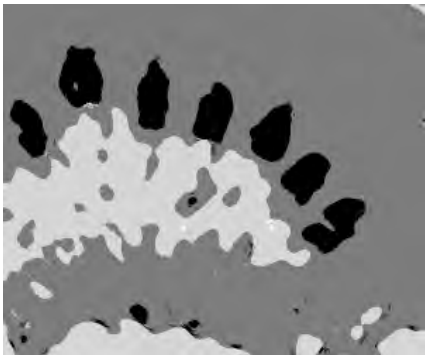


INTERDISCIPLINARY LIVELY APPLICATIONS PROJECT

AUTHORS:
James Dzwonchyk
Gerald Kobylski
Thomas Messervey
(Mathematical Sciences)

U.S. Military Academy
West Point, NY 10996

CONTENTS

1. Setting the Scene
2. Modeling the Problem
3. Solving the Problem
4. Requirements
5. A Write-Up Template
References
Sample Solution
About the Authors

Bridge Analysis

MATHEMATICS CLASSIFICATIONS:
Vector Analysis, Linear Algebra

DISCIPLINARY CLASSIFICATIONS:
Civil Engineering (Statics)

PREREQUISITE SKILLS:
Equilibrium, systems of equations

PHYSICAL CONCEPTS EXAMINED:
Forces, load transfer, structures

COMPUTING REQUIREMENTS:
Solving systems of linear equations

The UMAP Journal 28 (1) 27–40.

Contents

1. Setting the Scene
2. Modeling the Problem
3. Solving the Problem
4. Requirements
5. A Write-Up Template
References
Sample Solution
About the Authors

1. Setting the Scene

Solving systems of equations has applications in a multitude of disciplines. Once such application is in civil engineering. Although humor has been added to the scenario below, problems similar to this one are encountered by civil engineers on a daily basis.

While moving supplies for the upcoming Go Army Math tailgate party, a certain mathematics professor (for anonymity's sake, we will call him Dr. Jones) has precariously overloaded his cargo vehicle. While en route to the tailgate party, located somewhere behind the West Point ski slope, he comes upon a deteriorating old truss bridge that is not classified according to its maximum load capacity. Noticing the shark-infested waters below the bridge, he wonders if he should be concerned about the overloaded truck, which weighs about 10 tons (20,000 lb). He pulls out his cell phone and gives you a call for a quick analysis and recommendation. **Figure 1** depicts the precise location that causes the most internal force and stress on the truss structure.

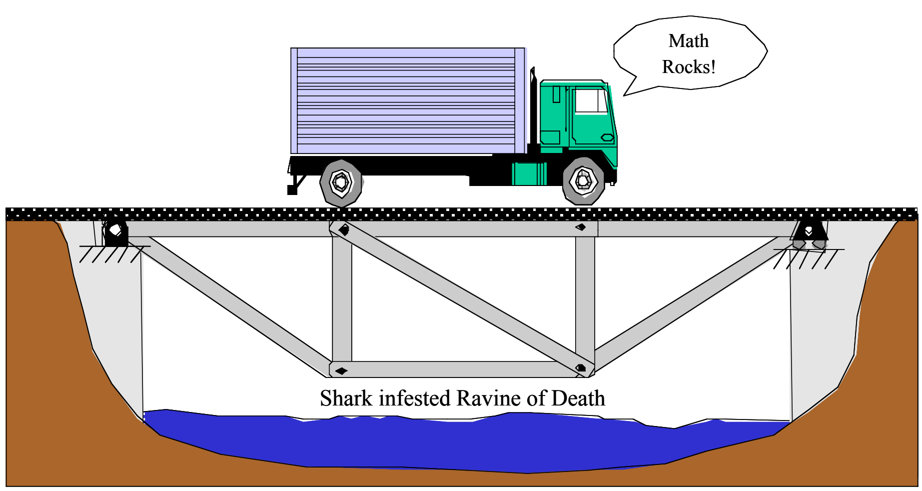


Figure 1. A precarious situation.

2. Modeling the Problem

Aside from being incredibly cool and esthetically pleasing, trusses are amazingly strong because they are formed by interlocking pin-connected triangles. As such, the truss members are never required to bend (deforming into a smile or frown shape) but instead stretch (tension) or squish (compression).

