

Structure and Drive

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A system is made up of several operations with flow passing between them. The structure of the system describes the flow paths from inputs to outputs. In this section we recognize three structure alternatives, line, tree and network. The drive option specifies the cause of flow through the system. Here we have two options, pull and push. For the pull option, flow is pulled from the outputs of operations. For the push option, flow is pushed into the inputs of the operations.

For each of the six combinations of structure and drive, given the flows into or out of the structure, one can compute the flows through the system operations. The remainder of this section describes the analysis.

Pull Line



The line structure, illustrated in Fig. 1, is the simplest because flow enters at the first operation and leaves from the last. Here we show the pull line where the flows are determined by the flow leaving the system at operation 5. When there is no change in flow caused by the operations, the flow that enters operation 1 also leaves operation 5 and all operations have the same flow. We analyze the case where some mechanism allows the flow to be increased or decreased as it passes through an operation.

We assume the operations are numbered in order from 1 to m , with m the last operation. η_m is the flow withdrawn from operation m . In general we will also allow flow to be withdrawn from the other operations as well, with η_i defined as the flow pulled from the output of operation i . We call the flows pulled from or pushed into the system *external flows* to distinguish them from the flows within the system. Our goal is to find the flow through each operation as a function of the external flows. Define the following notation.

Figure 1. The pull line

x_i = The flow passing *into* operation i

x_i' = The flow passing *out of* operation i

The flow ratio for an operation is the ratio between the flow leaving the operation and the flow entering. Then

$$r_i = \frac{x_i}{x_i}$$

There are many situations where the ratio is other than 1. Perhaps the operation does inspection in a manufacturing system and faulty items are removed from the flow. Here the ratio would be less than 1. In another situation, the operation may divide an item into two parts. Every entering item results in two leaving items, so the ratio is 2. We assume the ratios are given.

The value of x_i ' is entirely dependent on the pull flow withdrawn at operation i and the amount required by the following operation, $i + 1$.

$$x_i = \eta_i + x_{i+1}$$

$$x_i = \frac{x_i}{r_i} = \frac{\eta_i + x_{i+1}}{r_i} \quad (1)$$

Since the flow for an operation depends on the flow of its unique following operation, we can compute flows recursively, starting with the x_m and continuing for each operation with sequentially decreasing operation index.

An example based on Fig. 1 has operations 2 and 4 each causing a 10% loss in flow. The resultant operation flows are in Table 1. In order to pull 100 units from operation 5, more than 100 units must pass through operation 1 to accommodate for the losses.

Table 1. Pull line parameters and operation flows with $\eta_5 = 100$.

Index (i)	Pull Out (η_i)	Ratio (r_i)	Flow (x_i)
1	0	1	123.46
2	0	0.9	123.46
3	0	1	111.11
4	0	0.9	111.11
5	100	1	100

Push Line



The push line is illustrated in Fig. 2. The only difference between the push and the pull lines is the driver for flows. When all flow ratios are 1, there really is no difference between the two drive mechanisms because all operation flows are the same.

Again we assume the operations are numbered in order from 1 to m , with m the last operation. Flows enter operation 1 in the amount γ_1 . In addition to node 1, products may also be pushed into the network at other operations. The flow entering at operation i is γ_i . Note that push flow enters the process just before the operation, while pull flow leaves the process at the output of the operation. The notation for flow and flow ratios remain the same as for the pull line.

The flow into an operation is equal to the external flow added at the operation plus the flow withdrawn from the previous operation.

$$\begin{aligned} x_i &= \gamma_i + x_{i-1} \\ x_i &= \gamma_i + r_{i-1}x_{i-1} \end{aligned} \quad (2)$$

Figure 2. The push line

Starting with operation 1, Eq. 2 can be used sequentially to compute the flows on the line. The example for the pull line is repeated below with 100 units pushed into operation 1 instead of flow being pulled from operation 5.

Comparison of Tables 1 and 2 will show the difference between the two cases when non-unity flow ratios are present.

Table 2. Push line parameters and operation flows with $\eta_5 = 100$.

