UMAP

Modules in
Undergraduate
Mathematics
and Its
Applications

Published in
cooperation with

The Society for
Industrial and
Applied Mathematics,

The Mathematical
Association of America,

The National Council
of Teachers of
Mathematics,

The American
Mathematical
Association of
Two-Year Colleges,

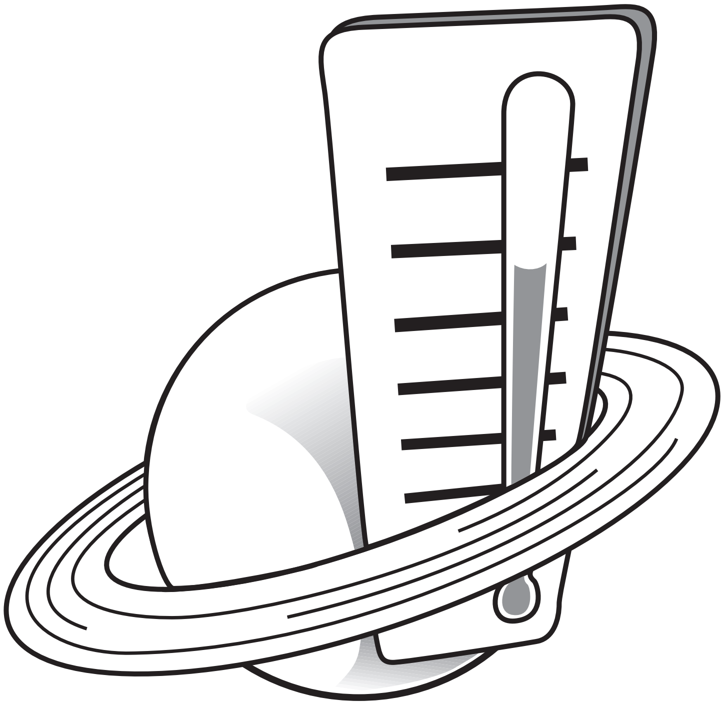
The Institute for
Operations Research
and the Management
Sciences, and

The American
Statistical Association.

Module 770

Climate Change and the Daily Temperature Cycle

Robert M. Gethner



Applications of Geophysics and
Climate Modeling

COMAP, Inc., Suite 210, 57 Bedford Street, Lexington, MA 02420 (781) 862–7878

INTERMODULAR DESCRIPTION SHEET:	UMAP Unit 770, corrected (12 / 98) 1st ed.
TITLE:	Climate Change and the Daily Temperature Cycle
AUTHOR:	Robert M. Gethner Mathematics Dept. Franklin & Marshall College Lancaster, PA 17604–3003 `r_gethner@acad.FandM.edu`
MATHEMATICAL FIELD:	Linear ordinary differential equations
APPLICATION FIELD:	Geophysics, climate modeling
TARGET AUDIENCE:	Students in a sophomore / junior-level mathematical models class.
ABSTRACT:	We construct differential equations models of the daily temperature cycle on atmosphere-free planets and on planets that have atmospheres. The models give some insight into the behavior, observed in recent decades, that nighttime lows have been increasing faster than daytime highs over the Earth's land surfaces.
PREREQUISITES:	Mastery of one-variable calculus and some familiarity with first-order differential equations, including knowing how to linearize them. Some optional computer exercises require the use of a graphing calculator, or, better, computer software, for plotting functions and determining coordinates on the plots.

This UMAP Module originally appeared in *The UMAP Journal* 19 (1) (1998) 33–86.

COMAP, Inc., Suite 210, 57 Bedford Street, Lexington, MA 02173
(800) 77-COMAP = (800) 772-6627, or (781) 862-7878; `http://www.comap.com`

Climate Change and the Daily Temperature Cycle

Robert M. Gethner
Mathematics Dept.
Franklin & Marshall College
Lancaster, PA 17604–3003
`r_gethner@acad.FandM.edu`

Table of Contents

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS (UMAP) PROJECT

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications, to be used to supplement existing courses and from which complete courses may eventually be built.

The Project was guided by a National Advisory Board of mathematicians, scientists, and educators. UMAP was funded by a grant from the National Science Foundation and now is supported by the Consortium for Mathematics and Its Applications (COMAP), Inc., a nonprofit corporation engaged in research and development in mathematics education.

Paul J. Campbell	Editor
Solomon Garfunkel	Executive Director, COMAP

1. Introduction

During the last few decades, air temperatures at ground level over the Earth's land surfaces have been going up. Curiously, the increase has been unequally distributed between night and day, with nighttime lows rising three times as fast as daytime highs. The *diurnal temperature range (DTR)*—the difference between highs and lows—has thus been decreasing. In this Module, we'll construct a mathematical model of the Earth and its atmosphere in order to gain insight into that behavior.

Temperatures fluctuate, of course, from day to day and location to location; but, on average, near-surface air temperatures over land rose by about 1° F during the period from 1951 to 1990. During the same period (chosen because reliable data is available for that time), nighttime lows went up by about 1.5° F, while daytime highs rose by only 0.5° F. Thus, the diurnal temperature range fell and the average temperature rose at about the same rate. (The figures are based on a study of 37% of the Earth's land mass [Karl et al. 1993, 1007, 1009].)

The model that we'll study, though extremely simple as atmospheric models go, is nonetheless mathematically interesting and sophisticated enough to simulate the effects on the daily temperature cycle of two influences that warm the surface:

- increases in the Sun's intensity, which increases the rate at which energy enters the Earth / atmosphere system; and
- increases in the concentration of greenhouse gases in the atmosphere, which makes the atmosphere more efficient at absorbing energy from the surface and at emitting energy down to the surface.

The most prevalent greenhouse gas in the Earth's atmosphere, after water vapor, is carbon dioxide (CO_2). There is always some natural variation in the amount of atmospheric CO_2; for example, the atmosphere absorbs CO_2 from volcanic eruptions and gives up CO_2 to the oceans. But during the thousand years prior to the industrial revolution, the variation was small, about 4% [Houghton et al. 1996, 18]. In contrast, the concentration of CO_2 has increased by about 25% since 1850 [Peixoto and Oort 1992, 435; Houghton et al. 1996, 18], with most of the increase probably due to human activities such as burning of fossil fuels. Depending on the future pattern of fuel consumption, the amount of atmospheric CO_2 is likely to become double its 1850 value at some time in the next 50 to 150 years [Peixoto and Oort 1992, 436; Houghton et al. 1996, 8]. Concentrations of other greenhouse gases have been rising, too [Houghton et al. 1996, 19]. Thus, it's reasonable to conjecture that the observed rise in temperature is caused by increases in atmospheric greenhouse gases. Although the case for that conjecture is good, there's some uncertainty, in part because good historical data isn't available on variations in the intensity of the Sun [Baliunas and Soon 1996].

Qualitatively, the daily temperature cycle in the model that we're going to study exhibits strikingly different responses to changes in solar intensity

Table 1.
Symbol table.

A	constant = $\sigma T_e^4/c$
A	absorptivity of the atmosphere, assumed equal to ϵ
ATR	annual temperature range
a	co-albedo, the fraction of sunlight absorbed
α	albedo, the fraction of sunlight intercepted but not absorbed
B	constant = $AP/2$ (Sections 6, 7); constant = $Pq\sigma T_e^4/2c_0$ (Section 9)
β	constant
C	heat capacity of the surface
c	heat capacity per square meter
ΔT_e	rise in equilibrium surface temperature
DTR	diurnal (daily) temperature range $u_{\max} - u_{\min}$
ϵ	emissivity of the atmosphere = absorptivity of the atmosphere = $\epsilon_c + \epsilon_w$
ϵ_c	fraction of surface-emitted longwave radiation absorbed by atmospheric CO_2
ϵ_w	fraction of surface-emitted longwave radiation absorbed by atmospheric H_2O
G	greenhouse function, the factor by which an atmosphere increases surface temperature
γ	constant
H	constant
J	symbol for joules, a unit of energy
K	symbol for Kelvins (Kelvin degrees), a unit of temperature
k	constant = $4\sigma T_e^3/c$
k	exponential decay constant; $k = 1/\tau$
λ	wavelength
K.E.	kinetic energy = $mv^2/2$
Ω, Ω_E	solar constant for Earth = 1372 $\mathrm{W/m^2}$
Ω_V	solar constant for Venus
ω	constant = $kP/2$ = $P/2\tau$ (Sections 6, 7); constant = $2P\sigma\widehat{T}_0^3/c_0$ (Section 9)
P	period of solar flux
ψ	piecewise-continuous function, periodic of period 2, with mean value 0
q	fraction of shortwave flux absorbed by the surface
S	Earth's surface area
s	seconds; dummy variable for integration; time measured in units of half a planetary day, $s = 2t/P$
σ	Stefan–Boltzmann constant = 5.67×10^{-8} $\mathrm{W/m^2K^4}$
T	temperature in Kelvins
T	surface temperature of an airless planet
$\widehat{T}$	equilibrium surface temperature of an airless planet
T_C	temperature in Celsius degrees
T_e	effective temperature
T_F	temperature in Fahrenheit degrees
T_0	surface temperature of a planet that has an atmosphere
$\widehat{T}_0$	equilibrium surface temperature of a planet that has an atmosphere
T_1	atmospheric temperature
$\widehat{T}_1$	equilibrium atmospheric temperature
t	time
τ	e-folding time, the time for the value of u to diminish by a factor of e; $\tau = 1/k$
u	displacement of surface temperature from its equilibrium value
W	symbol for units of Watts

versus changes in concentrations of greenhouse gases. Neither response is large enough to match the observed response, however, which suggests that factors are at work that our model doesn't account for—not surprising in a model that essentially consists of a single linear ordinary differential equation—and we'll discuss at the end what those factors might be. In the meantime, we'll develop a feeling for how mathematics can be used to begin to understand a complex part of nature, and for how even simple mathematics can lead us to conjectures that we'd have been unlikely to make by thinking purely qualitatively.

Simple models, in fact, though their predictions are only suggestive of how reality might behave, and far from the final word on it, have some advantages over complex ones. Reality is hard to analyze because phenomena are intertwined, with multiple causes leading to multiple effects. For the same reason, models sophisticated enough to mimic the reality convincingly are hard to analyze. The most sophisticated climate models, the Global Climate Models (GCMs), simulate winds, temperatures, pressures, clouds, and precipitation over the whole globe and at all vertical levels of the atmosphere, along with oceanic phenomena such as currents, salinity effects, and air-sea interactions, and "[t]he artificial climates generated by these models are typically as complicated and inscrutable as the Earth's climate" [North et al. 1981, 91].

The model that we'll study, on the other hand, effectively views the Earth's surface as a single point and the atmosphere as another point; and it takes into account two physical processes, namely, storage of heat energy by the surface and atmosphere, and transfer of energy between the two via electromagnetic radiation. (Models that account only for those processes are called *energy balance models*.) It will therefore be relatively easy to investigate the effect on model temperatures of changing a single factor. We'll begin, in fact (after a review of the physics that we need) with a planet without an atmosphere. That will allow us to build up, in a simple setting, the mathematics that we'll also use later when we include an atmosphere; it will also help us to isolate the role that the atmosphere plays in controlling the daily temperature cycle of the surface.

2. Temperature, Energy, and Power

Almost everything that we will do involves the ideas of energy and temperature. The only two forms of energy that we'll need to deal with are radiant energy (**Section 3**) and heat energy (described below), but it's helpful to be able to compare those forms with *kinetic energy*: the energy associated with an object's motion. (Regarding the names for the different types of energy, see Feynman et al. [1963, Chapter 4].) The kinetic energy of an object of mass m moving at speed v is K.E. $= mv^2/2$. For example, let's estimate the kinetic energy of a bicycle and its rider, coasting at 10 miles per hour, if together they weigh 200 lbs. One purpose of this example is to ease into the metric system. If you're not fluent in metric units, the following very rough conversions, accurate to within about 10% except where noted, are useful to know (precise conversions appear in **Appendix II**): 1 meter (m) is about 1 yard; 1 kilometer (km) is about half a mile (accurate to within about 25%); 1 kilogram (1 kg) is about 2 pounds (mass); 1 meter per second (m / s) is about 2 miles per hour.

Thus, the bike and rider are coasting at about 5 m / s, and their combined mass is about 100 kg. Their combined kinetic energy is about

$$(100\text{ kg}) \times (5\text{ m / s})^2/2 = 1{,}250\text{ kg m}^2\text{ / s}^2.$$

We'll measure energy in *joules* (J): 1 J = 1 kg m^2/s^2. The bike and rider have kinetic energy 1,250 J. A 100-watt lightbulb uses 100 J of electrical energy per second, so the same amount of energy that was required to pedal the bike up to its coasting speed (ignoring the energy used to overcome friction and air resistance) would keep the bulb lit for about 12 seconds.

Another form of energy is *heat energy*: the sum of the kinetic energies of all the molecules that make up the object. Objects that have higher heat energies also have higher temperatures, and the two ideas are related to each other by the notions of specific heat and heat capacity.

The *specific heat* of a substance is the amount of energy that it takes to raise the temperature of one unit mass of the substance by one degree.

The *heat capacity* of an object is the amount of energy it takes to raise the temperature of the object by one degree.

The "degrees" that we'll use are on the Kelvin scale, which is related to the Fahrenheit and Celsius scales by the formulas $T_F = 32 + 9T_C/5$ and $T_C = T - 273.15$, where T_F, T_C, and T are the temperatures in Fahrenheit, Celsius, and Kelvin. In particular, a temperature *rise* of one degree Kelvin (1° K) or just "one Kelvin" (1 K) is also an increase of one degree Celsius (1° C), or, roughly, 2° F.

Now, suppose that you draw 0.05 m^3 of hot water into your bathtub, and that during the bath the water cools by 10 K (about 20° F), thereby losing heat energy (which goes into heating the walls of the tub and the air in the bathroom). To calculate the change in energy, we first multiply the mass of water in the tub (volume times density; water's density is about 1.00×10^3 kg / m^3) by the specific heat of water. That gives the heat capacity of the water in the tub. Then we multiply the heat capacity by the change in temperature:

$$\Delta\text{Energy} = \left(0.05\text{ m}^3\right)\left(1.00 \times 10^3\ \frac{\text{kg}}{\text{m}^3}\right)\left(4{,}184\ \frac{\text{J}}{\text{kg K}}\right)(-10\text{ K}) = -2.1 \times 10^6\text{ J}.$$

If the energy lost could be captured and used to power a 100-watt lightbulb, the bulb would burn for about 2×10^4 s, or nearly 6 hours.

Power is rate at which energy is generated or consumed per unit time. The unit that we'll use for power is the *Watt* (W): 1 W = 1 J / s. A 100-watt lightbulb consumes energy at the rate of 100 joules of electrical energy each second, or 100 Watts. A 60-watt lightbulb, if left on for an hour, would use energy equal to (60 W)(1 hour) = (60 J / s)(3600 s) = 2.16×10^5 J.

To predict the Earth's temperature, we need to know the rate at which the Earth receives *radiant* energy from the Sun. We'll calculate that in the next section.

Exercises

To do many of the exercises throughout this Module, you'll need **Appendix II**.

1. Convert the following temperatures (given in degrees Fahrenheit) to Celsius and to Kelvin: 32° F, 64° F, 90° F.

2. Calculate the mass, in kilograms, of a kindergartner who weighs 40 lbs.

3. If a house is a cube of side 10 m, how many kilograms of air does it hold, assuming that no room is taken up by walls, furniture, people, etc.? (Assume a day of low humidity.)

4. The Earth's atmosphere is a roughly spherical shell of air, very thin compared to the size of the Earth; that is, we can imagine the space taken up by the atmosphere as the space between two spheres, one coinciding with the Earth's surface and the other concentric with it and of only slightly larger radius. ("If proportions were preserved, the thickness of the atmosphere would be represented on an ordinary office globe by scarcely more than the thickness of a coat of paint" [Peixoto and Oort 1992, 14].) Now imagine allowing the outer sphere to shrink in such a way that the atmosphere still remains in the space between the Earth's surface and the outer sphere. The more the outer sphere shrinks, the denser the enclosed air becomes. If you continue to compress the outer sphere until the enclosed air has the density of water, how thick will the atmosphere be? That is, what will be the difference in radii between the outer and inner spheres?