Two identical trusses symmetrically support the bridge deck, one on each side of the bridge (in **Figure 1**, the second lies unseen behind the first). Half of the weight of the truck goes to each truss. Of the 20,000 lb weight of the truck plus cargo, 10,000 lb is assigned to each truss; of that, assuming that the front of the truck is heavier, 5720 lb to each front tire and 4280 lb is assigned to each rear tire, as shown in **Figure 2**. At joint A, the bridge is pinned such that it cannot fall into the gap when someone applies his/her brakes. At joint D, the bridge is free to roll so that it can expand on a hot day. With some trigonometry, you can determine the lengths of the diagonals.

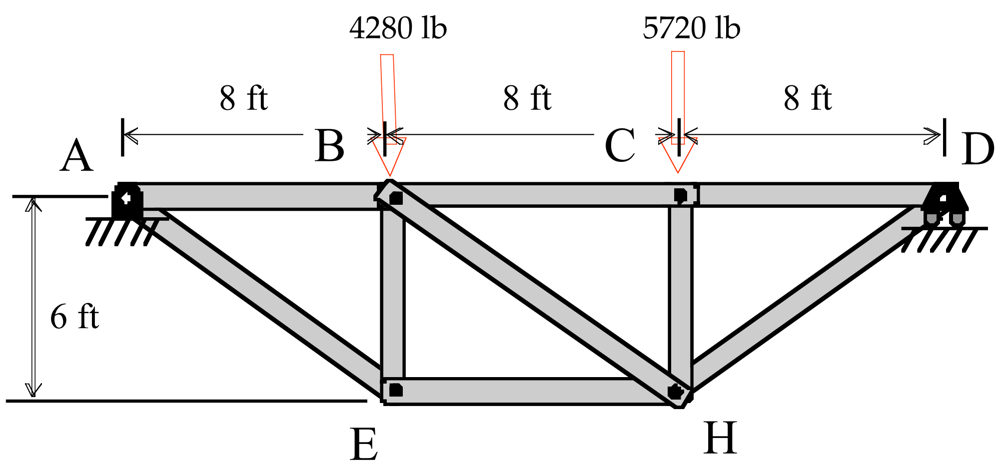


Figure 2. Schematic of one of the trusses.

3. Solving the Problem

To determine if the bridge is going to fail, you will use the *method of joints*. You will assume that if any one member of the bridge truss fails, the entire bridge fails. The method of joints enforces the condition of equilibrium at each joint in the truss. Equilibrium at each joint demands that all internal and external forces "balance" each other or else the truss would move. The method of joints was developed by Squire Whipple (1804–1888), one of America's most prominent pre-Civil War bridge engineers. You can see photos of some of the seven remaining famous Whipple truss bridges at American Society of Civil Engineers [n.d.], Historic Bridge Foundation [n.d.], and Whipple Website [n.d.]. An unattributed sketch of him appears at Janberg [2007], and a brief summary of contributions is at Whipple [1998].

In this problem, there are three unknown external forces A_x, A_y, and D_y, as shown in **Figure 3**. These forces occur at the points where the truss "touches" the ground. These forces, together with the 4280 lb and 5270 lb loads, are external forces that induce internal forces within the truss members. The internal forces within each member of the truss can be determined by using the method of joints sequentially at each joint. **Figure 3** shows the method of joints applied at each joint. Notice that each member has been "cut." The arrows indicate assumed directions for the internal forces. All forces where members have been "cut" are shown in tension, that is, the force is pulling on the members. To transfer the weight of the truck from the middle of the bridge to the supports at the ends, some members must be in compression and some must be in tension. Therefore, a resulting negative force indicates that the initial assumption of tension was incorrect and that that particular member is in fact in compression. In **Figure 3**, the arrows for members with negative forces would point inwards (getting squished).

For the truss to satisfy equilibrium, at each joint the sum of the forces in the x-direction must equal 0 and the sum of the forces in the y-direction must equal 0. In mathematical terms: $\sum F_X = 0$ and $\sum F_y = 0$. The 4280 lbs at joint B and the 5720 lbs at joint C indicate that the truck weight is applied directly at the joint. Let's take a look at how one would analyze one of the joints utilizing the method of joints.