Index (i)	Push in (γ_i)	Ratio (r_i)	Flow (x_i)
1	100	1	100
2	0	0.9	100
3	0	1	90
4	0	0.9	90
5	0	1	81

Pull Tree

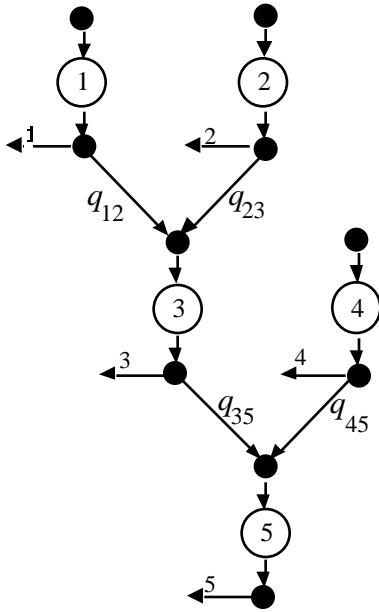


Figure 3. The pull tree

The generic pull tree process is illustrated in Fig. 3. For this structure the flow through each operation goes to a unique following operation, while each operation may have several input flows from other operations. This structure is appropriate for modeling manufacturing processes where raw materials are combined or mixed to produce a single product. Product is withdrawn or *pulled* from the operation with the greatest index, operation 5 for the example, in the amount η_5 .

In addition to the final operation of the process, our models also allow flow to be pulled from the other operations. These flows represent intermediate products. In general, we identify the amount pulled from the output of operation i as η_i , the *pull flow* at operation i .

For the tree structures we require that the operations be indexed so that when flow passes from operation i to operation j , $i < j$. The greatest index is m .

For the pull tree we identify the proportion, q_{ij} , as the amount of the output of operation i required for each unit of product passing through operation j . The value of q_{ij} may be any positive amount to represent a variety of manufacturing situations. An assembly operation that requires one unit of each input to be combined to produce one unit of a subassembly would have the proportions equal to 1 for each input. A mixing operation that combines inputs into a mixture would have input proportions that sum to 1. An operation that requires more than one unit of some input would be modeled with a proportion greater than 1 on the associated input.

Tabular Representation

We can represent the data for a pull tree of Fig. 3 with a two-dimensional table as illustrated below.

Table 3. Pull tree parameters

Index	Next	Pull Out	Ratio	Proportion
1	3	0	0.9	0.5
2	3	0	0.9	0.5
3	5	0	0.9	0.5
4	5	0	0.9	0.5
5	0	100	0.9	1

For this illustration, we are assuming that 10% of the units passing through each operation are scrapped. We also assume proportions of 0.5 for operations 1 through 4. This means that operation 3 receives half of its input from each of operations 1 and 2. Further, operation 5 receives half of its input from each of operations 3 and 4. For the example, 100 units are pulled from operation 5.

For the pull tree, an operation can send its output to no more than one other operation, so the column labeled *Next* is sufficient to describe the tree structure. The column labeled *Proportion* gives the proportion of the input of the next-operation that is obtained from the operation. The number (0.5) in the row for operation 1, holds the value of q_{13} .

For the pull tree structure we define the following notation. We use i for the general operation index.

a_i = the index of the operation following (or *after*) operation i .
This is the number in the *Next* column.

γ_i = the flow pulled from the output of operation i .

r_i = the ratio between the output and input flows for operation i .

q_{ia} = the proportion of the input of operation a_i that is obtained from operation i .

We use the symbol a as the second subscript on q_{ia} to indicate that it is the proportion of the input of the following operation that must come from operation i . When an operation has no following operation we assign the value 0 to a_i , and the value of q_{ia} will have no effect.

Flow

Given pull flows for the operations, we want to compute the flow through each operation. We use the notation x_i , x_i' and r_i as previously defined.

To illustrate the computation of the unit flows we use an example with three operations as in Fig. 4.

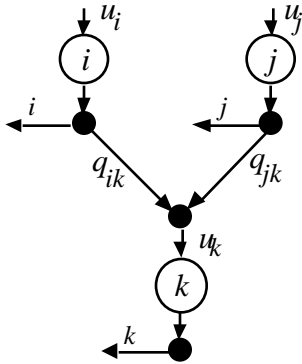


Figure 4. Portion of a pull tree

The value of the flow out of operation *i*, x_i , is entirely dependent on the pull flow withdrawn at operation *i* and the amount required by the following operation *k*. Since the amount required is $q_{ik}x_k$, we have the relation between the flows at *i* and *k*.

$$x_i = \eta_i + q_{ik}x_k$$

$$x_i = \frac{x_i}{r_i} = \frac{\eta_i + q_{ik}x_k}{r_i}$$

Notice that the flow for an operation depends on the flow of its unique following operation. For the pull tree process, the unit flows can be computed recursively, starting with x_m and continuing for each operation with sequentially decreasing operation index.