5. A baseball has mass 150 g [de Mestre 1990, 137]. Compute the kinetic energy of a baseball moving at 40 m / s.

6. How much energy does it take to raise the temperature of a house from 64° F to 68° F? The answer depends on the house. But assume the following:

- the house is air-tight,
- all the energy goes into heating the air (and not into running a circulating pump or heating the floor, walls, furniture, etc.), and
- the house has the dimensions given in **Exercise 3**.

3. Radiation, Flux, and the Solar Constant

The Sun puts out *radiant energy* carried by electromagnetic radiation, a fancy name for "light." It propagates through space as a wave (at a speed of 3 ×

10^8 m / sec); and, as with water waves, the *wavelength* is the distance between peaks of adjacent waves.

Some light is visible, some not. The shortest wavelength of visible light is 0.390×10^{-6} m (purple light), while the longest wavelength is 0.760×10^{-6} m (red light). Invisible light includes ultraviolet light, X-rays, and gamma rays (the latter emitted by radioactive substances), all with wavelengths shorter than those of visible light; and infrared light, microwaves, and television and radio waves (all with wavelengths longer those that of visible light). If our eyes were sensitive to infrared light in addition to the usual visible light, we would see a broader rainbow, with an infrared arc above the red arc [Greenler 1980, 18–21].

The radiation emitted by the Sun travels outward in ever-expanding spheres. Some of that radiation reaches the Earth / atmosphere system and is absorbed by it, which provides energy to keep the atmosphere, oceans, and continents warm, and to make wind and storms.

We need a measure of how much power solar power the Earth intercepts. In most of our work, it will be convenient to measure the power per unit area, called *flux*. Imagine a large transparent sphere—a soap bubble—centered at the Sun, which just touches the top of the Earth's atmosphere (see **Figure 1**).

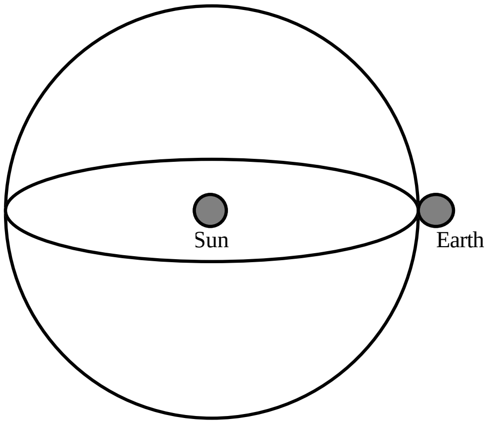


Figure 1. A Sun-centered "soap bubble" tangent to the top of the Earth's atmosphere.

The flux of electromagnetic radiation crossing the bubble (i.e., the amount of energy passing through the bubble per second per unit surface area of the bubble) is called the *solar constant*, denoted by Ω. Values given for Ω vary somewhat from book to book; we'll take $\Omega = 1372$ W / m^2 [Harte 1988, 69] . The sunlight crossing each square meter of the bubble provides nearly enough power to illuminate fourteen 100-watt lightbulbs.

That value of Ω pertains to the Earth: the flux crossing a Sun-centered bubble just touching the top of Venus's atmosphere, for instance, is larger because the radiation crosses a smaller bubble, each square meter of which therefore receives a larger fraction of the total. Thus, Venus's constant, Ω_V, is bigger than the Earth's constant $\Omega_E = \Omega$.

To compute Ω_V, first notice that the total solar power passing through the

Earth's bubble is 1372 W times the bubble's area:

$$4\pi(149.6 \times 10^9 \text{ m})^2(1372 \text{ W/m}^2) = 3.86 \times 10^{26} \text{ W}.$$

Then, assuming that no energy is lost on the way from Venus to the Earth, this last figure is also the total power crossing Venus's bubble; so, to compute Venus's solar constant, we divide the figure by that bubble's area:

$$\Omega_V = \frac{3.86 \times 10^{26} \text{ W}}{4\pi(108.20 \times 10^9 \text{ m})^2} = 2623 \text{ W/m}^2.$$

In studying the Earth's (or any planet's) climate we need, not the power per unit area of the *bubble,* but rather, the power intercepting the Earth per unit area of the *Earth's surface.* Different locations on the Earth receive different amounts of radiation, and those amounts vary with the time of day and the time of year, but we can easily compute an average flux as follows. Imagine that, as part of some inscrutable cosmic experiment, extraterrestrial beings have placed a gigantic cardboard screen close to the Earth, on the opposite side of the Earth from the Sun, and oriented perpendicular to the line joining the Sun and the Earth (see **Figure 2**).

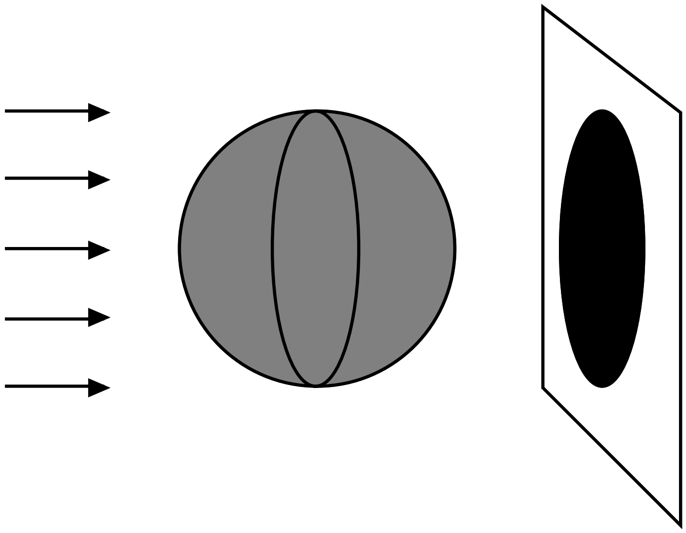


Figure 2. Sunlight, coming in from the left, puts Earth on the Big Screen.

The Sun casts a disk-shaped shadow of the Earth on the screen; because the Earth / Sun distance is large compared to the size of the Earth, the disk's radius equals the radius of the Earth. So the shadow's area is πr^2, where r is the Earth's radius. Each square meter of the screen not blocked by the Earth intercepts solar power Ω. The power intercepted by the Earth is the power *not* received by the screen, namely, $\pi r^2\Omega$. The average solar power received by the Earth and its atmosphere, per unit area of the Earth's surface, is thus $\pi r^2\Omega/4\pi r^2 = \Omega/4$.

Not all of the power received is *absorbed*, however. A planet's (global average) *albedo*, denoted by α, is the proportion of incoming sunlight that is intercepted, but not absorbed, by the planet (by "planet" here is meant the whole system: planet and atmosphere). The Earth's albedo is currently about .3, meaning that 30% of the solar energy intercepted is reflected away. The energy is reflected by air, clouds, oceans, land, and ice and snow. The planet's *co-albedo*, denoted by a, is the proportion of incoming sunlight that is absorbed by the planet and its atmosphere. Then $a = 1 - \alpha$; for the Earth, $a = .7$. The Earth and its atmosphere together thus absorb an average solar flux equal to $a\Omega/4$.

To practice the ideas of this section, let's calculate how much energy an average square meter of the Earth's surface absorbs in the form of solar radiation each year and relate that to temperature changes.

Of all the incoming solar radiation absorbed by the Earth / atmosphere system, about one-third is absorbed by the atmosphere (some of it after first being reflected from the surface). About two-thirds is absorbed by the surface (some after first being reflected up and down several times between the surface and the atmosphere) [Harte 1988, 165]. It follows that a typical one-meter-square patch of surface absorbs solar energy at an average rate of

$$(2/3)(a\Omega/4)(1\ \mathrm{m}^2) \approx 160\ \mathrm{J/s}.$$

Therefore, during the course of a year, the energy absorbed by the patch is

$$(160\ \mathrm{J/s})(31{,}536{,}000\ \mathrm{s}) \approx 5 \times 10^9\ \mathrm{J}.$$

Does that energy raise the surface temperature? Let's estimate the surface's heat capacity. Two-thirds of the surface is covered with ocean. The sunlight that falls on the ocean penetrates to only a few meters below the surface, but convection and wind-generated turbulence mix those sun-warmed surface waters with a layer of water, about 50–100 m deep, called the *mixed layer*. Over the course of a year, that's as deep as the energy gets. (Over longer time scales, other, slower, processes mix the mixed layer with the deep ocean. In paleoclimatology, the study of the climate's ancient past, time scales of thousands or millions of years must be dealt with, and the entire ocean volume taken into account.)

What about land? Land surfaces have a density roughly comparable to that of water, but their specific heats are only about one-quarter that of water, and, over the course of a year, the solar energy that strikes land is only communicated, by conduction, a few meters deep [Peixoto and Oort 1992, 221]. We are therefore safe, when making a rough estimate, in neglecting land in comparison to water. Taking the heat capacity of the Earth's land surfaces to be zero, then, and taking the depth of the mixed layer to be 75 m, we have for the heat capacity of a typical one-meter-square patch of surface,

$$\begin{aligned}
&(2/3)(\text{heat capacity of oceanic mixed layer}) + (1/3)(\text{heat capacity of land})\\
&\quad = (2/3)(\text{specific heat of water})(\text{mass of water in patch}) + (1/3)(0)\\
&\quad = (2/3)(4{,}184\ \mathrm{J/kgK})(75\ \mathrm{m}^3)(1000\ \mathrm{kg/m}^3) \approx 2 \times 10^8\ \mathrm{J/K}.
\end{aligned}$$

So the surface receives enough solar energy each year to raise the surface temperature by

$$\frac{\text{solar energy absorbed}}{\text{energy required to raise surface temperature by 1 K}} = \frac{5 \times 10^9\ \text{J}}{2 \times 10^8\ \text{J/K}} = 25\ \text{K}.$$

If the surface temperature were really rising at 25 K/yr, the Earth would soon be uninhabitable. In reality, the annual global average surface temperature is observed to be approximately constant. What keeps the temperature from rising is that the surface not only absorbs radiation, but also emits it. We'll discuss emission in the next section.

Exercises

7. Compute Mars's solar constant.

8. How much power does the Earth's surface (the entire surface, not just a one-meter-square patch) absorb from the Sun? How does this figure compare to world energy consumption? A typical nuclear or coal-fired power plant generates about 10^9 Watts [Harte 1988, 68]. How many power plants worth of power does the Earth's surface absorb from the Sun?

9. Determine the heat capacity of the Earth's atmosphere per square meter of the Earth's surface. (Use the specific heat of air at constant pressure.) How large is your answer relative to the heat capacity of the mixed layer per square meter of the Earth's surface?

4. Blackbodies and Effective Temperature

Any object at non-zero absolute (Kelvin) temperature emits electromagnetic radiation at a variety of wavelengths. The total rate of emission, and the distribution of emission among the various wavelengths, depends on various characteristics of the object, including its temperature.

Because the Sun is very hot, most of the radiation that it emits has a short wavelength; its maximum emission is around 0.5 millionths of a meter [Peixoto and Oort 1992, 92], which is in the visible range—presumably because our eyes are adapted to see sunlight. The Earth, too, emits radiation—it glows!—but mostly at a much longer wavelength (mostly infrared, about 4 to 60 millionths of a meter) because the Earth is much cooler than the Sun; since our eyes aren't sensitive to radiation at that wavelength, we don't see it. (What we do see when we look at the ground is reflected sunlight.) If you hold your hand near (not too near) the burner of an electric stove, you'll experience a sensation of heat before you see any change in the burner: Your hand is able to detect changes in emission at lower wavelengths than your eye is. The sensation of heat will become more and more intense as the burner gets hotter and hotter: this is because, the hotter an object is, the more power it emits. The burner will start to glow orange when it gets hot enough to emit a significant amount of radiation in the visible range.

The Sun emits mostly *shortwave* radiation, defined as radiation whose wavelength is less than $4{\times}10^{-6}$ m; the Earth emits mostly *longwave* radiation, with wavelength greater than $4{\times}10^{-6}$ m.

A *blackbody* [Wallace and Hobbs 1977, 287–289] is a hypothetical object, at uniform temperature, that absorbs all the radiation that it intercepts (hence reflects none) in all wavelengths, and that emits radiation at the maximum possible rate in each wavelength. Such an object emits flux from its boundary surface at a rate given by the *Stefan–Boltzmann Law*, which says that, for a blackbody at temperature T,

$$\text{Flux Out} = \sigma T^4.$$

Here σ is the *Stefan–Boltzmann constant*,

$$\sigma = 5.67 \times 10^{-8}\ \text{W}/\text{m}^2\text{K}^4. \tag{1}$$

The Stefan–Boltzmann Law forms the basis for our first, very crude, estimate of the Earth's temperature. A planet's *effective temperature* [Goody and Walker 1972, 46–49], denoted by T_e, is the temperature the planet would have if

1. it were at uniform temperature;
2. it were in radiative equilibrium with solar radiation (i.e., the power emitted by the planet equaled the solar power absorbed by the planet; and
3. it emitted power at the same rate that a blackbody would at the same temperature.

From these conditions, we can derive a formula for the effective temperature. The radiative equilibrium condition (2) can be written Flux In $=$ Flux Out. We've already seen that Flux In $= a\Omega/4$. By conditions (1) and (3), Flux Out $= \sigma T^4$. Thus,

$$a\Omega/4 = \sigma T_e^4, \tag{2}$$

so that

$$T_e = (a\Omega/4\sigma)^{1/4}. \tag{3}$$

For the Earth, this last formula gives

$$T_e = \left(\frac{0.7 \times 1372\ \text{W}/\text{m}^2}{4 \times 5.67 \times 10^{-8}\ \text{W}/\text{m}^2\text{K}^4}\right)^{1/4} = 255\ \text{K} = -1^\circ\ \text{F}.$$

In the case of a planet that has *no* atmosphere, a heuristic argument can be given that suggests that the effective temperature should equal (or at least be close to) its surface temperature. The amount of solar energy absorbed by a planet varies from location to location and from season to season and between day and night, but we argued in **Section 2** that the average solar flux absorbed per square meter of a planet's surface is $a\Omega/4$. The amount of radiation that the planet emits into space varies also. But observations show that, if these fluxes are averaged over a planet's surface, and over the course of a few years, then the averages are approximately equal and show little variation over time. Thus, as a first approximation, planets in the solar system are in radiative equilibrium. Now, a planet's surface is certainly not a blackbody—it reflects much of the shortwave radiation that intercepts it—but the materials that make up the surface behave approximately like blackbodies in the longwave range. Most of the radiation emitted by the surface is in that range (see **Figure 3**), so, if we imagine the surface to have uniform temperature T, then it seems reasonable to assume that the flux emitted by the surface is σT^4.

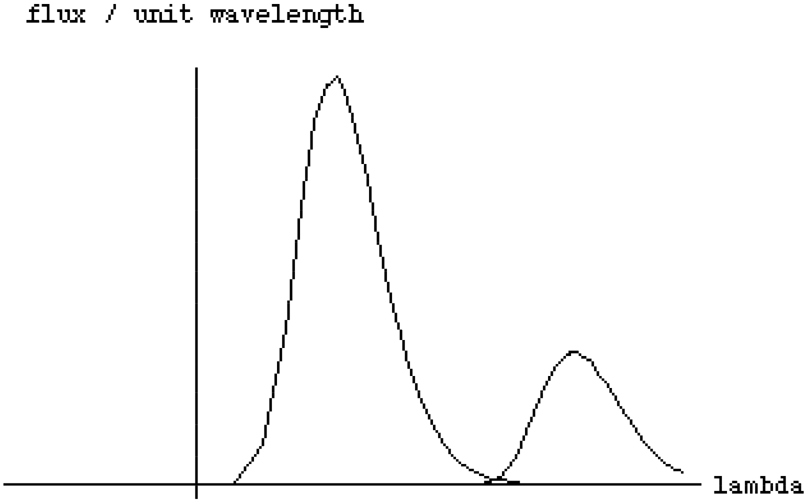


Figure 3. Blackbody curves. A blackbody having the Sun's temperature would emit flux per unit wavelength shown by the curve on the left; a blackbody having the Earth's temperature would emit flux per unit wavelength shown by the curve on the right. The Earth and the Sun are not blackbodies, but the curves give a sense of the wavelength ranges in which the two objects do most of their emission. The two graphs intersect at approximately $\lambda = 4 \times 10^{-6}$ m: nearly all solar radiation is emitted in the shortwave range, and nearly all terrestrial emission in the longwave range. For legibility, the wavelength scale is distorted and the terrestrial curve's height is exaggerated.