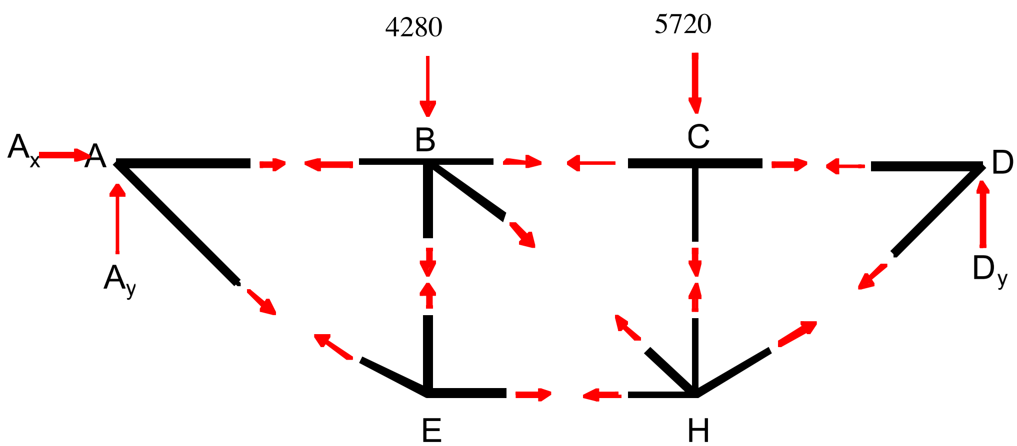


Figure 3. An application of the method of joints.

Joint A

The truss members are labeled using the letters on the truss. As can be seen in **Figure 3**, there are four forces at joint A, A_x, A_y, F_{AB}, and F_{AE}:

- A_x can be described as a force on joint A acting in the x-direction. item A_y can be described as a force on joint A acting in the y-direction.
- F_{AB} is described as the force from joint A towards joint B.

- F_{AE} is described as the force from joint A towards joint E.

Note that $F_{AE} = F_{EA}$. Also note that when analyzing joint A, a portion of F_{AE} is acting in the x-direction while the remainder is acting in the y-direction. When summing the forces in the x- and y-directions, you must take this into account. How to do this is explained in the following paragraphs.

To determine what portion of the force is acting in the in the x- and y-directions, you must resolve the force into its x- and y-components. When you resolve a force, you are determining what portion of the force is acting in the x-direction and what portion is acting in the y-direction.

Figure 4 is a "blow-up" of F_{AE} and demonstrates how we resolve this force. The magnitude of F_{AE} can be considered as the hypotenuse of a right triangle (the one shown on the left); it can be can be resolved into the base and height of this right triangle. The length of the base is the magnitude of the x-component of F_{AE} and the height is the magnitude of the y-component of F_{AE}.

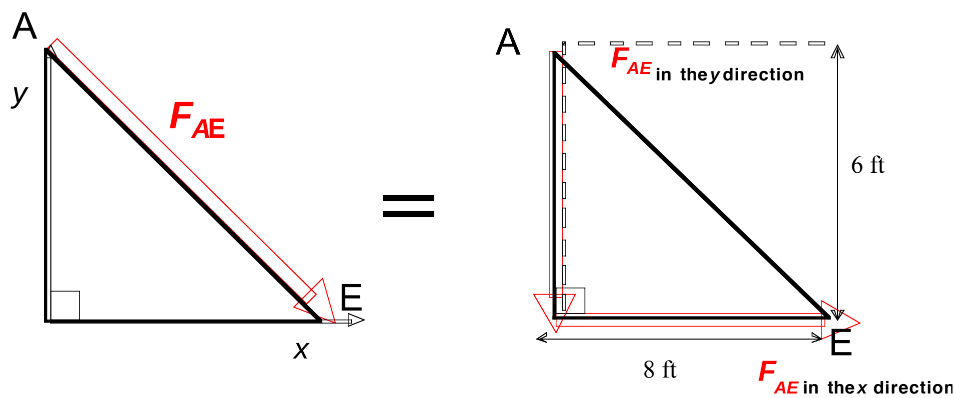


Figure 4. F_{AE} represented as the hypotenuse of a right triangle.

