gd)
    d1=max(d1,abs(fa-ga),abs(fb-gb),abs(fc-gc),abs(fd-gd))
    For both the sample and the model, the distribution is integrated in each of four quadrants, and the maximum difference is saved.
  enddo 11
  call pearsn(x1,y1,n1,r1,dum,dumm) Get the linear correlation coefficient r1.
  sqen=sqrt(float(n1))
  rr=sqrt(1.0-r1**2)
  Estimate the probability using the K-S probability function probks.
  prob=probks(d1*sqen/(1.0+rr*(0.25-0.75/sqen)))
  return
END
```

```
SUBROUTINE quadct(x,y,xx,yy,nn,fa,fb,fc,fd)
  INTEGER nn
  REAL fa,fb,fc,fd,x,y,xx(nn),yy(nn)
```

Given an origin  $(x,y)$ , and an array of  $nn$  points with coordinates  $xx$  and  $yy$ , count how many of them are in each quadrant around the origin, and return the normalized fractions. Quadrants are labeled alphabetically, counterclockwise from the upper right. Used by  $ks2d1s$  and  $ks2d2s$ .

```
  INTEGER k,na,nb,nc,nd
  REAL ff
  na=0
  nb=0
  nc=0
  nd=0
  do 11 k=1,nn
    if(yy(k).gt.y)then
      if(xx(k).gt.x)then
        na=na+1
      else
        nb=nb+1
      endif
    else
      if(xx(k).gt.x)then
        nd=nd+1
      else
        nc=nc+1
      endif
    endif
  enddo 11
  ff=1.0/nn
  fa=ff*na
  fb=ff*nb
```

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```

fc=ff*nc
fd=ff*nd
return
END

```

```

SUBROUTINE quadv1(x,y,fa,fb,fc,fd)
REAL fa,fb,fc,fd,x,y

```

This is a sample of a user-supplied routine to be used with `ks2d1s`. In this case, the model distribution is uniform inside the square  $-1 < x < 1$ ,  $-1 < y < 1$ . In general this routine should return, for any point  $(x,y)$ , the fraction of the total distribution in each of the four quadrants around that point. The fractions, `fa`, `fb`, `fc`, and `fd`, must add up to 1. Quadrants are alphabetical, counterclockwise from the upper right.

```

REAL qa,qb,qc,qd
qa=min(2.,max(0.,1.-x))
qb=min(2.,max(0.,1.-y))
qc=min(2.,max(0.,x+1.))
qd=min(2.,max(0.,y+1.))
fa=0.25*qa*qb
fb=0.25*qb*qc
fc=0.25*qc*qd
fd=0.25*qd*qa
return
END

```

The routine `ks2d2s` is the two-sample case of the two-dimensional K-S test. It also calls `quadct`, `pearsn`, and `probks`. Being a two-sample test, it does not need an analytic model.

```

SUBROUTINE ks2d2s(x1,y1,n1,x2,y2,n2,d,prob)
INTEGER n1,n2

```

```

REAL d,prob,x1(n1),x2(n2),y1(n1),y2(n2)

```

*C* *USES* `pearsn`, `probks`, `quadct`

Two-dimensional Kolmogorov-Smirnov test on two samples. Given the  $x$  and  $y$  coordinates of the first sample as  $n1$  values in arrays `x1(1:n1)` and `y1(1:n1)`, and likewise for the second sample,  $n2$  values in arrays `x2` and `y2`, this routine returns the two-dimensional, two-sample K-S statistic as `d`, and its significance level as `prob`. Small values of `prob` show that the two samples are significantly different. Note that the test is slightly distribution-dependent, so `prob` is only an estimate.

```

INTEGER j
REAL d1,d2,dum,dumm,fa,fb,fc,fd,ga,gb,gc,gd,r1,r2,rr,
* sqen,probks

```

```

d1=0.0
do 11 j=1,n1 First, use points in the first sample as origins.
call quadct(x1(j),y1(j),x1,y1,n1,fa,fb,fc,fd)
call quadct(x1(j),y1(j),x2,y2,n2,ga,gb,gc,gd)
d1=max(d1,abs(fa-ga),abs(fb-gb),abs(fc-gc),abs(fd-gd))