Thus, the surface temperature T satisfies $a\Omega/4 = \sigma T^4$, which is simply **(2)** with the effective temperature T_e replaced by the surface temperature T.

Do airless planets really have surface temperatures close to their effective temperatures? Mars, whose atmosphere is very thin, has effective temperature 216.9 K (**Exercise 10**) and average surface temperature 218 K [Schneider 1996, 582], an excellent agreement. Mercury, which has virtually no atmosphere, has effective temperature 441.6 K (**Exercise 10**) and surface temperature 395 K [Schneider 1996, 582], not such a good agreement. A possible source of the disagreement is simply uncertainty in the observed surface temperature. Another book [Lang 1992, 50] gives the value 440 K for Mercury's surface temperature, while [Houghton 1986, 1] gives a Martian surface temperature of 240 K. Another source of the disagreement may be an overly simplistic aspect of the argument. We assumed that the planet radiates from all parts of its surface; on a slowly rotating planet like Mercury, it may be more appropriate to assume that it only radiates from its sunlit hemisphere [Henderson-Sellers 1983, 31–32, 87–88]. When we examine Martian daily temperature changes in **Sections 5** and **7**, we'll cheerfully assume that the equilibrium surface temperature equals the effective temperature.

Although the heuristic argument pertained to an airless planet, the same reasoning suggests that the effective temperature is a first approximation to the temperature that the Earth's *surface* would have if it absorbed as much radiation as the surface and atmosphere *together* do now (namely, $a\Omega/4$). Since the Earth has effective temperature 255 K and average surface temperature 290 K (62° F) [Harte 1988, 164], the Earth can loosely be said to be 35 K warmer than it would be without its atmosphere. That statement is an oversimplification, because without its atmosphere the system would have a different albedo and would therefore absorb shortwave radiation at a different rate. But the concept of effective temperature is useful in that it isolates the role that the atmosphere plays in warming the surface simply by virtue of being an absorber and emitter of longwave radiation.

We'll explore that role in detail when we build a more sophisticated model, in **Section 8**, which includes the surface and atmosphere as separate entities. Before doing that, though, we'll analyze, in **Sections 5–7**, a model of planets that have no atmosphere.

Exercises

10. Verify the values given in the text for the effective temperatures of Mars and Mercury.

11. Two spherical blackbodies, one having three times the radius of the other, are at the same temperature. Calculate the ratio of the power emitted by the larger sphere to that emitted by the smaller sphere.

12. Planet X has no atmosphere, and its axis of rotation is perpendicular to the plane of its orbit. Also, its day has the same length as its year. Thus only one hemisphere ever receives sunlight. Derive a formula for the temperature of that hemisphere. State clearly any assumptions that you have to make.

13. Calculate the flux emitted by a blackbody that has the Earth's average surface temperature. Compare your answer to the observed terrestrial surface emission of 398 W/m^2 [Grotjahn 1993, 44].

5. Temperatures on an Airless Planet

The heuristic argument in the last section leads us to consider a model planet that is atmosphere-free and whose surface is at temperature T, constant over the whole planet, and whose surface absorbs flux $a\Omega/4$ from the Sun and emits flux σT^4. Denote by $\widehat{T}$ the planet's equilibrium surface temperature, the one for which Flux In $=$ Flux Out. Then $a\Omega/4 = \sigma\widehat{T}^4$, so that $\widehat{T} = (a\Omega/4\sigma)^{1/4}$. In other words, by **(3)**, the equilibrium surface temperature is simply the effective temperature: $\widehat{T} = T_e$. Since the two temperatures are equal, we'll use the notation T_e for both. (That will change in **Sections 8–9**, when we introduce an atmosphere into our model; the equilibrium surface temperature then will no longer equal the effective temperature.)

For the planet just described, let's consider three questions:

1. How does the equilibrium surface temperature respond to changes in the solar constant?
2. If the solar constant suddenly jumps to a new value, how long does T_e take to reach its new equilibrium value?
3. How large is the diurnal temperature range?

In this section, we'll answer Question 1, and make progress toward answering Questions 2 and 3 by deriving differential equations for the time-dependent non-equilibrium temperatures. We'll solve the equations in **Section 6** and return to Questions 2 and 3 in **Section 7**.

5.1 How Much Will the Temperature Rise?

By the definition of derivative and **(3)**,

$$\Delta T_e \approx \frac{dT_e}{d\Omega}\,\Delta\Omega = \left(\frac{a}{1024\sigma\Omega^3}\right)^{1/4}\Delta\Omega.$$

For Mars, this formula, along with the answer to **Exercise 7**, gives, for a 1% increase in the (Martian) solar constant,

$$\begin{aligned}\Delta T_e &\approx \left(\frac{.85}{1024\left(5.67\times10^{-8}\ \mathrm{W/m^2K^4}\right)\left(591.0\ \mathrm{W/m^2}\right)^3}\right)^{1/4}(.01)(591.0\ \mathrm{W/m^2})\\ &= 0.54\ \mathrm{K}.\end{aligned}$$

In the model, then, the Martian equilibrium surface temperature would rise by about half a degree Kelvin.

5.2 How Long Until the Surface Reaches Equilibrium?

Before the solar constant increased, the surface temperature was at its old equilibrium of 216.9 K. [This is the effective temperature of Mars (**Exercise 10**), which we're assuming to be equal to the equilibrium surface temperature.] If the solar constant now suddenly increases to a new constant value, the surface will be absorbing more power than it's emitting and will no longer be in equilibrium. Its net energy will rise and so will its temperature T. As T rises, the rate of emission will rise, too, thus slowing down the temperature rise as T approaches its new equilibrium value. The temperature graph levels off fast enough, as we'll see later, that the new equilibrium value will never be reached. So the answer to Question 2 is, "Forever." A more fruitful question is, "How long will the temperature take to come 95% closer to its new equilibrium than it was at the moment the solar constant changed?" Even that question is difficult to answer satisfactorily using our model, but the attempt which we now begin will give us physical insight that will be helpful when we study the daily temperature cycle.

First we derive a differential equation for T, valid whether or not Flux In $= a\Omega/4$ and Flux Out $= \sigma T^4$. Recall from **Section 4** that flux is the power absorbed or emitted per unit area. During a short time interval $[t, t + \Delta t]$, the planet's surface will absorb energy given approximately by

$$(\text{Power In})\Delta t, \tag{4}$$

where Power In is the power at time t, the beginning of the interval. Because the power may change during the interval, **(4)** is only an approximation to the energy absorbed during the time interval. But for Δt small, the power has little time to change, so the approximation is a good one and gets better as Δt gets smaller.

Similarly, the amount of energy lost due to emission during the interval is $(\text{Power Out})\Delta t$. Thus the net change in energy is $(\text{Power In} - \text{Power Out})\Delta t$.

On the other hand, if we assume that all the energy gained by the surface goes into increasing its temperature (and not into, say, stirring up winds and ocean currents), then the net change in energy is also $C \cdot \Delta T$, where C is the surface's heat capacity. Thus $C\Delta T \approx (\text{Power In} - \text{Power Out})\Delta t$, which becomes, after we divide by Δt and allow Δt to approach 0,

$$C\,\frac{dT}{dt} = \text{Power In} - \text{Power Out}; \tag{5}$$

this is our differential equation for T. The left-hand side is a *heat storage term*: it represents the amount of energy added to the surface per unit time.

Specifically, in the case Flux In $= a\Omega/4$ and Flux Out $= \sigma T^4$,

$$cS\,\frac{dT}{dt} = \frac{Sa\Omega}{4} - S\sigma T^4,$$

where S is the planet's surface area and $c \equiv C/S$ is the heat capacity *per square meter* of the planet's surface. Thus, by **(2)**,

$$c\,\frac{dT}{dt} = \sigma T_e^4 - \sigma T^4, \tag{6}$$

a nonlinear equation that has a messy solution (see **Exercise 17** for a simple special case). To get a nice solution, we linearize the T^4 in the equation about the equilibrium value of T: We first set

$$u = T - T_e;$$

that is, we let u be the *displacement* of surface temperature from its equilibrium value: the distance from the T to T_e, with a negative sign if $T < T_e$. Then, after replacing T by $T_e + u$ in **(6)**, expanding the $(T_e + u)^4$, and throwing away the resulting terms that involve powers of u higher than the first power, we obtain (**Exercise 14**) a new differential equation in the unknown u:

$$c\,\frac{du}{dt} = -4\sigma T_e^3 u. \tag{7}$$

This is a homogeneous linear differential equation whose solution should be close to the displacement of surface temperature from its equilibrium value T_e, as long as that displacement isn't too large. We'll solve the equation in **Section 6** and interpret the solution physically in the **Section 7**.

5.3 How Large Is the Diurnal Temperature Range?

We now visualize our planet as a flat piece of land (scientific progress at its best) that alternately experiences day and night. That isn't quite as crazy as it sounds: the "planet" is simply a large patch of land, a hemisphere, perhaps, on a rotating planet. Since there are no oceans and no atmosphere, energy can only be transported from that hemisphere to the other via conduction through the soil and rock that make up the surface, which happens extremely slowly. Thus we can think of our hemisphere as effectively isolated from the other hemisphere—a planet in its own right. We denote the planet's rotation period by P, and, in order to simplify the calculations that occur later, we assume that day and night last equal amounts of time, $P/2$.

As before, we assume that the surface emits flux σT^4. It no longer absorbs flux $a\Omega/4$, though: the flux varies depending on the time of day. Instead we assume that the flux absorbed has *mean* value $a\Omega/4$ over the course of a day, i.e., over a time of length P. By definition, the *mean* of a piecewise continuous function f over an interval $[a, b]$ is $(b-a)^{-1}\int_a^b f(x)\,dx$, so we define

$$\text{Flux In} = \sigma T_e^4\left[1 + \psi\left(\frac{2t}{P}\right)\right], \tag{8}$$

where ψ is a piecewise continuous function, periodic of period 2, such that

$$\int_0^2 \psi(s)\; ds = 0. \qquad \textbf{(9)}$$

Then Flux In is periodic with period P (**Exercise 19**). By **(9)** and **(2)**, its mean value over the interval $[0, P]$ is

$$P^{-1}\int_0^P \sigma T_e^4 \left[1+\psi\left(\frac{2t}{P}\right)\right]\; dt = 2^{-1}\sigma T_e^4 \int_0^2 \left[1+\psi\left(s\right)\right]\; ds = \frac{a\Omega}{4},$$

as desired.

Now we multiply the fluxes from the last paragraph by S and put the products into **(5)**, which gives

$$c\,\frac{dT}{dt} = \sigma T_e^4 \left[1+\psi\left(\frac{2t}{P}\right)\right] - \sigma T^4. \qquad \textbf{(10)}$$

After linearizing **(10)** in the same way that we did **(6)**, we obtain (**Exercise 15**)

$$c\,\frac{du}{dt} + 4\sigma T_e^3 u = \sigma T_e^4 \psi\left(\frac{2t}{P}\right). \qquad \textbf{(11)}$$

This is a nonhomogeneous linear differential equation whose solution $u = u(t)$ should be close to the displacement of surface temperature from T_e, as long as that displacement isn't too large.

What formula should we use for ψ? Probably the best choice would be the period-two extension of

$$\psi(s) = \begin{cases} \pi\sin(\pi s) - 1, & \text{if } 0 \le s \le 1; \\ -1, & \text{if } 1 \le s \le 2. \end{cases} \qquad \textbf{(12)}$$

The Flux In would then be zero at night, and would, during the day, gradually rise to a maximum and gradually fall back to zero (see the top pair of graphs in **Figure 4**). With that choice, unfortunately, there's no closed-form expression for the diurnal temperature range. An exercise in **Section 7** leads you through a computer analysis.

A much more tractable function is

$$\psi(s) = H\sin(\pi s), \qquad \textbf{(13)}$$

where H is a constant, $0 \le H \le 1$ (see the middle pair of graphs in **Figure 4**). With this ψ, the Sun never sets (hence suitable for modeling the British Empire); it will be useful in **Section 9**, in exercises on the annual temperature cycle.

In most of our work, however, we'll use a square wave, namely, the period-2 extension of

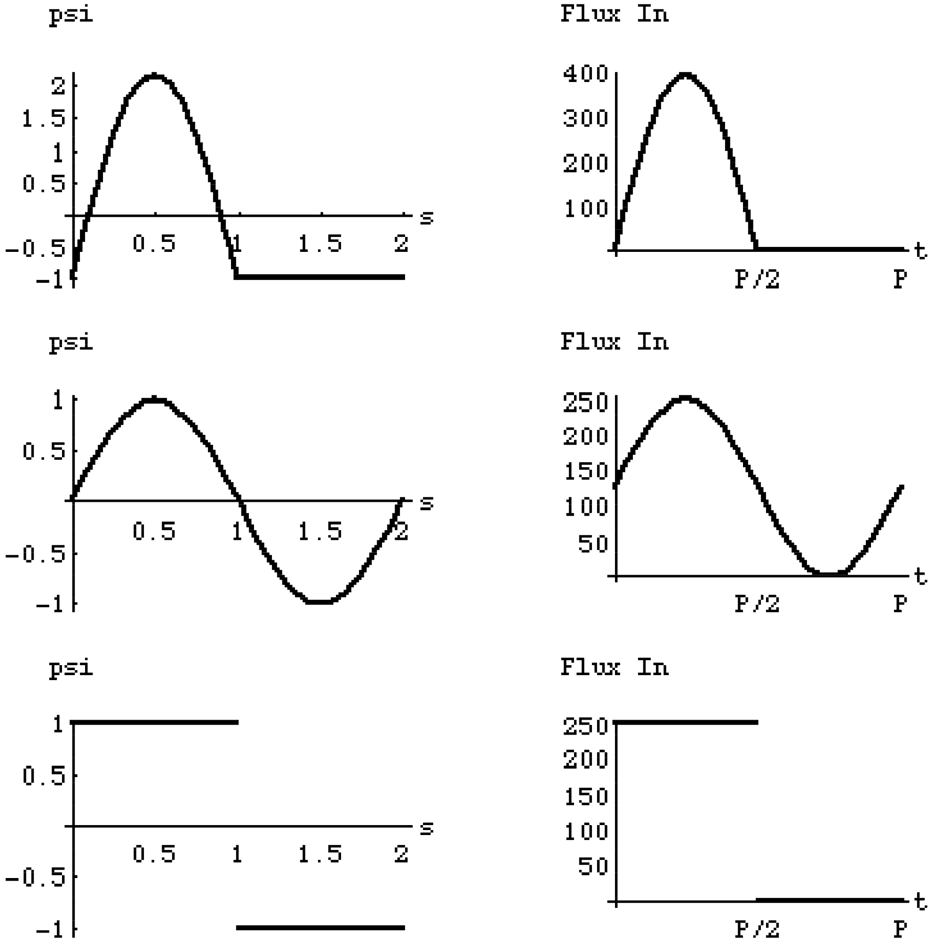


Figure 4. Solar fluxes. Left column, top to bottom, shows the functions ψ given by **(12, 13, 14)**, respectively. On the right are the corresponding Flux In functions **(8)**, with the effective temperature $T_e = 216.9$ K (appropriate to Mars).

$$\psi(s) = \begin{cases} 1, & \text{if } 0 \le s \le 1; \\ -1, & \text{if } 1 \le s \le 2, \end{cases} \qquad \textbf{(14)}$$

a function that isn't much harder to analyze than **(13)** and which, though it doesn't allow the solar intensity to change during the day, does provide both night and day (see the bottom pair of graphs in **Figure 4**). And it leads to a reasonable mean value, namely $a\Omega/2$, for Flux In during the day.