The right triangle on the left has a height and a base of 6 ft and 8 ft, respectively, lengths that came from **Figure 2**. Using the Pythagorean Theorem, the hypotenuse can be determined to be 10 ft. Using similar triangles, you can compare the $y/x/F_{AE}$ triangle on the left with the 6/8/10 triangle on the right. In **Figure 4**, x and y are the components of F_{AE} and are also the base and height of the right triangle, respectively. The y-component of F_{AE} (or the height of the triangle) can then be determined using the relationship

$$\frac{6}{10} = \frac{y}{F_{AE}} \quad \text{so that} \quad y = \frac{6}{10} F_{AE}.$$

Similarly, the magnitude of F_{AE} in the x-direction can be determined to be:

$$\frac{8}{10} F_{AE} \quad \text{so that} \quad x = \frac{8}{10} F_{AE}.$$

Up to this point, you have accounted for the magnitude of F_{AE} in the x- and y-directions. Since forces also contain direction, you must now account for the direction of F_{AE}. To do this, a consistent sign convention must be assigned to identify which direction is positive and which direction is negative. The usual convention is that the x-component is positive if it is pointing to the right and the y-component is positive if it is pointing up.

You are now ready to begin summing forces at joint A. Recall that for this joint to be in equilibrium, the sum of the forces in the x- and y-directions must be zero. The following equation sums the forces in the y-direction at joint A:

$$A_y - \frac{6}{10} F_{AE} = 0.$$

Next, you should sum the forces at this joint in the x-direction. By repeating this process at each of the other joints, you can establish a system of equations to determine the unknown forces within each of the members. A unique solution exists for this system (because we have the same number of equations as unknowns). This sounds like a great opportunity to set up a matrix to help solve the system of equations!

4. Requirements

1. Determine all of the internal and external forces acting on the truss.[1]
2. To evaluate the safety of the truss, you will need to determine how much force the individual members can withstand without failing. You check with your squad leader, a civil engineering major, who does some quick calculations and reports that each member can carry approximately 9000 lb in tension and 6500 lb in compression.[2] Determine if the burgers make it to the tailgate party.
3. Write up your analysis and findings in a technical report. A write-up template is provided for you below. In the analysis portion, your writing should explain how you went about your analysis so that the reader knows exactly how you obtained your results.

5. A Write-up Template

Since for many of you this may be your first technical report, we have given you the template below to assist you in this endeavor. The comments that are in square brackets are specific instructions for your writing. The other text contains points that you should include in your report.

[1] If you would like some supplemental on-line instruction in truss analysis, navigate to a USMA website for CE-300 at `http://www.west-point.org/academy/ce300/lesson_6.htm` and run the Flash "e-Lecture." If you dont have Flash installed on your computer, you can run the "Self-Executing e-Lecture" instead.

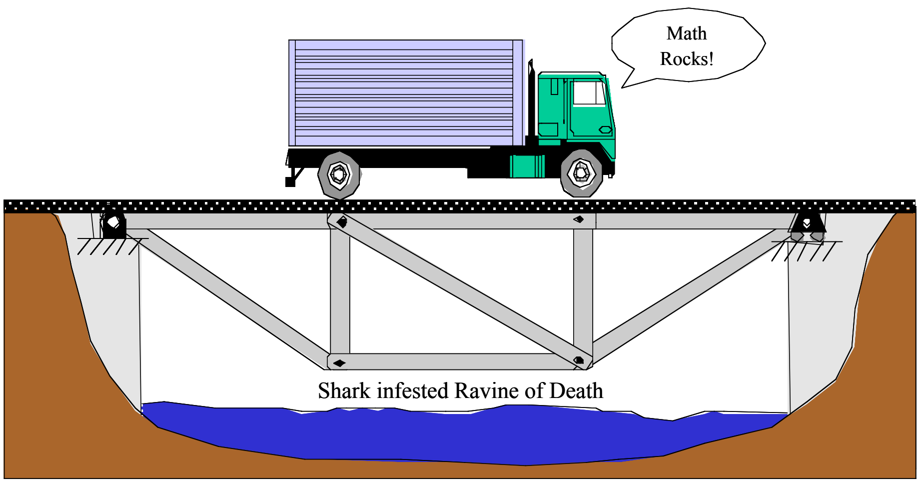
[2] The force that a long slender member can withstand under compression is usually much less than the load that it can withstand under tension.