```

```

enddo 11
d2=0.0
do 12 j=1,n2 Then, use points in the second sample as origins.
call quadct(x2(j),y2(j),x1,y1,n1,fa,fb,fc,fd)
call quadct(x2(j),y2(j),x2,y2,n2,ga,gb,gc,gd)
d2=max(d2,abs(fa-ga),abs(fb-gb),abs(fc-gc),abs(fd-gd))

```

```

enddo 12
d=0.5*(d1+d2) Average the K-S statistics.
sqen=sqrt(float(n1)*float(n2)/float(n1+n2))
call pearsn(x1,y1,n1,r1,dum,dumm) Get the linear correlation coefficient for each sample.
call pearsn(x2,y2,n2,r2,dum,dumm)
rr=sqrt(1.0-0.5*(r1**2+r2**2))

```

```

Estimate the probability using the K-S probability function probks.
prob=probks(d*sqen/(1.0+rr*(0.25-0.75/sqen)))
return
END

```

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## CITED REFERENCES AND FURTHER READING:

- Fasano, G. and Franceschini, A. 1987, *Monthly Notices of the Royal Astronomical Society*, vol. 225, pp. 155–170. [1]  
 Peacock, J.A. 1983, *Monthly Notices of the Royal Astronomical Society*, vol. 202, pp. 615–627. [2]  
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## 14.8 Savitzky-Golay Smoothing Filters

In §13.5 we learned something about the construction and application of digital filters, but little guidance was given on *which particular* filter to use. That, of course, depends on what you want to accomplish by filtering. One obvious use for *low-pass* filters is to smooth noisy data.

The premise of data smoothing is that one is measuring a variable that is both slowly varying and also corrupted by random noise. Then it can sometimes be useful to replace each data point by some kind of local average of surrounding data points. Since nearby points measure very nearly the same underlying value, averaging can reduce the level of noise without (much) biasing the value obtained.

We must comment editorially that the smoothing of data lies in a murky area, beyond the fringe of some better posed, and therefore more highly recommended, techniques that are discussed elsewhere in this book. If you are fitting data to a parametric model, for example (see Chapter 15), it is almost always better to use raw data than to use data that has been pre-processed by a smoothing procedure. Another alternative to blind smoothing is so-called “optimal” or Wiener filtering, as discussed in §13.3 and more generally in §13.6. Data smoothing is probably most justified when it is used simply as a graphical technique, to guide the eye through a forest of data points all with large error bars; or as a means of making initial *rough* estimates of simple parameters from a graph.

In this section we discuss a particular type of low-pass filter, well-adapted for data smoothing, and termed variously *Savitzky-Golay* [1], *least-squares* [2], or *DISPO* (Digital Smoothing Polynomial) [3] filters. Rather than having their properties defined in the Fourier domain, and then translated to the time domain, Savitzky-Golay filters derive directly from a particular formulation of the data smoothing problem in the time domain, as we will now see. Savitzky-Golay filters were initially (and are still often) used to render visible the relative widths and heights of spectral lines in noisy spectrometric data.

Recall that a digital filter is applied to a series of equally spaced data values  $f_i \equiv f(t_i)$ , where  $t_i \equiv t_0 + i\Delta$  for some constant sample spacing  $\Delta$  and  $i = \dots - 2, -1, 0, 1, 2, \dots$ . We have seen (§13.5) that the simplest type of digital filter (the nonrecursive or finite impulse response filter) replaces each data value  $f_i$  by a linear combination  $g_i$  of itself and some number of nearby neighbors,

$$g_i = \sum_{n=-n_L}^{n_R} c_n f_{i+n} \quad (14.8.1)$$

Here  $n_L$  is the number of points used “to the left” of a data point  $i$ , i.e., earlier than it, while  $n_R$  is the number used to the right, i.e., later. A so-called *causal* filter would have  $n_R = 0$ .

As a starting point for understanding Savitzky-Golay filters, consider the simplest possible averaging procedure: For some fixed  $n_L = n_R$ , compute each  $g_i$  as the average of the data points from  $f_{i-n_L}$  to  $f_{i+n_R}$ . This is sometimes called *moving window averaging* and corresponds to equation (14.8.1) with constant  $c_n = 1/(n_L + n_R + 1)$ . If the underlying function is constant, or is changing linearly with time (increasing or decreasing), then no bias is introduced into the result. Higher points at one end of the averaging interval are on

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