Exercises.

14. Derive the linearized differential equation **(7)**.

15. Derive the linearized differential equation **(11)**.

16. Suppose that the same extraterrestrial beings that put up the screen in **Figure 2** decide to pull Mars 1% farther from the Sun than it now is. Assuming that its albedo remains constant, by how much will Mars's equilibrium surface temperature fall?

17. A airless planet is revolving around a star that, at time $t = 0$, suddenly vanishes. The planet is now emitting radiation but not absorbing it, so the surface temperature T satisfies the (nonlinear) differential equation $c\,dT/dt = -\sigma T^4$.

- **a)** Show that the solution of the differential equation (as it stands; i.e., without linearizing) is

$$T(t) = \left\{ \frac{3\sigma t}{c} + \frac{1}{[T(0)]^3} \right\}^{-1/3}.$$

- **b)** Derive a formula for the amount of time τ that it takes for the surface temperature to descend to half of its original value.
- **c)** Check your formula from part **(b)** to make sure that τ is a decreasing function of $T(0)$: hotter planets cool off relatively faster. Explain physically why that makes sense.

18. Suppose that Power In $>$ Power Out. Does **(5)** imply that the surface temperature will increase, decrease, or remain constant? Give an answer based simply on the equation, not the physics. Does your answer agree with your physical intuition?

19. Verify that ψ in **(8)** has period P.

20. Check that each of the functions ψ in **(12–14)** satisfies **(9)**.

21. The formulas for Flux In in this section attempt to model the daily variation in influx of solar radiation under "average" conditions. They don't take into account seasonal variations, which affect the daily temperature cycle. (On land on the Earth, the diurnal temperature range tends to be larger in the summer than in the winter [Cao et al. 1992, 923]). Draw a graph of Flux In that takes into account both seasonal and diurnal variations, i.e., such that the planet receives less radiation during the day in the winter than it does in the summer.

6. A Mathematical Interlude

Equations **(7)** and **(11)** can be rewritten, respectively, as

$$\frac{du}{dt} + ku = 0 \tag{15}$$

and

$$\frac{du}{dt} + ku = A\psi\left(\frac{2t}{P}\right), \tag{16}$$

where ψ is a piecewise-continuous function periodic of period 2 and satisfying **(9)**, and where

$$k = \frac{4\sigma T_e^3}{c} \qquad \text{and} \qquad A = \frac{\sigma T_e^4}{c}. \tag{17}$$

We now solve **(15)** and **(16)** in general, for k and A positive constants not necessarily given by **(17)**. We need a solution general enough to cover a number of cases, so we won't specify a formula for ψ at this point.

6.1 Solution of the Homogeneous Equation

You may recall from calculus that **(15)** can be solved by separation of variables, leading to the general solution $u = Ce^{-kt}$, , where C is an arbitrary constant. By letting $t = 0$ in this last equation, we find that $C = u(0)$, so that

$$u = u(0)e^{-kt}. \tag{18}$$

Notice that as $t \to \infty$, the function $u(t)$ approaches zero (its equilibrium value) but never reaches it. Now, **(15)** and **(18)** are the same equations that arise in studying the decay of radioactive substances, so we could measure how quickly u approaches equilibrium by using the idea of *half-life*. It's more convenient, though, to use the related idea of *e-folding time*: the amount of time required for the value of u to diminish by a factor of e. To derive a formula for the e-folding time, denoted by τ, we note from **(18)** and the definition of τ that $e^{-1} = u(t+\tau)/u(t) = e^{-kt}$; thus

$$\tau = \frac{1}{k}. \tag{19}$$

After time 3τ has elapsed, u has been reduced by a factor of $e^3 \approx 20$; thus, the value of u is reduced by about 95% after three e-folding times have passed.

6.2 Solution of the Inhomogeneous Equation

To solve **(16)**, we first make a change of independent variables, defining

$$s = \frac{2t}{P}, \tag{20}$$

so that, by the chain rule,

$$\frac{du}{dt} = \frac{du}{ds} \cdot \frac{ds}{dt} = \frac{du}{ds} \cdot \frac{2}{P}.$$

In terms of the new variable s, **(16)** takes the form

$$\frac{du}{ds} + \omega u = B\psi(s), \tag{21}$$

where

$$\omega = \frac{kP}{2} = \frac{P}{2\tau} \qquad \text{and} \qquad B = \frac{AP}{2}. \tag{22}$$

By changing variables in this way, we're measuring time in units of half a period; this makes our calculations later more manageable.

To solve **(21)**, where ω and B are positive constants not necessarily given by **(22)**, we multiply both sides by the integrating factor $e^{\omega s}$, which gives

$$\frac{d}{ds}\left[e^{\omega s} u(s)\right] = e^{\omega s} B\psi(s).$$

Then, after replacing s by x and integrating both sides from 0 to s, we have

$$e^{\omega s} u(s) - u(0) = B \int_0^s e^{\omega x} \psi(x)\, dx,$$

so that the general solution[1] of **(21)** is

$$u(s) = e^{-\omega s} \left[u(0) + B \int_0^s e^{\omega x} \psi(x)\, dx \right]. \tag{23}$$

Since the right-hand side of **(21)** is periodic, it's reasonable to guess that **(21)** has a periodic solution u. (That equation ultimately came from the differential equation **(11)**, in which ψ is related to the periodic daily changes in incoming solar flux, which induce periodic daily changes in the displacement u of temperature from its equilibrium value.) We'll show that there is exactly one periodic function, of period 2, of the form **(23)**. To find it, note that such a function u must satisfy $u(2) = u(0)$. Putting $s = 2$ in **(23)** thus gives

$$u(0) = e^{-2\omega} \left[u(0) + B \int_0^2 e^{\omega x} \psi(x)\, dx \right];$$

solving this last equation for $u(0)$ and plugging the result into **(23)** yields (**Exercise 22**)

$$u(s) = Be^{-\omega s} \left[\frac{1}{e^{2\omega} - 1} \int_0^2 e^{\omega x} \psi(x)\, dx + \int_0^s e^{\omega x} \psi(x)\, dx \right]. \tag{24}$$

---

[1] More precisely, **(23)** gives all functions u that are continuous for all reals and satisfy **(21)** on all intervals on which ψ is continuous. I'll use "solution" in that sense throughout the paper.

The condition $u(2) = u(0)$ is necessary for periodicity; by doing **Exercise 25**, you can show that it's also sufficient, i.e., that $u(s+2) = u(s)$ for *all* s, not just for $s = 0$. Thus, the u given by **(24)** is the unique periodic solution of **(21)**. Furthermore, because $\omega > 0$, all solutions of **(21)** approach the periodic solution when $s \to \infty$ (**Exercise 26**).

Let's see what **(24)** looks like, for $0 \le s \le 2$, when ψ has the form **(14)**. By **Exercise 23**,

$$\int_0^s e^{\omega x}\psi(x)\,dx = \begin{cases} \left(e^{\omega s}-1\right)/\omega, & \text{if } 0 \le s \le 1; \\ \left(2e^{\omega} - e^{\omega s} - 1\right)/\omega & \text{if } 1 \le s \le 2. \end{cases} \qquad \textbf{(25)}$$

Thus,

$$\frac{1}{e^{2\omega}-1}\int_0^2 e^{\omega x}\psi(x)\,dx =$$
$$\frac{2e^{\omega} - 2e^{2\omega} + e^{2\omega} - 1}{\omega\left(e^{2\omega}-1\right)} = \frac{1}{\omega}\left[\frac{-2e^{\omega}\left(1-e^{\omega}\right)}{\left(1-e^{\omega}\right)\left(1+e^{\omega}\right)} + 1\right],$$

i.e.,

$$\left(e^{2\omega}-1\right)^{-1}\int_0^2 e^{\omega x}\psi(x)\,dx = \frac{1}{\omega}\left[1 - \frac{2e^{\omega}}{1+e^{\omega}}\right]. \qquad \textbf{(26)}$$

Now we plug **(25)** and **(26)** into **(24)**: When $s \le 1$, we have

$$u(s) = \frac{B}{\omega}e^{-\omega s}\left[1 - \frac{2e^{\omega}}{1+e^{\omega}} + e^{\omega s} - 1\right] = \frac{B}{\omega}\left[1 - \frac{2e^{\omega}e^{-\omega x}}{1+e^{\omega}}\right]. \qquad \textbf{(27)}$$

Similarly (**Exercise 24**), when $s \ge 1$,

$$u(s) = \frac{B}{\omega}\left[-1 + \frac{2e^{2\omega}e^{-\omega x}}{1+e^{\omega}}\right]. \qquad \textbf{(28)}$$

We could now use the definition **(20)** of s in terms of t to cast **(27)** and **(28)** in terms of t, thus solving the differential equation **(16)**, but that isn't necessary. The motivation for deriving these expressions for u was to find the diurnal temperature range in the case of an atmosphere-free planet, and this can be done using **(27)** and **(28)** as they stand. Convince yourself that u is increasing for $0 \le s \le 1$ and decreasing for $1 \le s \le 2$ (see if you can do this without taking derivatives). Thus, by **(27)**, the diurnal temperature range is

$$u_{\max} - u_{\min} = u(1) - u(0) = \frac{2B}{\omega}\left[1 - \frac{2}{1+e^{\omega}}\right]. \qquad \textbf{(29)}$$

In the next section, we'll explore this formula's physical significance.

Exercises

22. Derive the formula **(24)** for $u(s)$.

23. Derive the expression **(25)** for $\int_0^s e^{\omega x}\psi(x)\,dx$, where ψ is given by **(14)**.

24. Derive the expression **(28)** for $u(s)$ in the case $s \geq 1$.

25. To show that the function u given by **(24)** is periodic, do the following.

a) Show that

$$u(s+2) = Be^{-\omega s}e^{-2\omega} \times$$
$$\left[\left(e^{2\omega}-1\right)^{-1}\int_0^2 e^{\omega x}\psi(x)\,dx + \int_0^2 e^{\omega x}\psi(x)\,dx + \int_2^{s+2} e^{\omega x}\psi(x)\,dx\right].$$

b) Make the substitution $y = x - 2$ in the third integral in part **a)**; then use the periodicity of ψ and some algebra to show that $u(s+2) = u(s)$.

26. Show that when $\omega > 0$, every solution of **(21)** approaches its unique periodic solution. Hint: The general solution u satisfies **(23)**. The unique periodic solution u satisfies **(24)**; for the purposes of this exercise, denote it by u_P. Show that for each u of the form **(23)**, we have $[u(s) - u_P(s)] \to 0$ as $s \to \infty$.

7. Temperatures on an Airless Planet, Continued

In **Section 5**, we used our model of an atmosphere-free planet to estimate that, if the solar constant were to suddenly rise by 1%, Mars's equilibrium surface temperature would jump by about half a degree Kelvin, from 216.9 K to 217.4 K.

Now we'll try to answer **Question 2** of that section: Compute how quickly the surface temperature would approach its new equilibrium value.

To do this, we imagine that the change in the solar constant occurred at time $t = 0$; then $T(0) = 216.9$ K, and for $t > 0$, the effective temperature is $T_e = 217.4$ K. For $t > 0$, we define $u(t) = T(t) - T_e = T(t) - 217.4$ K; that is, $u(t)$ is the displacement of the surface temperature from its *new* equilibrium value. Then u satisfies the differential equation **(7)**, i.e., $du/dt + ku = 0$ with $k = 4\sigma T_e^3/c$. So, by **(19)**, u has e-folding time

$$\tau = \frac{c}{4\sigma T_e^3}. \qquad \textbf{(30)}$$

To compute a numerical value for τ, we need one for c. Here we encounter a difficulty. Let's get a crude estimate of c, for the Earth, and hope that a similar value might pertain to Mars. Wallace and Hobbs give the following facts for the

Earth. The materials—soil, rock, sand, and clay—that make up land conduct heat very slowly, so that, during the course of a day, the solar energy absorbed by the land surface penetrates to a depth of less than a meter. Let's take the penetration depth during the day to be 1 m. (Martian and terrestrial days have about the same length—Mars's is 24.6 hours [Beatty and Chaikin 1990, 289]—so it seems reasonable to guess that the penetration depth might be similar on Mars.) Furthermore, the specific heat of those materials is only about one-quarter of that of water [Wallace and Hobbs 1977, 338–339]. Also, we can estimate [Peixoto and Oort 1992, 221] that land and water have about the same density. Recall that c is the surface's heat capacity per square meter of surface. Thus,

$$\begin{aligned} c &\approx \frac{(.25 \cdot \text{specific heat of water})(\text{land density})(\text{volume of land 1 m deep})}{\text{area of land}} \\ &= (1{,}046\ \mathrm{J/kg\,K})(1{,}000\ \mathrm{kg/m^3})(1\ \mathrm{m}) \approx 10^6\ \mathrm{J/m^2K}, \end{aligned} \qquad \textbf{(31)}$$

and so, by **(30)** and **(31)**, the e-folding time is

$$\tau = \frac{10^6\ \mathrm{J/m^2K}}{4\left(5.67\times10^{-8}\ \mathrm{W/m^2K^4}\right)(217.4\ \mathrm{K})^3} = 4.3\times10^5\ \mathrm{s} \approx 5\ \text{days}.$$

It might appear, then, that, 15 days after the solar output jumped to its new value, the surface temperature would have come within 95% of its new equilibrium value. In fact, though, there's a flaw in the reasoning (besides the obvious one that the calculation really applies to the Earth, not to Mars) that shows up a problem with our model. We assumed a penetration depth of 1 m based on a time scale of one day. If, however, we had used another time scale, we would have gotten a different answer. According to Wallace and Hobbs, the solar energy absorbed by a land surface on Earth is conducted, during the course of a year, to a depth of a few meters. Let's say “a few” means “five.” Then the e-folding time becomes 25 days instead of 5. The problem is that the value of c depends on the penetration depth, which in turn depends on the time scale, which depends on c. The “surface” isn't really a single entity, but, rather, a collection of layers. A more refined model would take into account the rate at which energy moves from the top layer down to lower levels. Our model doesn't allow us to say how quickly the temperature approaches its new equilibrium.

7.1 The Diurnal Temperature Range

We now turn to **Question 3** of **Section 5**, the problem of determining the *diurnal temperature range (DTR)*. Here the time scale is unambiguous; we have to use a value of c appropriate to the surface layer that the Sun warms during one day.

At the Martian equator, daytime highs are “about 300° K, not very different from temperatures in the Earth's tropics”; but at night the temperature ”drops to a frigid 160° K, much colder than any place on the surface of the Earth." [Goody and Walker 1972, 66] This results in a DTR of 140 K. Now by **(17)** and **(22)**,

$$\frac{B}{\omega} = \frac{T_e}{4}. \tag{32}$$

Thus, if ψ is as in **(14)**, then **(29)** shows that the diurnal temperature range is

$$\text{DTR} = \frac{T_e}{2}\left[1 - \frac{2}{1+e^{\omega}}\right]. \tag{33}$$

Figure 5 is a graph of **(33)** with $T_e = 216.9$ K.

Figure 5. Mars's DTR as a function of ω.

From **(33)**, we can see that for fixed T_e, the DTR is an increasing function of ω such that $\lim_{\omega\to\infty} \text{DTR}(\omega) = T_e/2$. Thus, the model temperatures can't exhibit DTR's larger than half the effective temperature—108 K in the case of Mars. This is smaller than the Goody-Walker value of 140 K, but at least it's the right order of magnitude. (Allowing the solar intensity to vary during the day allows for a realistic DTR; see **Exercise 29**.) The model achieves DTR's close to 108 K when ω is large. Recall from **(22)** that

$$\omega = \frac{P}{2\tau}. \tag{34}$$

By **(34)** and **(30)**,

$$\omega = \frac{2\sigma T_e^3 P}{c}, \tag{35}$$

so that large ωs correspond to small cs, and, indeed:

> The thermal conductivity of the Martian surface is believed to be small, with only a thin layer taking part in the diurnal oscillations. Since a thin layer of dry surface material can hold very little heat, large temperature changes result from the diurnal cycle of heating and cooling. (We can observe this phenomenon on Earth, where diurnal surface temperature changes are much larger in dry desert areas a long way from the sea than they are on or near the oceans or on land covered with vegetation. Observations on Mars indicate that the surface does indeed resemble light, dry desert sand. [Goody and Walker 1977, 66].