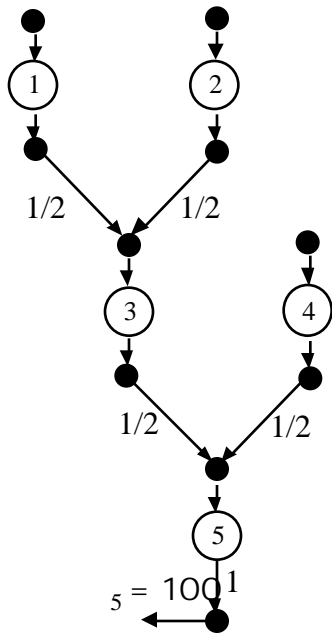


Figure 5. Example pull tree

To illustrate a specific case, consider the pull tree process shown in Fig. 5. The numbers on the arcs are proportions. One unit is pulled from node 5.

For the illustration we use the data of Table 3. We start with the operation with the greatest index. By definition, this operation will have no followers so:

$$x_5 = \eta_5 / r_5 = 100 / 0.9 = 111.1$$

Considering the operation with the next lower index, we compute

$$x_4 = (\eta_4 + q_{45}x_5) / r_4 = (0.5)(111.1) / 0.9 = 61.73$$

Continuing in order of decreasing index,

$$x_3 = (\eta_3 + q_{35}x_5) / r_3 = (0.5)(111.1) / 0.9 = 61.73$$

$$x_2 = (\eta_2 + q_{23}x_3) / r_2 = (0.5)(61.73) / 0.9 = 34.29$$

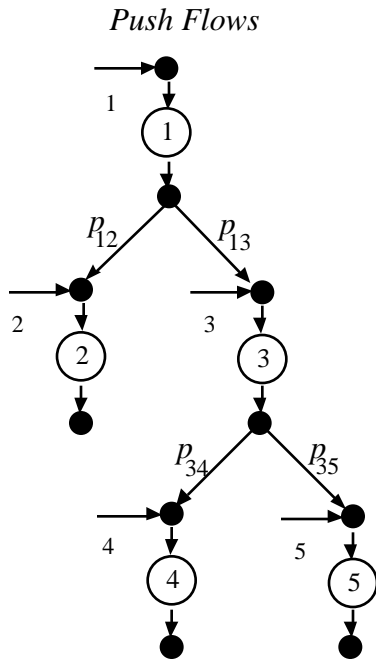
$$x_1 = (\eta_1 + q_{13}x_3) / r_1 = (0.5)(61.73) / 0.9 = 34.29$$

The results are shown in Table 4

Table 4. Pull tree parameters and operation flows with $\eta_5 = 100$.

i	a_i	η_i	q_{ia}	r_i	x_i
1	3	0	0.5	0.9	34.29
2	3	0	0.5	0.9	34.29
3	5	0	0.5	0.9	61.73
4	5	0	0.5	0.9	61.73
5	0	1	1	0.9	111.11

Push Tree



The generic push tree is illustrated in Fig. 6. For this structure the flow into an operation comes from a unique preceding operation, while the operation may have several output flows going to other operations. This structure is appropriate for modeling service systems where customers arrive at a source node, node 1 in the example in the amount γ_1 . In addition to node 1, flow may also be pushed into the network at other operations. The push flow entering at operation i is γ_i . Note that push flow enters the process just before the operation, while pull flow leaves the process after passing through the operation.

Figure 6. The push tree

The flow that passes through an operation may be split to go to other operations to receive different types of processing. Units pass through the tree until finally they are withdrawn to the nodes that have no successors, nodes 2, 4 and 5 in the figure. For the tree structures we require that the operations be numbered so that when flow passes from operation i to operation j , $i < j$. The greatest index is m .

For the push tree we identify the proportion, p_{ij} as the proportion of the output of operation i that is passed to operation j . The value of p_{ij} may be any positive amount to represent a variety of manufacturing situations. For a splitting operation that separates the total flow passing through operation i into several paths, the sum of the proportions leaving i would equal 1.

Tabular Representation

We can represent the data for a push tree with a two-dimensional table as illustrated for the example below.

Table 5. Push tree parameters

Index	Previous	Push Flow	Ratio	Proportion
1	0	100	0.9	1
2	1	0	0.9	0.5
3	1	0	0.9	0.5
4	3	0	0.9	0.5
5	3	0	0.9	0.5

For this illustration, we are assuming that 10% of the units passing through each operation are scrapped. We also assume proportions of 0.5 for operations 2 through 5. This means that the output of operation 1 is split with half going to operation 2 and half going to operation 3. Further, the operation of operation 3 is split with half going to operation 4 and half to operation 5. For the example, 100 units are pushed into operation 1.

For the push tree, an operation receives its input from no more than one other operation, so the column labeled *Previous* is sufficient to describe the tree structure. For row i , the column labeled *Proportion* gives the proportion of the output of the previous operation that goes to operation i . The number (0.5) in the row for operation 2, holds the value of p_{13} .

For the pull tree structure we define the following notation. We use i for the general operation index.

b_i = the index of the operation preceding (or *before*) operation i .

This index is appears in the *Previous* column of the table.

γ_i = the flow pushed into the input of operation i .

r_i = the ratio between the output and input flows for operation i .

p_{bi} = the proportion of the output of operation b_i that goes to operation i .

For p_{bi} we use the b to represent the previous operation. When an operation has no preceding operation we assign the value 0 to b_i , and the value of p_{bi} has no effect.

Flow

To illustrate the computation of the unit flows we use an example with three operations as in Fig. 7.

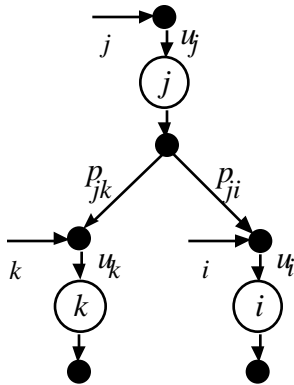


Figure 7. Portion of a push tree

For operation i in Fig. 7, the value of x_i is entirely dependent on the push flow added at operation i and the amount passed from operation j . Since the amount from operation j is $p_{ji}x_j$, we have the relation between the flows at i and j .

$$\begin{aligned} x_i &= \gamma_i + p_{ji}x_j \\ x_i &= \gamma_i + p_{ji}r_jx_j \end{aligned} \quad (2)$$

For the push tree, the flows can be computed recursively, starting with the x_1 and continuing for each operation with sequentially increasing indexes.

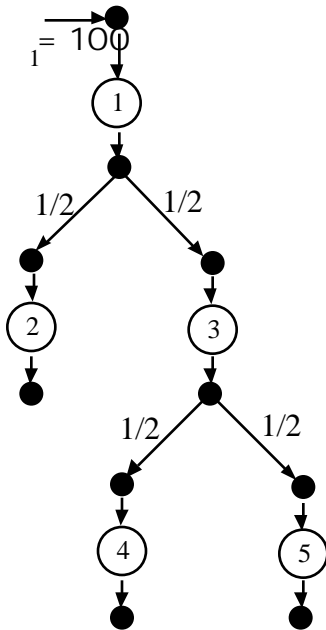


Figure 8. Example push tree

To illustrate a specific case, consider the push tree shown in Fig. 8. The numbers on the arcs are proportions. The flow pushed into node 1 is 100.

For the illustration we use the data of Table 5. We start with the operation with the smallest index. By definition, this operation will have no predecessor operations so:

$$x_1 = \gamma_1 = 100$$

Considering the operation with the next greater index we compute

$$x_2 = \gamma_2 + p_{21}r_1x_1 = (0.5)(0.9)(100) = 45$$

Continuing in order of increasing indices,

$$x_3 = \gamma_3 + p_{31}r_1x_1 = (0.5)(0.9)(100) = 45$$

$$x_4 = \gamma_4 + p_{43}r_3x_3 = (0.5)(0.9)(45) = 20.25$$

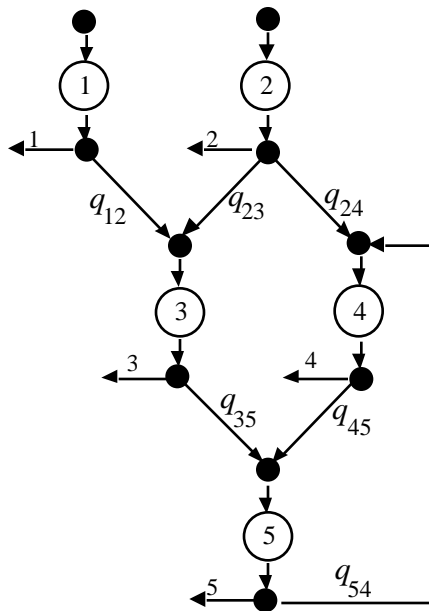
$$x_5 = \gamma_5 + p_{53}r_3x_3 = (0.5)(0.9)(45) = 20.25$$

The results are shown in Table 6.

Table 6. Push tree parameters and flows with $\gamma_1 = 100$.

i	b_i	γ_i	r_i	x_i
1	0	100	0.9	100
2	1	0	0.9	45
3	1	0	0.9	45
4	3	0	0.9	20.25
5	3	0	0.9	20.25

Pull Network



The pull network is illustrated in Fig. 9. For this structure the flow through each operation may go to more than one operation, and each operation may have several input flows from other operations. This is a more general structure than the pull tree structure. Product is withdrawn or *pulled* from any of the operations. Again we use η_i as the amount pulled from operation i . Indices are assigned to the operations arbitrarily, however, it is often convenient to assign the indices to be increasing in the direction of primary product flow.

Figure 9. The pull network

For the pull network we identify the proportion, q_{ij} , as the amount of the output of operation i required for each unit of product passing through operation j . The value of q_{ij} may be any nonnegative amount to represent a variety of manufacturing situations. An assembly operation that requires one unit of each input to be combined to produce one unit of a subassembly would have the proportions equal to 1 for each input. A mixing operation that combines inputs into a mixture would have input proportions that sum to 1. An operation that requires more than one unit of some input would be modeled with a proportion greater than 1 on the associated input.

The example shows an arc passing from operation 5 *back* to operation 4. In a practical instance, this might represent the reworking

of some part. Although we might be tempted to identify q_{54} as the proportion of the output of operation 5 returned to operation 4, this is not correct for a pull network. q_{54} is the proportion of the flow through operation 4 that comes from operation 5. Similarly, q_{24} is the proportion of the flow through operation 4 that comes from operation 2. We use the term proportion here in a slightly different sense than normal. When we identify the flow entering operation 4 as x_4 , the flow on the arc from 5 to 4 is $q_{54}x_4$. The flow on the arc from 2 to 4 is $q_{24}x_4$. There is no requirement that the proportions add to 1 as is usually the case.

Matrix Representation

Although the external flows and ratios can be represented in a table, it is necessary to represent the proportions on a square matrix. We call this proportion matrix \mathbf{Q} . The matrix \mathbf{Q} describes both following nodes and proportions.

$$\mathbf{Q} = \begin{matrix} & q_{11} & q_{12} & \cdots & q_{1m} \\ q_{21} & q_{22} & \cdots & q_{2m} \\ \vdots & \vdots & & \vdots \\ q_{m1} & q_{m2} & \cdots & q_{mm} \end{matrix}$$

For the pull network structure we define the following notation. We use i for the general operation index.

η_i = the flow pulled from the output of operation i .

r_i = the ratio between the output and input flows for operation i .

q_{ij} = the proportion of the input of operation j that is obtained from operation i .

Flow

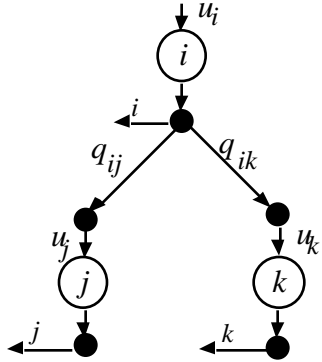


Figure 10. Portion of a pull network

To compute the unit flows for a general network, we will construct and solve a system of linear equations. To illustrate, consider the example in Fig. 10. The value of x_i , the output flow from operation i is dependent on the pull flow withdrawn at operation i and the amounts required by the following operations, j and k .

$$x_i = \eta_i + q_{ij}x_j + q_{ik}x_k$$

We write the equations entirely in terms of the input flows by using the flow ratio.

$$r_i x_i = r_i x_i = \eta_i + q_{ij}x_j + q_{ik}x_k$$

$$r_i x_i - q_{ij}x_j - q_{ik}x_k = \eta_i$$

This generalizes to the expression that must hold for each operation.