It makes physical sense that the DTR should be an increasing function of ω. By **(34)**, ω is proportional to the ratio of the planetary rotation period P to the e-folding time τ.

- If ω is large, then the surface gains and loses heat quickly in comparison to the rotation period, so that there's plenty of time during the day for the surface to heat up and plenty of time at night for the surface to cool off. Thus, we expect a large DTR.
- If ω is small, the days and nights don't provide adequate time for large temperature changes, and the DTR should be small.

From **(33)**, we see that for fixed ω, the DTR is also an increasing function of T_e, and this, too, makes physical sense. A hot planet, i.e., one with a large T_e, emits longwave radiation at a high rate and thus cools off quickly at night when the Sun is down. Such a planet warms up quickly during the day, because it absorbs solar energy at a high rate; if it didn't, the effective temperature wouldn't be large (see **(2)**). Conversely, a cool planet experiences less rapid temperature changes over the course of a rotation period.

Next we graph the temperature as a function of time when ψ is as in **(14)**: Using **(32)**, we rewrite **(27)** and **(28)** as

$$u(s) = \begin{cases} \dfrac{T_e}{4}\left[1 - \dfrac{2e^{\omega}e^{-\omega s}}{1+e^{\omega}}\right], & \text{if } 0 \le s \le 1; \\ \\ \dfrac{T_e}{4}\left[-1 + \dfrac{2e^{2\omega}e^{-\omega s}}{1+e^{\omega}}\right], & \text{if } 1 \le s \le 2. \end{cases} \qquad \textbf{(36)}$$

Recall that $u(s)$ is the displacement of temperature from its equilibrium value as a function of time, where time is measured in units of half a period. **Figure 6** is a graph of one day's worth of $u(s)$, for $T_e = 216.9$ K and $\omega = 4$.

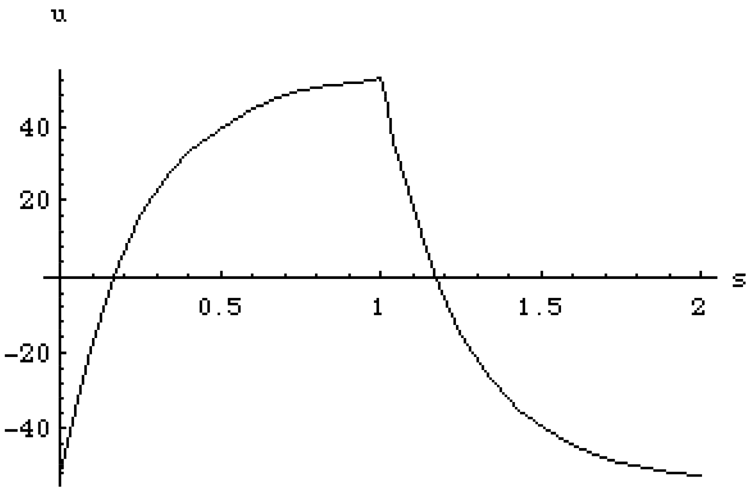


Figure 6. Temperature as a function of time during a Martian day.

The displacement u is increasing and concave down for $0 < s < 1$. Physically, u is increasing because the surface is warming up during the day. Here's why u is concave down. At dawn, the planet is cool and therefore is radiating away energy at a slow rate; in the meantime, it's absorbing energy rapidly from the Sun. The surface's net gain in energy thus occurs rapidly, so the temperature graph rises sharply. Later in the day, the surface is absorbing energy at the same rate as earlier (recall that Flux In is constant in this model), but, because the surface is hotter, it radiates energy away more quickly than before. Thus the temperature graph is still rising, but at a lower rate than before.

A similar argument (**Exercise 27**) explains why u is decreasing and concave up for $1 < s < 2$. The discontinuity in our square-wave function ψ is reflected in the nonsmoothness (nondifferentiability) of u at $s = 1$ (and, if we continue u periodically, also at $s = 2$ and all integral values of s).

In the **Introduction**, I stated that, during the last few decades, the DTR over the Earth's land surfaces has decreased at about the same rate that the average temperature has increased. On an airless planet, what happens to the DTR when, in response to an increase in the solar constant, the effective temperature goes up? The DTR must *increase*, because the DTR **(33)** is an increasing function of T_e and ω, while, by **(35)**, ω is an increasing function of T_e. Furthermore (**Exercise 28**), we have

$$\frac{d}{dT_e}\,\mathrm{DTR} = \frac{1}{2}\left[1 - \frac{2}{1+e^{\omega}} + \frac{6\omega e^{\omega}}{(1+e^{\omega})^2}\right], \qquad \textbf{(37)}$$

and thus the rate of change of the DTR depends only on ω. The graph of the right-hand side of **(37)** shown in **Figure 7** suggests that for large ω, the DTR on Mars would rise by about a half a degree per one-degree rise in T_e.

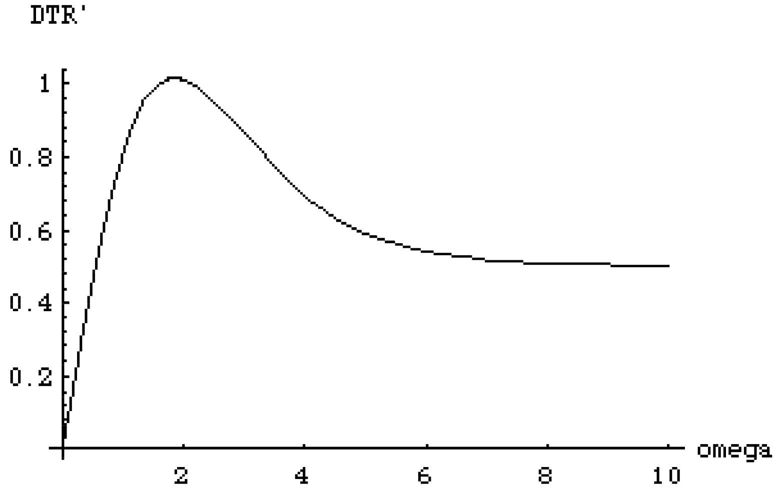


Figure 7. The derivative of the diurnal temperature range with respect to T_e as a function of ω for an airless planet.

Exercises

27. Explain physically why u given by **(36)** is decreasing and concave up for $1 < s < 2$.

28. **a)** Show that $T_e \cdot d\omega/dT_e = 3\omega$.

b) Derive the formula **(37)** for (d/dT_e)DTR.

29. We're going to study a more realistic model of Mars in which sunlight is more intense in the middle of the day than in the morning and the evening. Suppose throughout this exercise that ψ has the form **(12)**. For such ψ, it can be shown that the unique periodic solution of **(21)** is

$$u(s) = \begin{cases} B\left\{-\dfrac{1}{\omega} + \dfrac{\sin\left(\pi s - \tan^{-1}(\pi/\omega)\right)}{\left[1+(\omega/\pi)^2\right]^{1/2}} \right. \\ \qquad \left. + \dfrac{e^{-\omega s}}{\left[1+(\omega/\pi)^2\right](1-e^{-\omega})}\right\}, & \text{if } 0 \le s \le 1; \\ \\ B\left\{\dfrac{-1}{\omega} + \dfrac{e^{\omega}e^{-\omega s}}{\left[1+(\omega/\pi)^2\right](1-e^{-\omega})}\right\}, & \text{if } 1 \le s \le 2, \end{cases} \qquad \textbf{(38)}$$

where B satisfies **(22)**. In particular, if u is the periodic solution of the differential equation **(11)** that we derived for an atmosphere-free planet, then, by **(32)**, we have $B = T_e\omega/4$. Do the following with electronic help.

a) Plot $u(s)$ given by **(38)**, using this last expression for B, with $T_e = 216.9\,\text{K}$. Do this for several values of ω. Is the DTR a decreasing function of ω? As ω increases, the value of s for which the temperature is maximum moves to the left, getting closer and closer to 0.5. Explain physically why this should be so.

b) Find a value of ω that allows the model to match well the true Martian DTR. For that value, at about what time of day will the maximum temperature be achieved?

c) Use **(35)** to determine the heat capacity c for the value of ω that you found in part **b)**; how does it compare to the c in **(31)**? Watch your units; the figure **(1)** for σ involves measuring time in seconds—why?

d) Suppose that the solar constant increases. For the value of ω that you found in part **b)**, estimate the increase in DTR per degree increase in the equilibrium surface temperature.

e) Graph $u(s)$ (as in part **a)**) for a very large value of ω. The graph is nearly horizontal for $1 \leq s \leq 2$. Does that make physical sense?

8. Equilibrium Temperatures on a Planet with an Atmosphere

In **Section 4**, we saw that the Earth's average surface temperature, 290 K, is 35 K warmer than its effective temperature of 255 K. The difference is produced by the atmosphere, which absorbs longwave radiation emitted by the surface, and is thus kept warm; being warm, the atmosphere also emits longwave radiation, some of which the surface absorbs, in addition to the shortwave radiation it's already absorbing from the Sun. The surface is thus hotter than it would be if no atmosphere overlay it.

To quantify the exchange of radiation, we assume as before that every point on the surface is at the same (time-dependent) temperature $T_0 = T_0(t)$. Similarly, we assume that every point of the atmosphere is at the same temperature $T_1 = T_1(t)$. We have to account for three types of power:

- shortwave power,
- longwave power, and
- power not due to radiation.

8.1 Shortwave Power

We still assume that the Earth / atmosphere system absorbs flux $a\Omega/4$ from the Sun, but now a portion of the flux is absorbed by the surface and a portion by the atmosphere. The details of the apportionment are rather complicated.

Of the flux $\Omega/4$ that reaches the top of the atmosphere, before any absorption, part is reflected back into space, part is absorbed by the atmosphere, and part reaches the surface. Part of the flux that reaches the surface is reflected back to the atmosphere, and part of *that* is reflected back to the surface. Some of the latter is absorbed by the surface. Thus, the total shortwave flux absorbed by the surface is the result of multiple reflections of sunlight between surface and atmosphere, and similarly for the total shortwave flux absorbed by the atmosphere.

Now, the surface and atmosphere together absorb shortwave flux $a\Omega/4$. Let's denote by q the fraction of that flux absorbed by the surface, after the multiple bounces; then the total flux absorbed by the surface is $qa\Omega/4$. The rest of the $a\Omega/4$ is absorbed by the atmosphere (again after multiple bounces), so the total shortwave flux absorbed by the atmosphere is $(1-q)a\Omega/4$.

This notation conveniently allows us to avoid dealing with the complexities of the multiple reflections, but we have to use it with care. The parameter q is a function of several other quantities, including the fraction of sunlight absorbed by the surface on the first bounce and the reflectivities of the surface and atmosphere. In other words, q depends on the albedo, so that, if we wanted to use our model to predict the effect of changing the albedo, we would have to find out how the value of q is affected. That would lead to an elegant application of geometric series [Harte 1988, 89–94].

We have now shown, in view of **(2)**,

$$\begin{aligned}\text{shortwave power absorbed by surface} &= Sq\sigma T_e^4,\\ \text{shortwave power absorbed by atmosphere} &= S(1-q)\sigma T_e^4,\end{aligned} \qquad \textbf{(39)}$$

where S is the Earth's surface area.

8.2 Longwave Power

Recall that a blackbody, by definition, absorbs all the radiation it intercepts in all wavelengths, and emits radiation from its boundary at the maximum possible rate in each wavelength. We noted in **Sections 4** and **5** that the Earth's surface behaves like a blackbody, to a good approximation, in the longwave range, and that, at the temperatures typical of the surface (and atmosphere), nearly all of the radiation is emitted in the longwave range. That led us to assume that our model surface emits flux σT_0^4, all in the longwave range. If the atmosphere were a blackbody, it would emit flux σT_1^4, nearly all of it in the longwave range.

In fact, the atmosphere does *not* behave like a blackbody, even in the longwave range; it doesn't absorb all the longwave radiation incident on it, nor does it emit with maximal efficiency.

Let's suppose that the atmosphere absorbs a fraction A, where $0 \leq A \leq 1$, of the flux σT_0^4 that it receives from the surface, and emits flux $\epsilon\sigma T_1^4$, where $0 \leq \epsilon \leq 1$. Then A and ϵ are the atmosphere's *absorptivity* and *emissivity*, respectively.

Now, a physical principle called *Kirchoff's law* [Wallace and Hobbs 1977, 291–292], which is valid in most of the atmosphere, states that the efficiency of a body at absorbing radiation *at a given wavelength* equals its efficiency at emitting radiation at that wavelength. Suppose, for example, that an object absorbs 10% of all the flux of red light incident on it, and 20% of all the flux of blue light incident on it. Then it will emit flux, in the form of red light, at 10% of the rate that a blackbody at its temperature would; and it will emit flux, in the form of red light, at 20% of the rate that a blackbody at its temperature would.

Inspired by Kirchoff's law, we'll assume that

$$A = \epsilon$$

—the atmosphere's emissivity equals its absorptivity—so that the flux that the model atmosphere absorbs from the surface is approximately $\epsilon\sigma T_0^4$. This lumping together of all the different wavelengths of longwave wavelengths in Kirchoff's law makes for a rather crude approximation, but perhaps no cruder than our assumption that the surface behaves like a blackbody in the longwave range. (See **Appendix I** for a more complete discussion of this.)

The flux is the power emitted per square meter of the atmosphere's boundary, which, in our model, consists of two spheres: one contiguous with the planet's surface, and the other exposed to space, each having surface area S. The atmosphere emits from both spheres, but only absorbs through the bottom sphere since the only longwave radiation available to be absorbed is what's being emitted by the planet's surface. Similarly, the surface absorbs all the longwave power emitted by the bottom sphere. We therefore have

$$\begin{aligned}
\text{longwave power absorbed by surface} &= S\epsilon\sigma T_1^4, \\
\text{longwave power emitted by surface} &= S\sigma T_0^4, \\
\text{longwave power absorbed by atmosphere} &= S\epsilon\sigma T_0^4, \\
\text{longwave power emitted by atmosphere} &= 2S\epsilon\sigma T_1^4.
\end{aligned} \qquad \textbf{(40)}$$

8.3 Nonradiative Power

On the real Earth, there are two nonradiative means by which energy is transferred from the surface to the atmosphere.

- First, heat moves, by conduction, from the surface to the bottom layer of the atmosphere, and from there, by turbulence and convection, to higher levels.
- Second, some of the surface's energy is expended in evaporating water. When the water vapor that results from the evaporation reaches a high enough altitude to condense, it releases that energy into the atmosphere.

Rather than deal with the physical details of those processes, which we'll group collectively under the title of "mechanical heat transfer," we'll simply assume that the transfer of energy obeys Newton's law of cooling, that is, that energy is transferred at a rate proportional to the difference between the surface and atmospheric temperatures:

$$\begin{aligned}\text{power leaving surface by mechanical means} &= S\gamma(T_0 - T_1),\\ \text{power entering atmosphere by mechanical means} &= S\gamma(T_0 - T_1);\end{aligned} \tag{41}$$

where γ is a positive constant. In most of our work, we'll neglect this term, since including it seems not to make much of a difference in the model we're studying.

Figure 8 shows all of the elements in the exchange of energy among the surface, the atmosphere, and space.