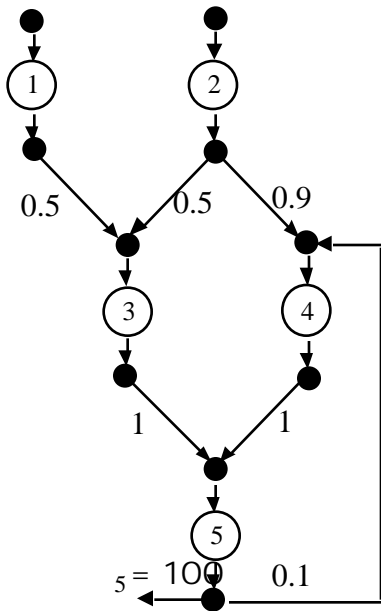
$$r_i x_i - \sum_{j=1}^m q_{ij}x_j = \eta_i \text{ for } i = 1 \dots m$$

We define the augmented proportion matrix as

$$\mathbf{Q}_a = \begin{matrix} r_1 - q_{11} & -q_{12} & -q_{13} & \cdots & -q_{1m} \\ -q_{21} & r_2 - q_{22} & -q_{23} & \cdots & -q_{2m} \\ -q_{31} & -q_{32} & r_3 - q_{33} & \cdots & -q_{3m} \\ \vdots & \vdots & \vdots & & \vdots \\ -q_{m1} & -q_{m2} & -q_{m3} & \cdots & r_m - q_{mm} \end{matrix}$$

Also define the column vector \mathbf{x} of flows. Then the unit flows are the solution to the linear set of equations

$$\mathbf{Q}_a \mathbf{x} = \boldsymbol{\eta} \text{ or } \mathbf{x} = \mathbf{Q}_a^{-1} \boldsymbol{\eta}$$



To illustrate a specific case, consider the pull network shown in Fig. 11. The numbers on the arcs are proportions. 100 units are pulled from node 5. The flow ratios are all 1.

The proportions $q_{13} = q_{23} = 0.5$ indicate that for each unit passing through operation 3, 0.5 units must pass through each of operations 1 and 2. The proportions entering operation 4 indicate that for each unit passing through operation 4, 0.9 units must pass through operation 2 and 0.1 must pass through operation 5. The flow transfers for the network of Fig.11 are represented by the \mathbf{Q} matrix. The pull flows and operation flows are the column matrices $\boldsymbol{\eta}$ and \mathbf{x} .

Figure 11. Example pull network

$$\mathbf{Q} = \begin{matrix} & & 0 & 0 & 0.5 & 0 & 0 & & 0 \\ & & 0 & 0 & 0.5 & 0.9 & 0 & & 0 \\ & 0 & 0 & 0 & 0 & 0 & 1 & & \boldsymbol{\eta} = 0 \\ & 0 & 0 & 0 & 0 & 0 & 1 & & 0 \\ & 0 & 0 & 0 & 0.1 & 0 & 0 & & 100 \end{matrix} \quad \mathbf{x} = \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix}$$

The augmented matrix is:

$$\mathbf{Q}_a = \begin{matrix} 1 & 0 & -0.5 & 0 & 0 \\ 0 & 1 & -0.5 & -0.9 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -0.1 & 1 \end{matrix}$$

The set of linear equations determining the unit flows is:

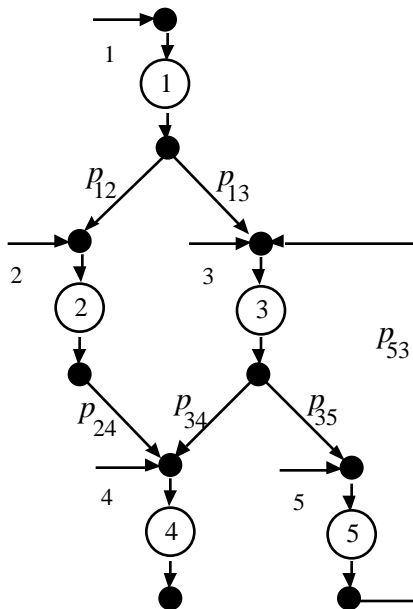
$$\mathbf{Q}_a \mathbf{x} = \boldsymbol{\eta}$$

$$\begin{array}{cccccc} 1 & 0 & -0.5 & 0 & 0 & x_1 & 0 \\ 0 & 1 & -0.5 & -0.9 & 0 & x_2 & 0 \\ 0 & 0 & 1 & 0 & -1 & x_3 & = & 0 \\ 0 & 0 & 0 & 1 & -1 & x_4 & 0 \\ 0 & 0 & 0 & -0.1 & 1 & x_5 & 100 \end{array}$$

Solving we find:

$$x_1 = 55.56, x_2 = 155.56, x_3 = 111.11, x_4 = 111.11, x_5 = 111.11$$

Push Network



The push network is illustrated in Fig. 12. For this structure the flow through each operation may go to more than one operation, and each operation may have several input flows from other operations. This is a more general structure than the push tree. Flow is inserted or *pushed* into any of the operations. We use γ_i as the amount pushed into operation i . Indices are assigned to the operations arbitrarily, however, it is often convenient to assign the indices to be increasing in the direction of primary product flow.

Figure 12. The Push Network

For the push network we identify the proportion, p_{ij} , as the amount of the output of operation i that is passed to operation j for each unit passing through operation i . The value of p_{ij} may be any nonnegative amount. Typically for a service system, the sum of the proportions leaving an operation is equal to 1. This means that the flow is split among the several following operations. It may be necessary to use other combinations of proportions to represent different systems.

The example shows an arc passing from operation 5 *back* to operation 3. In a practical instance, this might represent the reworking of some part. p_{53} as the proportion of the output of operation 5 returned to operation 3. It is not necessary to define a proportion for the flow leaving the system at operation 5.

Matrix Representation

Although we can represent much of the data for a push network with a two-dimensional table, it is necessary to represent the proportions on a square matrix. We call this proportion matrix \mathbf{P} . Again, proportions have different meanings for push and pull networks.

$$\mathbf{P} = \begin{matrix} & p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \vdots & & \vdots \\ p_{m1} & p_{m2} & \cdots & & p_{mm} \end{matrix}$$

For the push network structure we define the following notation. We use i for the general operation index.

γ_i = the flow pushed to the input of operation i .

r_i = the ratio between the output and input flows for operation i .

p_{ij} = the proportion of the flow through of operation i that is passed to operation j .

Flow

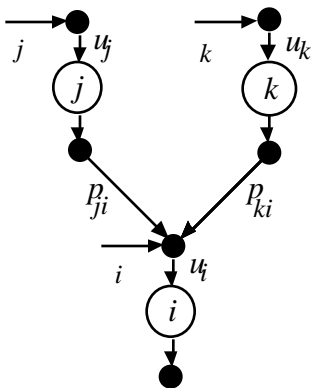


Figure 13. Portion of a push network

To compute the unit flows for a general network, we will construct and solve a system of linear equations. To illustrate, consider the example in Fig. 13. The value of x_i , the input flow from operation i depends on the push flow at operation i and the amounts provided by the preceding operations, j and k .

$$x_i = \gamma_i + p_{ji}x_j + p_{ki}x_k$$

We write the equations entirely in terms of the operation flows by using the flow ratios.

$$x_i = \gamma_i + p_{ji}r_jx_j + p_{ki}r_kx_k$$

$$x_i - p_{ji}r_jx_j - p_{ki}r_kx_k = \gamma_i$$

This generalizes to the expression that must hold for each operation.