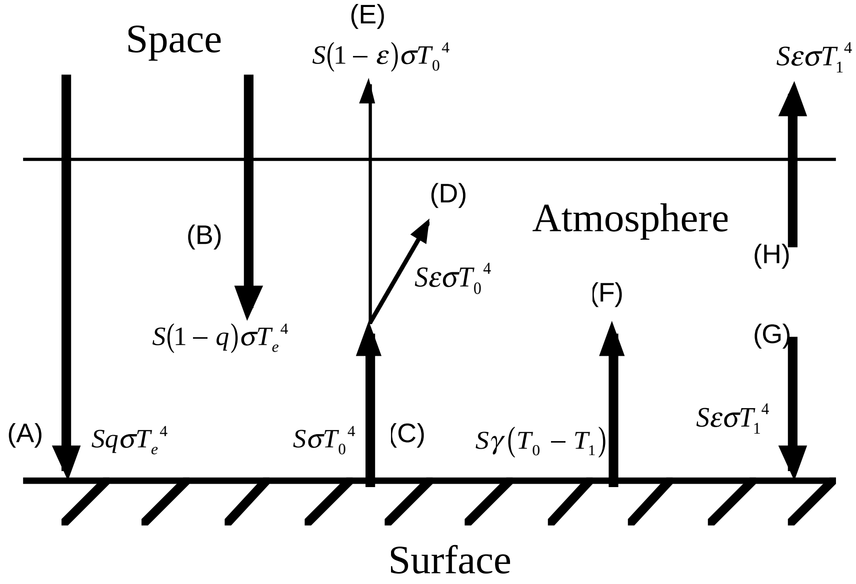


Figure 8. Exchange of energy. The Earth / atmosphere system absorbs solar power $S\sigma T_e^4$, some (A) at the surface, and the rest (B) in the atmosphere. Meanwhile the surface is emitting power (C), some of which (D) the atmosphere absorbs, the rest of which (E) goes out into space. Power is also transmitted from surface to atmosphere via mechanical heat transfer (F). Finally, the atmosphere emits power from its bottom boundary [(G), all absorbed by the surface] and its top boundary [(H), lost in space].

Now we'll concentrate on the *equilibrium* temperatures, denoted by $\widehat{T}_0$ and $\widehat{T}_1$. For each of the surface and atmosphere, respectively, we equate the sum of the Powers In **(39–41)** to the sum of the Powers Out; the result, after dividing by the Earth's surface area S, is a pair of equations of the form Flux In $=$ Flux Out satisfied by $\widehat{T}_0$ and $\widehat{T}_1$:

$$q\sigma T_e^4 + \epsilon\sigma\widehat{T}_1^4 = \sigma\widehat{T}_0^4 + \gamma\left(\widehat{T}_0 - \widehat{T}_1\right); \qquad \text{(Surface Balance)} \tag{42}$$

$$(1-q)\sigma T_e^4 + \epsilon\sigma\widehat{T}_0^4 + \gamma\left(\widehat{T}_0 - \widehat{T}_1\right) = 2\epsilon\sigma\widehat{T}_1^4. \qquad \text{(Atmospheric Balance)} \tag{43}$$

In our model of an atmosphere-free planet, the equilibrium surface temperature equaled the effective temperature. Now that we have an atmosphere that's no longer the case; in fact (**Exercise 33**), $\widehat{T}_0 > T_e$ for the numerical values of ϵ and q that we'll shortly be using.

I mentioned in the **Introduction** that the amount of atmospheric CO_2 is likely to become double its 1850 value at some time in the next 50 to 150 years. The energy balance equations **(42–43)** allow us to give an answer to a standard question in climate modeling, namely, what effect would that have on the global average surface temperature? First, by adding **(42)** to **(43)** and rearranging,

$$\sigma T_e^4 = (1-\epsilon)\sigma \widehat{T}_0^4 + \epsilon\sigma\widehat{T}_1^4, \tag{44}$$

so that

$$\epsilon = \frac{\widehat{T}_0^4 - T_e^4}{\widehat{T}_0^4 - \widehat{T}_1^4}.$$

Currently[2], for the Earth,

$$T_e = 255\ \mathrm{K}, \qquad \widehat{T}_0 = 290\ \mathrm{K}, \qquad \text{and} \qquad \widehat{T}_1 = 250\ \mathrm{K},$$

so

$$\epsilon = .8983. \tag{45}$$

Next, solving **(42)** for q and γ gives

$$q = \frac{\widehat{T}_0^4 - \epsilon\widehat{T}_1^4 + \gamma\sigma^{-1}\left(\widehat{T}_0 - \widehat{T}_1\right)}{T_e^4} \qquad \text{and} \qquad \gamma = \frac{\sigma\left(qT_e^4 + \epsilon\widehat{T}_1^4 - \widehat{T}_0^4\right)}{\widehat{T}_0 - \widehat{T}_1} \tag{46}$$

We'll assume that $\gamma = 0$, thus neglecting mechanical heat transfer, an assumption that simplifies our computations and makes virtually no difference to our conclusion (see **Exercise 32**). Eliminating $\widehat{T}_1$ from **(42)** and **(43)** and solving for $\widehat{T}_0$ gives (**Exercise 31**)

$$\widehat{T}_0 = T_e G(\epsilon, q), \qquad \text{where} \qquad G(\epsilon,q) = \left(\frac{1+q}{2-\epsilon}\right)^{1/4}, \qquad \text{if } \gamma = 0. \tag{47}$$

The presence of an atmosphere increases the surface temperature by a factor of $G(\epsilon, q)$. (The factor can be less than one; see **Exercise 33**). We'll call the function G the *greenhouse function*. From **(45)**, the first half of **(46)**, and our assumption that $\gamma = 0$, we find that

$$q = .8428 \qquad \text{if } \gamma = 0. \tag{48}$$

---

[2] The figure for $\widehat{T}_1$ comes about by taking a density-weighted mean of the altitude-dependent temperatures given in Holton [1992, 487].

Recall from **Section 3** that the real Earth's surface absorbs about two-thirds of the total solar radiation absorbed by the Earth / atmosphere system. The model overestimates that fraction.

Because carbon dioxide is a greenhouse gas, a doubling of the concentration of atmospheric carbon dioxide would make the atmosphere more efficient at absorbing and emitting longwave radiation. The greater efficiency corresponds to a bigger ϵ. Let's estimate the new ϵ; we'll then use **(47)** to determine the future surface temperature. Similarly, we'll estimate the surface temperature prior to the industrial revolution. The total warming will be the difference between the future and old surface temperatures. All three values of ϵ are independent of γ and q, and thus are also applicable to the case $\gamma > 0$, $q = 1$ considered in **Exercise 32**.

Prior to the Industrial Revolution, the concentration of carbon dioxide in the atmosphere was about 280 ppm(v), where "ppm(v)" stands for "parts per million by volume"; i.e., of every one million air molecules, 280 were carbon dioxide. Twice that concentration would be 560 ppm(v). In 1992, the concentration was about 350 ppm(v) [Peixoto and Oort 1992, 434][3]; so we need to know the value of ϵ that would result from increasing the current concentration of CO_2 by a factor of $560/350 = 1.6$.

The principal greenhouse gases are water vapor and carbon dioxide; we'll neglect the others. We'll assume that the abilities of the two gases to absorb longwave radiation can be separated, so that $\epsilon = \epsilon_w + \epsilon_c$, where ϵ_w and ϵ_c are the fractions of surface-emitted longwave radiation absorbed by atmospheric water vapor and by atmospheric carbon dioxide. We'll also assume that the effectiveness of water vapor at absorbing incoming longwave radiation is proportional to the concentration of water vapor: $\epsilon_w = \beta \times (\text{concentration of water vapor})$, where β is a constant. We assume similarly for carbon dioxide; but, to a very crude first approximation, a molecule of carbon dioxide is only about one-quarter as effective as a molecule of water vapor at absorbing longwave radiation [Harte 1988, 184, Exercise 4], so that $\epsilon_c = \beta(\text{concentration of } CO_2)/4$. Also, the calculation in Harte [1988, 179] shows that there are currently 14.6 times as many water vapor molecules as carbon dioxide molecules in the atmosphere, and thus the concentration of water vapor is currently 14.6×350 ppm(v) $= 5{,}110$ ppm(v). Thus, assuming that the concentration of water vapor remains constant over time, the ratio of the "future epsilon" (after doubling of carbon dioxide) to the "current epsilon" (the present-day value in **(45)**] is

$$\begin{aligned}\frac{\text{future } \epsilon}{\text{current } \epsilon} &= \frac{\beta[(\text{current water vapor}) + 0.25(\text{future } CO_2)]}{\beta[(\text{current water vapor}) + 0.25(\text{current } CO_2)]} \\ &= \frac{5110 \text{ ppm(v)} + 0.25[560 \text{ ppm(v)}]}{5110 \text{ ppm(v)} + 0.25[350 \text{ ppm(v)}]} = 1.0101\end{aligned}$$

---

[3] Monthly or more frequent readings at various locations from 1979 on are available from the Climate Monitoring Diagnostic Lab of the National Oceanic and Atmospheric Administration, at `http://www.cmdl.noaa.gov/ccg/co2/GLOBALVIEW` . The data files are updated annually in August. By 1997, the mean concentration was about 360 ppm(v).

and thus, by **(45)**,

$$\text{future } \epsilon = 1.0101(\text{current } \epsilon) = 1.0101(.8983) = .9074.$$

Then, assuming the value of q is unchanged, we find from **(47)** that the future equilibrium surface temperature will be 290.6 K.

Similarly,

$$\begin{aligned}\frac{\text{old } \epsilon}{\text{current } \epsilon} &= \frac{\beta[(\text{current water vapor}) + 0.25(\text{old CO}_2)]}{\beta[(\text{current water vapor}) + 0.25(\text{current CO}_2)]} \\ &= \frac{5110 \text{ ppm(v)} + 0.25[280 \text{ ppm(v)}]}{5110 \text{ ppm(v)} + 0.25[350 \text{ ppm(v)}]} = 0.9966,\end{aligned}$$

and old $\epsilon = 0.9966(.8983) = .8953$, leading to an old surface temperature of 289.8 K and

$$\boxed{\textit{a total predicted rise of 0.8 K.}}$$

This increase is small compared to those predicted by the Global Climate Models, which average about 3 K [Peixoto and Oort 1992, 477]. Those models incorporate the physics in a more realistic way than we have and take into account many factors we've neglected. In particular, our analysis neglected *feedback effects*, processes whereby an initial change in a variable affects the values of other variables in such a way as to induce further changes in the first variable. An important such process that operates in the atmosphere is the *water vapor feedback effect*: An initial rise in atmospheric carbon dioxide makes the Earth's surface temperature rise, which in turn increases the evaporation rate and thus the amount of water vapor in the atmosphere. But because water vapor is a greenhouse gas, the result is a further rise in surface temperature, hence an even higher evaporation rate, and so forth—a process that converges (we hope that it converges!) to a higher temperature than the initial rise in carbon dioxide would suggest.

Exercises

30. The left-hand side of **(44)** is, by **(2)**, the flux of energy into the Earth / atmosphere system. Show that the right-hand side is the flux out; to do this, give a physical interpretation of each of its terms.

31. Derive the formula **(47)** for $\widehat{T}_0$.

32. By following the steps outlined below, redo the calculation of temperature rise due to CO_2-doubling, this time neglecting atmospheric absorption of shortwave radiation but including mechanical heat transfer.

a) Show that, when $q = 1$ and $\gamma > 0$, the surface temperature is a zero of the function

$$f(x) = 2\sigma T_e^4 - (2-\epsilon)\sigma x^4 - \gamma \left[x - \left(\frac{T_e^4 - (1-\epsilon)x^4}{\epsilon} \right)^{1/4} \right].$$

Hint: Solve **(44)** for $\widehat{T}_1$ and substitute into **(43)** (with $q = 1$).

b) Show that the function f in part **a)** has only one root. Hint: Prove that f is a decreasing function, which isn't hard to do if you avoid the temptation to compute f'.

c) Determine a numerical value of γ using **(46)**. Then use a computer or graphing calculator to determine the root of f for the "future" and "old" ϵs; the change in surface temperature is the difference between the two.

33. a) Verify the following facts about the greenhouse function $G(\epsilon, q)$ and interpret each fact physically:

1. $G(\epsilon, q) > 1$ if and only if $\epsilon + q > 1$;
2. G is an increasing function of ϵ;
3. G is an increasing function of q.

b) Use part **a)** to verify that $\widehat{T}_0 > T_e$ when $\gamma = 0$ and ϵ and q satisfy **(45)** and **(48)**. (Of course, we already know that $\widehat{T}_0 > T_e$: We chose those values of ϵ and q in order to match the observed temperatures.)

34. a) Derive a formula, in terms of T_e, for the maximum possible surface temperature of our model planet. (Assume that $\gamma = 0$.)

b) Venus has surface temperature 730 K [Beatty and Chaikin 1990, 93]. Use part **a)** to show that our model doesn't apply to Venus. (Venus can be modeled by dividing the atmosphere into several concentric spherical shells, each at uniform temperature; a single shell, as in our model, doesn't provide a big enough greenhouse effect.)

35. In the middle of the Cretaceous Period, about 100 million years ago, the Earth's climate was considerably warmer than it is now—so warm that Antarctica was probably free of permanent ice, and alligators lived near the Arctic Circle. "Geochemical models suggest that the atmosphere then may have contained between 5 and 10 times more carbon dioxide than it does now" [Schneider 1987, 76–77]. Use the model of this section (with $\gamma = 0$) to estimate the temperature at that time. (Keep it mind, though, that the model is likely to underestimate the real temperature.)

36. "Changes of several tenths of one percent in the Sun's brightness ... could cause temperature changes of 0.5° C on Earth" [Baliunas and Soon 96, 41]. Use the model of this section to evaluate that claim. (The authors argue that variations in the solar constant must be thoroughly understood in order to evaluate the effect of carbon dioxide on global temperature changes. They are examining Sun-like stars in order to gain insight into solar variations.)

9. Daily Temperature Cycle on a Planet with an Atmosphere

To model the daily temperature cycle on a planets that have atmospheres, we convert the equations **(42–43)** for the equilibrium temperatures into a pair of differential equations for the time-dependent temperatures, in the same way that we went from **(2)** to **(10)** in our study of atmosphere-free planets. Here and throughout this section we'll assume that $\gamma = 0$. The result is

$$c_0\,\frac{dT_0}{dt} = q\sigma T_e^4\left[1+\psi\left(\frac{2t}{P}\right)\right] + \epsilon\sigma T_1^4 - \sigma T_0^4; \tag{49}$$

$$c_1\,\frac{dT_1}{dt} = (1-q)\sigma T_e^4\left[1+\psi\left(\frac{2t}{P}\right)\right] + \epsilon\sigma T_0^4 - 2\epsilon\sigma T_1^4, \tag{50}$$

where ψ is a piecewise-continuous function, periodic of period 2, such that $\int_0^2 \psi(s)\ ds = 0$, and where c_0 and c_1 are the heat capacities, per unit area of the planet's surface, of the surface and atmosphere. As in the atmosphere-free case, we're assuming that the land-mass that we're studying is isolated from the rest of the planet and can therefore be thought of as a "planet" in its own right. This assumption is reasonable on the Earth as long as our land-mass is large enough—a continent, say —because typical Earthly wind speeds are about 5 m/s, i.e., about 400 km/day. Thus, temperatures on other parts of the globe will not have time, during the course of one day, to affect the inland regions that make up the vast majority of the continent.

Next we assume that T_1 is constant, i.e., $T_1(t) = \widehat{T}_1$ for all t. The justification for this is that

- it simplifies the mathematics considerably (always hard to argue with), and
- the atmosphere's diurnal temperature fluctuations are small compared to those of land.

In fact, "At the earth's surface, over land, the diurnal temperature range is on the order of 10 [Kelvin] at most locations, but it reaches values of more than 20 [Kelvin] over high-altitude desert locations." [Wallace and Hobbs 1977, 27]. In contrast, the atmosphere's heat capacity is high enough that temperatures there vary by about 0.68% [Goody and Walker 1972, 89], which, using our value of $\widehat{T}_1 = 250$ K, is 1.7 K. Thus, our assumption that T_1 is constant amounts to the statement that the atmosphere has infinite heat capacity.

So we replace T_1 by $\widehat{T}_1$ in **(49)**. We then define the displacement u of surface temperature from its equilibrium value $\widehat{T}_0$ by $u = T_0 - \widehat{T}_0$; then we linearize with the aid of **(42)** and change to the new independent variable $s = 2t/P$ defined by **(20)**. By **Exercise 37**, the resulting equation is

$$\frac{du}{ds} + \omega u = B\psi(s),$$

where

$$\omega = \frac{2P\sigma \widehat{T}_0^3}{c_0} \qquad \text{and} \qquad B = \frac{Pq\sigma T_e^4}{2c_0}. \tag{51}$$

These are a different ω and B from the ones in **Sections 6** and **7**. This last differential equation is just **(21)**, and it has a unique periodic solution given by **(24)**. Now, suppose that ψ has the form **(14)**. Then the periodic solution takes the form **(27–28)**, and the diurnal temperature range is **(29)**:

$$\text{DTR} = \frac{2B}{\omega} f(\omega), \qquad \text{where} \quad f(\omega) = 1 - \frac{2}{1+e^{\omega}}. \tag{52}$$

We can get a number of physical insights from **(52)**. First, we can see what effect the presence of an atmosphere has on the DTR. In reality an atmosphere probably affects the DTR in many ways, by modifying the planet's albedo, for example, but here let's isolate the role played by ϵ. That is, let's compare two planets that have the same rotation period, effective temperature, and heat capacity, one planet atmosphere-free, and the other having an atmosphere. Which has the larger DTR?

For the atmosphere-free planet: We have $\epsilon = 0$, $q = 1$. Denote by ω_0 the ω for that planet given by **(51)**. Then $\omega_0 = 2P\sigma T_e^3/c_0$.

For the planet with atmosphere: Equations **(51)** and **(47)** give

$$\omega = 2P\sigma \widehat{T}_0^3/c_0 = [G(\epsilon,q)]^3 \omega_0.$$

So, by **(52)**,

$$\frac{\text{DTR of planet with atmosphere}}{\text{DTR of atmosphere-free planet}} = [G(\epsilon,q)]^{-3} F(\omega_0), \tag{53}$$

where $F(x) = f\left([G(\epsilon,q)]^3 x\right)/f(x)$. Now suppose that $\epsilon + q > 1$. Then F has the following properties (**Exercise 38**):

$$F(x) \to [G(\epsilon,q)]^3 \text{ as } x \to 0^+; \tag{54a}$$

$$F \text{ is a decreasing function;} \tag{54b}$$

$$F(x) \to 1 \text{ as } x \to \infty. \tag{54c}$$

It follows from **(53)** and **(54ab)** that when $\epsilon + q > 1$, introducing an atmosphere compresses the DTR. This is a second way—besides raising the average surface temperature—in which the presence of an atmosphere moderates a planet's climate when $\epsilon + q > 1$. By **(53)** and **(54bc)**, the cube of the greenhouse function gives the maximum possible compression. For the Earth,

$$[G(\epsilon,q)]^3 = [G(.8983,.8428)]^3 = 1.47.$$

Thus, since the current Earth's land surfaces have DTR's of about 10 K, a fictitious Earthlike planet that had no atmosphere but that had the real Earth's rotation period, effective temperature, and heat capacity, would have a DTR of at most 15 K.

Since an atmosphere compresses the DTR, more atmosphere should compress it more. Let's check whether it does. If we fix

- the albedo α,
- the solar constant Ω (hence the effective temperature T_e),
- the surface heat capacity c_0,
- the fraction of solar radiation q absorbed by the surface, and
- the rotation period P,

will a rise in ϵ lead to a fall in the DTR? Yes: In **(53)**, ω_0 will remain constant while $[G(\epsilon, q)]^{-3}$ decreases.

Now we'll compute the rate of change of the DTR with respect to the equilibrium surface temperature. Again fix a, c_0, q, and P. We'll analyze two cases:

(I) Ω (hence T_e) remains constant while ϵ varies; and

(II) ϵ remains constant while Ω (hence T_e) varies.

Case I. By **(52)**, **(51)**, and **(47)**,

$$\begin{aligned}
\left.\frac{d}{d\widehat{T}_0}\text{DTR}\right|_{\epsilon \text{ varying, } \Omega \text{ fixed}} &= 2B\left[\frac{-f(\omega)}{\omega^2} + \frac{f'(\omega)}{\omega}\right]\frac{d\omega}{d\widehat{T}_0} \\
&= \frac{Pq\sigma T_e^4}{c_0}\left[\frac{-f(\omega)}{\omega^2} + \frac{f'(\omega)}{\omega}\right]\frac{3\omega}{\widehat{T}_0} \\
&= 1.5q\omega^2\left(\frac{T_e}{\widehat{T}_0}\right)^4\left[\frac{-f(\omega)}{\omega^2} + \frac{f'(\omega)}{\omega}\right] \\
&= \frac{q}{[G(\epsilon, q)]^4}\cdot 1.5\left[-f(\omega) + \omega f'(\omega)\right].
\end{aligned}$$

Thus,

$$\left.\frac{d}{d\widehat{T}_0}\text{DTR}\right|_{\epsilon \text{ varying, } \Omega \text{ fixed}} = \frac{q}{[G(\epsilon, q)]^4}\cdot H_{\text{more gas}}(\omega), \qquad \textbf{(55)}$$

where

$$H_{\text{more gas}}(\omega) = 3\left[\frac{-1}{2} + \frac{1}{1+e^\omega} + \frac{\omega e^\omega}{(1+e^\omega)^2}\right].$$

Case II. Again by **(52)**, **(51)**, and **(47)**,

$$\text{DTR} = \frac{q\widehat{T}_0 f(\omega)}{2[G(\epsilon, q)]^4}. \tag{56}$$

Now, $d\omega/d\widehat{T}_0 = 3\omega$ by **(51)**, so, by **(52)**,

$$\left.\frac{d}{d\widehat{T}_0}\text{DTR}\right|_{\Omega \text{ varying, } \epsilon \text{ fixed}} = \frac{f(\omega) + \widehat{T}_0 f'(\omega) \cdot 3\omega}{2[G(\epsilon, q)]^4}.$$

Thus,

$$\left.\frac{d}{d\widehat{T}_0}\text{DTR}\right|_{\Omega \text{ varying, } \epsilon \text{ fixed}} = \frac{q}{[G(\epsilon, q)]^4} \cdot H_{\text{hotter sun}}(\omega), \tag{57}$$

where

$$H_{\text{hotter sun}}(\omega) = \frac{1}{2} - \frac{1}{1+e^\omega} + \frac{3\omega e^\omega}{(1+e^\omega)^2}.$$

Figure 9 shows the graphs of $H_{\text{more gas}}$ and $H_{\text{hotter sun}}$. The model has the interesting property that enhanced greenhouse warming decreases the DTR, whereas enhanced solar warming increases it.

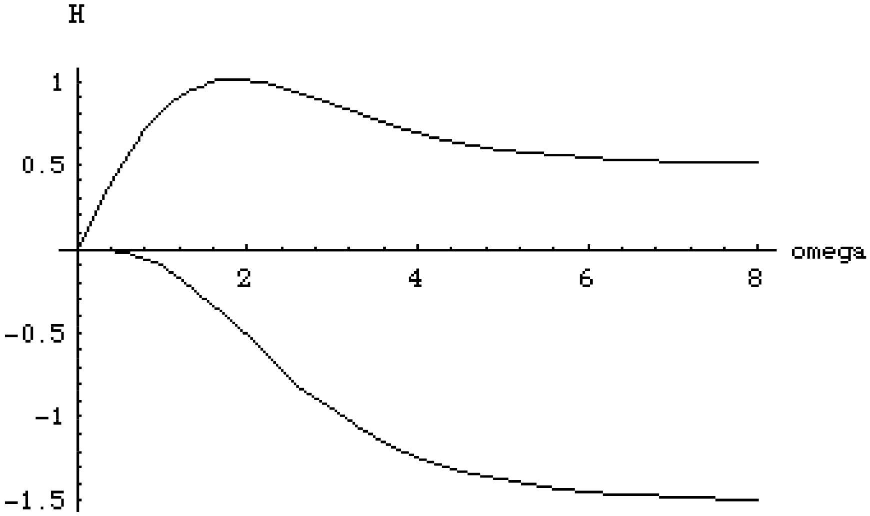


Figure 9. $H_{\text{more gas}}$ (lower curve) and $H_{\text{hotter sun}}$ (upper curve).

Using the values for the Earth that we computed in **Section 8**, we have $q[G(\epsilon, q)]^{-4} = .8428 \cdot [G(.8983, .8428)]^{-4} = 0.5$, so our model Earth can exhibit significant rates of change of the DTR if ω is large enough. From **(56)**, we can estimate (**Exercise 39**) that for the Earth, $.2 \leq \omega \leq .7$, roughly. For ω in that range, $(d/d\widehat{T}_0)$DTR is between about -0.0005 and -0.02 when the emissivity is allowed to vary. When instead Ω is allowed to vary, $(d/d\widehat{T}_0)$DTR is between about 0.1 and 0.3. In that range, then, our model DTR is quite insensitive to changes in the emissivity, but somewhat sensitive to changes in the solar constant.

Since the model's DTRs don't fall significantly, what is causing the observed decrease on the real Earth? Here are two possibilities (for these and others, see Beardsley [1992], Cao et al. [1992], Easterling et al. [1997], and Karl et al. [1993]):

- Increases in sulfate aerosols—microscopic particles produced by burning of fossil fuels—could be increasing the amount of sunlight reflected back into space during the day, thus slowing down the warming produced by the increase in greenhouse gases. No such slowing would occur at night.
- The water vapor feedback effect (**Section 8**) could be increasing the concentration of atmospheric water vapor. Since water vapor absorbs some of the shortwave radiation it intercepts, the atmosphere would become more efficient than before at absorbing the solar energy during the day, thus denying some of that energy to the surface. Again, the greenhouse-induced rise in surface temperature would be slowed down during the day only.

Our model's insensitivity to changes in atmospheric emissivity is shared by a more sophisticated model analyzed by Cao et al. [1992, 926–927], who performed computer experiments on a one-dimensional radiative model (i.e., a model that, like ours, considers only radiative, not mechanical, transfer of energy, but which allows temperature to depend on altitude and takes into account wavelength-dependent properties of gases' absorption and emission). They found that when atmospheric carbon dioxide is doubled, the DTR is reduced by 0.05 K. That reduction, though small, is larger than what we found, probably because their model took into account absorption of solar radiation by carbon dioxide. When they added a water vapor feedback to their model, the DTR decreased by 0.4 K. They also performed experiments on a global climate model and found increases in some regions and decreases in others, with a slight global mean decrease that they attributed to the water vapor feedback and related processes [Cao et al. 1992, 929].

Exercises

37. Verify that when $T_1(t) = \widehat{T}_1$ for all t, the linearized version of **(49)** can be put in the form $du/ds + \omega u = B\psi(s)$, where **(51)** holds.

38. Suppose that $\epsilon + q > 1$, let the function f be defined by **(52)**, and let F be defined following **(53)**.

a) Prove **(54ac)**.

b) Use a computer or graphing calculator to confirm **(54b)**.

39. **a)** Use **(56)** to verify that $0.2 < \omega < 0.7$, approximately, when $7 \leq \text{DTR} \leq 25$. (That's roughly the range of values for the DTR in the passage from Wallace and Hobbs in the text of this section.)

b) Calculate the corresponding values of the surface heat capacity c_0. Are they consistent with **(31)**?

40. In formulas **(55)** and **(57)**, $(d/d\widehat{T}_0)\text{DTR} = 0$ when $q = 0$. Give a physical explanation of why this happens.

41. Relative changes in the DTR are probably more important than absolute changes; Martians, with their 140 K DTR would find a one-degree change in the DTR less disconcerting than we Earthlings would with our 10 K DTR. Explore the size of the relative rate of change by plotting $\text{DTR}^{-1} \cdot (d/d\widehat{T}_0)\text{DTR}$ as a function of ω.

42. Let's explore the *annual temperature range (ATR)*: the average difference between highest summer temperatures and the lowest winter temperatures. On this time scale, we can no longer think of a continent as an isolated body, so we have to consider either the Northern or the Southern Hemisphere as our "planet". The surface is then dominated by the oceans. It's no longer appropriate to assume that the atmosphere has infinite heat capacity; in fact, the oceans' mixed layer (see **Section 2**), has heat capacity perhaps 20 times as large as that of the atmosphere (**Exercise 9**), and the annual variation in temperatures of the atmosphere and the Earth's surface are roughly comparable, about 5–10 K [Grotjahn 1993, 63, Figure 3.14]. (This variation is rather small because of the oceans' large heat capacity. Land regions experience much larger variation.) So we'll take the atmosphere's heat capacity to be *zero*.

a) Taking $\gamma = 0$ and $c_1 = 0$, solve **(50)** for $\epsilon\sigma T_1^4$. Substitute the result into **(49)** and linearize the resulting first-order differential equation in T_0 about $\widehat{T}_0$. Show that this leads to an equation of the form **(21)**, where $u = T_0 - \widehat{T}_0$, $s = 2t/P$, $\omega = P(2-\epsilon)\sigma\widehat{T}_0^3/c_0$, and $B = P(1+q)\sigma T_e^4/4c_0$.

b) Let $\psi(s) = H \sin \pi s$ and show that the unique periodic solution of **(21)** is $u(s) = BH\left(\omega^2 + \pi^2\right)^{-1/2} \sin \pi(s - \phi)$, where $\phi = \pi^{-1}\tan^{-1}(\pi/\omega)$. Deduce that the annual temperature range is

$$\text{ATR} = \frac{2BH}{\omega\left[1 + \left(\dfrac{\pi}{\omega}\right)^2\right]^{1/2}}.$$

c) Show that $\text{ATR} = \frac{1}{2}\widehat{T}_0 H\left(1 + (\pi/\omega)^2\right)^{-1/2}$.

d) Show that

$$\frac{d}{d\widehat{T}_0}\,\mathrm{ATR}\bigg|_{\Omega\text{ varying, }\epsilon\text{ fixed}} = \frac{H\left[1+4\left(\frac{\pi}{\omega}\right)^2\right]}{2\left[1+\left(\frac{\pi}{\omega}\right)^2\right]^{3/2}}.$$

e) For the Earth, $H \approx 0.4$. Determine a reasonable range of values of ω for the Earth and find

$$\frac{d}{d\widehat{T}_0}\,\mathrm{ATR}\bigg|_{\Omega\text{ varying, }\epsilon\text{ fixed}}$$

for those values.

f) In the model, how much time passes between the moment of maximum influx of solar radiation and the moment of highest surface temperature? (On the real Earth, the lag is about 6 weeks [Wallace and Hobbs 1977, 347].)

43. Let's revisit the annual temperature range (see **Exercise 42**). By solving linearized versions of **(49–50)**, with c_0 and c_1 both positive and with $\psi(s) = H\sin\pi s$, it can be shown that the annual temperature range is

$$\mathrm{ATR} = 2\sqrt{\frac{Q_1^2+Q_2^2}{L^2+M^2}}$$

where

$$\begin{aligned}
L &= k_1k_4 - k_2k_3 - \pi^2, & M &= \pi(k_1+k_4),\\
Q_1 &= H(k_2B_1 + k_4B_0), & Q_2 &= \pi H B_0,\\
B_0 &= \frac{Pq\sigma T_e^4}{2c_0}, & B_1 &= \frac{P(1-q)\sigma T_e^4}{2c_1},\\
k_1 &= \frac{2P\sigma\widehat{T}_0^3}{c_0}, & k_2 &= \frac{2P\epsilon\sigma\widehat{T}_1^3}{c_0},\\
k_3 &= \frac{2P\epsilon\sigma\widehat{T}_0^3}{c_1}, & k_4 &= \frac{4P\epsilon\sigma\widehat{T}_1^3}{c_1}.
\end{aligned}$$

Substitute the usual Earthly values of ϵ, q, T_e, $\widehat{T}_0$, and $\widehat{T}_1$ into these formulas, along with $H = 0.4$ and $P = 1$ year $= 365\times 24\times 60\times 60$ s $= 3.15\times 10^7$ s, and use a computer to plot the ATR as a function of c_0 and c_1. Determine ranges of values of c_0 and c_1 for which the ATR has reasonable values, about 5–10 K. For those values, make plots to estimate the rate of change of the ATR in the two cases "ϵ varies, Ω stays constant" and "Ω varies, ϵ stays constant." One way to estimate the rate of change in the first case is to recompute the

ATR with a new value of ϵ and corresponding new values of $\widehat{T}_0$ and $\widehat{T}_1$. The difference between the ATRs with the new and old values, divided by the difference in the new and old surface temperatures, is an approximation to $(d/d\widehat{T}_0)$ATR. You might consider repeating the exercise with $P = 1$ day to get a feeling for whether our simplifying assumption that the atmosphere has infinite heat capacity was realistic.