$$x_i - \sum_{j=1}^m p_{ji}r_jx_j = \gamma_i \text{ for } i = 1 \dots m$$

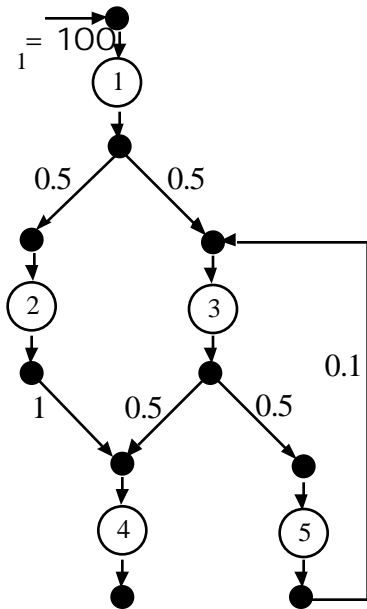
We define the augmented proportion matrix as

$$\mathbf{P}_a = \begin{bmatrix} 1 - r_1p_{11} & -r_1p_{12} & -r_1p_{13} & \cdots & -r_1p_{1m} \\ -r_2p_{21} & 1 - r_2p_{22} & -r_2p_{23} & \cdots & -r_2p_{2m} \\ -r_3p_{31} & -r_3p_{32} & 1 - r_3p_{33} & \cdots & -r_3p_{3m} \\ \vdots & \vdots & \vdots & & \vdots \\ -r_m p_{m1} & -r_m p_{m2} & -r_m p_{m3} & \cdots & 1 - r_m p_{mm} \end{bmatrix}$$

Also define \mathbf{x} , the column vector of operation flows, and γ , the column vector of push flows. Then the unit flows are the solution to the set of linear set of equations

$$\mathbf{x}^T \mathbf{P}_a = \gamma^T \text{ or } \mathbf{x}^T = \gamma^T \mathbf{P}_a^{-1}$$

The T superscript indicates the matrix transpose operation.



To illustrate a specific case, consider the push network shown in Fig. 14. The numbers on the arcs are proportions. 100 units are pushed into node 1. All flow ratios are all 1.

The proportions $p_{13} = p_{23} = 0.5$ indicate that for each unit passing through operation 1, 0.5 units must be sent to each of operations 2 and 3. The proportions leaving operation 3 indicate that for each unit passing through operation 3, 0.5 units will go to operation 4 and 0.5 will go to operation 5. The value of $p_{53} = 0.1$ means that 10% of the output of operation 5 will return to 3. The remainder of the flow out of operation 5 leaves the network. It is not necessary to define specific flows to leave the network.

The network of Fig.14 is represented by the \mathbf{P} matrix below.

Figure 14. Example push network

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 0.5 & 0.5 & 0 & 0 \end{matrix} \\ \begin{matrix} 100 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} \end{matrix} \quad \gamma = \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} \quad \mathbf{x} = \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix}$$

$$\mathbf{P}_a = \begin{matrix} \begin{matrix} 1 & -0.5 & -0.5 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -0.5 & -0.5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -0.1 & 0 & 1 \end{matrix} \end{matrix}$$

$$\mathbf{x}^T \mathbf{P}_a = \gamma^T$$

$$\begin{array}{rcccccc}
x_1 & 1 & -0.5 & -0.5 & 0 & 0 & 100 \\
x_2 & 0 & 1 & 0 & -1 & 0 & 0 \\
x_3 & 0 & 0 & 1 & -0.5 & -0.5 & 0 \\
x_4 & 0 & 0 & 0 & 1 & 0 & 0 \\
x_5 & 0 & 0 & -0.1 & 0 & 1 & 0
\end{array} = 0$$

Solving we find:

$$x_1 = 100, x_2 = 50, x_3 = 52.63, x_4 = 76.32, x_5 = 26.32$$

Use in the Add-ins

Three add-ins use the structure and drive options described in this paper. The *Inventory* add-in uses all three structures and the two drive options as they are described above. The values of the flow ratios are provided directly as data.

The *Process Flow Analysis* add-in uses the tree and network structures and both drive options. The line structure can be handled as a special case of the tree structure. The flow ratio is not input directly, but it is computed from other operation parameters. This add-in has the parameters:

w_i = the proportion of flow removed at operation i .

g_i = the number of items grouped at operation i .

The flow ratio is computed with the equation

$$r_i = \frac{u_i}{u_i} = \frac{(1 - w_i)}{g_i}$$

This add-in computes *unit flow*, the flow in each operation per unit of the driving flow. For the pull drive, we set $\eta_m = 1$. The other pull flows are relative to the pull flow at operation m . For the push drive we set $\gamma_1 = 1$. The other push flows are relative to the push flow at operation 1. Otherwise all formulas described above are used for this add-in.

The *Queue* add-in uses only the line and the push network structures. All ratios are set to 1 for this add-in.