Appendix I. Kirchoff's Law

In **Section 8**, I stated that, as a crude approximation, the atmosphere's absorptivity A equals its emissivity ϵ. To see why the approximation is plausible, we need the *Planck function*,

$$B(\lambda, T) = \frac{c_1}{\lambda^5 \left(e^{c_2/\lambda T} - 1\right)},$$

where c_1 and c_2 are positive constants [Wallace and Hobbs 1977, 287]. The flux emitted by a blackbody at temperature T in the wavelength range $\lambda_1 < \lambda < \lambda_2$ is gotten by integrating the Planck function from λ_1 to λ_2. In particular, it can be shown that $\int_0^\infty B(\lambda, T)\, d\lambda = \sigma T^4$. From **Figure A.1**, it appears that most of the emission at the Earth's and its atmosphere's temperatures is in the range $\lambda_L \le \lambda \le \lambda_U$, where (rather arbitrarily) $\lambda_L = 7 \times 10^{-6}$ and $\lambda_U = 17 \times 10^{-6}$, and it appears that for λ in that range, $B(\lambda, 290) \approx 2B(\lambda, 250)$.

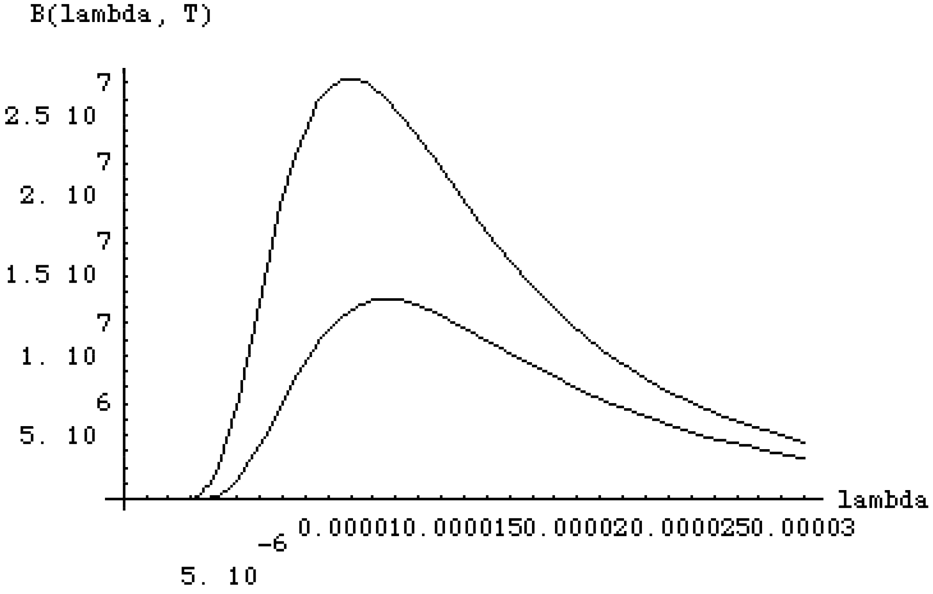


Figure A.1. Planck function for $T = 290$ K (upper curve) and $T = 250$ K (lower curve).

Denote by $\epsilon(\lambda)$, where $0 \le \epsilon(\lambda) \le 1$, the model atmosphere's efficiency at emitting radiation at wavelength λ; that is, for $\Delta\lambda$ small, the atmosphere emits a flux in the wavelength range $[\lambda, \lambda+\Delta\lambda]$ approximately equal to $\epsilon(\lambda)B(\lambda, T_1)\Delta\lambda$. Then the total flux emitted by the atmosphere over all wavelengths is

$$\int_0^\infty \epsilon(\lambda)B(\lambda, 250)\; d\lambda \approx \int_{\lambda_L}^{\lambda_U} \epsilon(\lambda)B(\lambda, 250)\; d\lambda.$$

Similarly, let $A(\lambda)$, where $0 \le A(\lambda) \le 1$, denote the model atmosphere's efficiency at absorbing surface-emitted flux of wavelength λ. The surface emits like a blackbody in the range $\lambda_L \le \lambda \le \lambda_U$, where it does most of its emitting, so the total flux from the surface absorbed by the atmosphere is approximately $\int_{\lambda_L}^{\lambda_U} \epsilon(\lambda)B(\lambda, 290)\; d\lambda$.

The overall efficiency ϵ of the atmosphere at emitting radiation (the emissivity) is the ratio of the flux it actually emits to the flux that a blackbody at the same temperature would emit. Using Kirchoff's law, we therefore have

$$\epsilon = \frac{\int_0^\infty \epsilon(\lambda)B(\lambda, 250)\; d\lambda}{\sigma \cdot 250^4} \approx \frac{1}{2}\left(\frac{290}{250}\right)^4 \frac{\int_{\lambda_L}^{\lambda_U} \epsilon(\lambda)B(\lambda, 290)\; d\lambda}{\sigma \cdot 290^4} \approx 0.9A,$$

i.e., $A \approx \epsilon$ as claimed!

The approximation is pretty rough—so is the reasoning—but probably no worse than lumping together all parts of the atmosphere and pretending they're at the same temperature, or than assuming that the surface is a blackbody. (Approximate emissivities for various components of the surface are [Peixoto and Oort 1992, 105]: .82 for land, .98 for vegetation, and .96 for water, none as close to 1 as might be hoped.)

Appendix II. Physical Constants and Conversions

From Resnick and Halliday [1977]:

1 m = 1 meter ≈ 39.4 in = 3.28 ft
1 km = 1000 m ≈ 0.6214 mi
1 kg = 1 kilogram = 1000 g ≈ 2.21 lb (mass)

Density of dry air at 20° C and 14.70 lb / in^2 (i.e., at roughly average atmospheric conditions near the Earth's surface): 1.29 kg / m^3

Density of water: 1.00×10^3 kg / m^3

$T_F = 32 + 9T_C/5$ and $T_C = T - 273.15$,
where T_F, T_C, and T are temperatures in Fahrenheit, Celsius, and Kelvin

From Harte [1988, 229–263]:

Specific heat of water: 4,184 J / kg K
Specific heat of air at constant volume: 719.6 J / kg K
Specific heat of air at constant pressure: 1,004.2 J / kgK
World energy consumption circa 1980: 10^{13} W
Mass of Earth's atmosphere: 5.14×10^{18} kg
Earth's surface area: 5.10×10^{14} m^2

From Houghton [1986, 2]:

Earth's global albedo: .3
Mars's global albedo: .15
Venus's global albedo: .77

From Goody and Walker [1972, 47]:

Mercury's global albedo: .058

From Beatty and Chaikin [1990, 289]:

Mean distance from the Earth to the Sun: 149.60×10^6 km
Mean distance from Venus to the Sun: 108.20×10^6 km
Mean distance from Mars to the Sun: 227.94×10^6 km
Mean distance from Mercury to the Sun: 57.91×10^6 km

Appendix III. Glossary

Absorptivity Fraction of incident energy flux that is absorbed.

ATR (annual temperature range) The difference between highest summer temperature and lowest winter temperature.

Albedo Fraction not absorbed of sunlight incident on a planet.

Blackbody A hypothetical object, at uniform temperature, that absorbs all the radiation that it intercepts (hence reflects none) in all wavelengths, and that emits radiation at the maximum possible rate in each wavelength. Such an object emits flux from its boundary surface at a rate given by the Stefan–Boltzmann Law.

Co-albedo Fraction absorbed of sunlight incident on a surface.

e**-folding time** Time that it takes for a quantity to diminish by a factor of e.

Effective temperature The temperature that an object would have if if it were at uniform temperature, in radiative equilibrium (i.e., the power emitted equaled power absorbed), and emitted power at the rate that a blackbody would at the same temperature.

Emissivity Fraction of incident energy flux that is emitted relative to a blackbody at the same temperature.

Equilibrium temperature Temperature that a body tends toward with increasing time.

Feedback effect Process whereby an initial change in a variable affects the values of other variables in such a way as to induce further changes in the first variable.

Flux Rate of flow of energy.

Greenhouse function The factor by which the presence of an atmosphere increases the surface temperature of a planet.

Greenhouse gas A gas that make the atmosphere more efficient at absorbing and emitting longwave radiation, thereby leading to a higher surface temperature. The principal greenhouse gases, in terms of total effect, are water vapor and carbon dioxide, because such large quantities are present.

Half-life Time that it takes for a quantity to diminish to half its previous size.

Heat capacity Energy required to raise the temperature of an object by one Kelvin.

Kinetic energy The energy associated with an object's motion.

Kirchoff's law The efficiency of a body at absorbing radiation at a given wavelength equals its efficiency at emitting radiation at that wavelength.

Longwave radiation Electromagnetic radiation with wavelength greater than 4×10^{-6} m.

Power Rate at which energy is generated or consumed per unit time.

Radiant energy The energy carried by all wavelengths of the electromagnetic spectrum.

Shortwave radiation Electromagnetic radiation of wavelength smaller than 4×10^{-6} m.

Solar constant For a point at a fixed distance from the Sun, the rate per unit area at which solar energy passes through the point; the farther from the Sun, the smaller the solar constant.

Specific heat Energy required to raise the temperature of a unit mass of a substance by one kelvin.

Stefan–Boltzmann law A blackbody at temperature T emits radiation at rate σT^4, where σ is the Stefan–Boltzmann constant.

Sulfate aerosols Microscopic sulfur-containing particles in the atmosphere, produced by burning of fossil fuels.

Water vapor feedback effect An initial rise in atmospheric carbon dioxide makes the Earth's surface temperature rise, which in turn increases the evaporation rate and thus the amount of water vapor in the atmosphere. But because water vapor is a greenhouse gas, the result is a further rise in surface temperature, hence an even higher evaporation rate, and so forth—a process that results in a higher temperature than the initial rise in carbon dioxide would suggest. (And the effect would work in reverse for an initial dip in atmospheric carbon dioxide.)

Answers and Solutions to Selected Exercises

1. 0° C, 17.8° C, 32.2° C; 273.2 K, 290.9 K, 305.4 K.

2. 18.1 kg.

3. 1290 kg.

4. Denote by H the compressed shell's thickness. The compressed shell's density would be (atmosphere's mass)/ ($H \cdot$ (Earth's surface area)); the compressed shell's density would also be that of water. Equating the two gives 10.1 m.

5. 120 J.

6. Multiply the mass of air in the house (from **Exercise 3**) by the specific heat of air at constant volume (the house is air-tight). This gives the heat capacity of the house (i.e., the amount of energy it takes to raise the temperature by 1 K). Then multiply the heat capacity by the change in temperature:

$$(1290 \text{ kg}) \left(719.6 \, \frac{\text{J}}{\text{kg K}}\right) \left(\frac{4}{1.8} \text{ K}\right) = 2.1 \times 10^6 \text{ J}.$$

7. 591.0 W/m^2.

8. Total power absorbed is 8.16×10^{16} W.

9. 10^7 $\text{J}/\text{m}^2\text{K}$; the atmosphere's heat capacity is one-twentieth that of the mixed layer.

11. 9.

12. The temperature is $(a\Omega/2\sigma)^{1/4}$, assuming that the dark hemisphere is too cold to emit radiation, that the sunlit hemisphere is in radiative equilibrium, and that temperature, absorption, and emission are uniform over the sunlit hemisphere.

13. $\sigma T^4 = (5.67 \times 10^{-8}\ \mathrm{W/m^2K^4})(290\ \mathrm{K})^4 = 401\ \mathrm{W/m^2}$.

16. 1.09 K.

29. **a)** Yes, the DTR is a decreasing function of ω.

b) $\omega = 4$; about 2:30 P.M.

c) $25{,}000\ \mathrm{J/m^2K}$.

d) About 1.25 K.

30. Look at **Figure 8**.

32. **c)** Temperature rise is 0.8 K.

34. **a)** $2^{1/4}T_e$.

35. 294 K for a five-fold increase in CO_2; model isn't applicable to a ten-fold increase.

39. **b)** Yes.

42. **f)** 12 weeks.

References

Baliunas, Sallie, and Willie Soon. 1996. The sun–climate connection. *Sky and Telescope* 92 (12) (December): 38–41.

Beardsley, Tim. 1992. Night heat. *Scientific American* 266 (2) (February): 21–24.

Beatty, J. Kelly, and Andrew Chaikin, eds. 1990. *The New Solar System*. 3rd ed. Cambridge, MA: Sky Publishing.

Cao, Hong Xing, et al. 1992. Simulated diurnal range and variability of surface temperature in a global climate model for present and doubled climates. *Journal of Climate* 5: 920–943.

Easterling, David R., et. al. 1997. Maximum and minimum temperature trends for the globe. *Science* 277: 364–367.

Few, Arthur. 1996. *System Behavior and System Modeling*. Sausalito, CA: University Science Books.

Feynman, Richard P., et al. 1963. *The Feynman Lectures on Physics*. Vol. I. Reading, MA: Addison-Wesley.

Goody, Richard M., and James C.G. Walker. 1972. *Atmospheres.* Englewood Cliffs, NJ: Prentice-Hall.

Greenler, Robert. 1980. *Rainbows, Halos, and Glories.* New York: Cambridge University Press.

Grotjahn, Richard. 1993. *Global Atmospheric Circulations: Observations and Theories.* New York: Oxford University Press.

Harte, John. 1988. *Consider a Spherical Cow: A Course in Environmental Problem Solving.* Sausalito, CA: University Science Books. 1994. Mill Valley, California: University Science Books.

Henderson-Sellers, Ann. 1983. *The Origin and Evolution of Planetary Atmospheres.* Bristol, UK: Adam Hilger Ltd.

Holton, James R. 1992. *An Introduction to Dynamic Meteorology.* 3rd ed. New York: Academic Press.

Houghton, John T. 1986. *The Physics of Atmospheres.* 2nd ed. New York: Cambridge University Press.

________, et al., eds. 1996. *Climate Change 1995: The Science of Climate Change.* New York: Cambridge University Press.

Karl, Thomas R., et al. 1993. Asymmetric trends of daily maximum and minimum temperature. *Bulletin of the American Meteorological Society* 74: 1007–1023.

Lang, Kenneth R. 1992. *Astrophysical Data: Planets and Stars.* New York: Springer Verlag.

de Mestre, Neville. 1990. *The Mathematics of Projectiles in Sport.* New York: Cambridge University Press.

North, Gerald R. et al. 1981. Energy Balance Climate Models. *Review of Geophysics and Space Physics* 19: 91–121.

Peixoto, José P., and Abraham H. Oort. 1992. *Physics of Climate.* New York: American Institute of Physics.

Resnick, Robert, and David Halliday. 1977. *Physics. Part I.* 3rd ed. New York: John Wiley and Sons.

Schneider, Stephen. 1987. Climate modeling. *Scientific American* 256 (5) (May): 72–80.

________, ed. 1996. *Encyclopedia of Weather and Climate.* New York: Oxford University Press.

Wallace, John M., and Peter V. Hobbs. 1977. *Atmospheric Science: An Introductory Survey.* New York: Academic Press.

Acknowledgments

Much of the work for this article was done while I was a Visiting Fellow at the Cooperative Institute for Environmental Sciences (CIRES) during Summer 1996. I am grateful to Howard Hanson, who suggested the project, answered my questions, and commented on drafts. I am also grateful to the students in my Mathematical Models class in Fall 1996, who cheerfully put up with an early version of this presentation—in particular, to Mohammed Anjarwala and Tamás Simon, who went on to do an independent study project with me on related material in Spring 1997. Finally, I thank my student Karen Weaver, who in Fall 1998 uncovered some errors that have been corrected in this edition.

About the Author

Robert M. Gethner did his Ph.D. work at the University of Wisconsin–Madison, with a thesis in complex analysis, under the direction of Simon Hellerstein. He is an associate professor of mathematics at Franklin & Marshall College.