Title

[A title page should be the first thing that your instructor sees; it should contain a title, the names of your team members, and the date.]

Background

[This section should be a brief introduction to the problem being analyzed. In two or three sentences, you should paraphrase the background that you were given on the assignment sheet in order to bring your reader into the loop. Be sure to write this in your own words. You may reproduce **Figure 1** if you like. Number and label all figures and tables (as we have in this ILAP Module) and attribute to their creators any figures that you did not create yourself.]



Template Figure 1. Situation of the problem statement [figure reproduced from problem statement].

Facts

Figure 1 shows a representation of one of the trusses of the bridge. We know the following:

- The truck and cargo weigh 10 tons (20,000 lb).
- There are two identical trusses symmetrically supporting the deck that the truck drives on.
- The bridge is pinned at joint A and free to roll at joint D.
- If at least one member of the truss exceeds its maximum load capacity, then the bridge will fail (causing the truck, the burgers, and our esteemed math professor to plunge to their doom in the shark-infested Ravine of Death.)
- Each horizontal member is 8 ft long. Each vertical member is 6 ft long.
- Each diagonal member is [insert value and units here].

- The method of joints can be used to determine the external and internal forces acting on a truss.
- The greatest force and stress occur if each tire of the truck sits directly above a center joint, e.g., joints B and C.

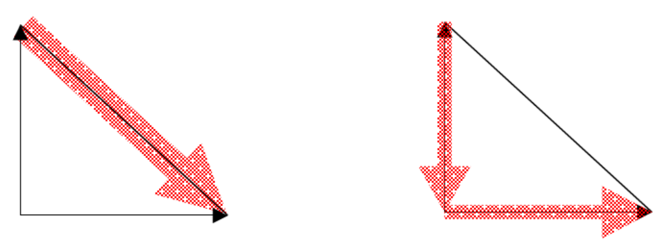
Assumptions

We make the following assumptions for our analysis:

- The maximum load capacity of each member is [insert value here] pounds of tension and [insert value here] pounds of compression.
- The only forces acting on the bridge are those caused by the truck.
- The truck's weight is uniformly distributed over its width. Therefore, the two rear tires will exert the same force. Similarly, the two front tires will exert the same force.
- We also assume the worst-case scenario, that is, that each tire of the truck is sitting directly above a center joint. In particular, a force of [insert value here] lb is being exerted on joint B and a force of [insert value here] lb is being exerted on joint C.
- [Optional: Any additional assumptions you think are valid and necessary may be included here.]

Analysis

[Thoroughly explain how to resolve the forces on joint B into their x- and y-components. Should you choose to include illustrations for your analysis of this joint, you may draw your own; use a computer tool (e.g., the Microsoft Drawing Tool), use the diagram shown below, and/or neatly sketch them in by hand.]



Template Figure 2. Resolution of forces [figure reproduced from problem statement].

Describe how to set up the equations for the sum of the forces in the x- and y-directions for joint B.

[Use an equation editor to put your sum of forces equations for joint B here.]

[Write out the complete system of equations for the sum of the forces for all the joints. (You do not have to repeat the thorough analysis that you did with joint B.)]

[Put your complete system of equations here.]

[Write down the matrix-vector equation to solve by using Mathematica.]
[Put your matrix-vector equation here.]
[You may copy and paste from Mathematica or use an equation editor.]

[Explain how you use Mathematica to solve your matrix-vector equation.]

Conclusion

[Here is where you should report the internal and external forces acting on the truss under your assumptions. Give the force acting on each member and be sure to note whether that force is in tension or compression.]

Recommendations

[Here is where you report your findings on the safety of the truss and make your recommendations to your esteemed math professor. (And what about the fate of those burgers?) Look back to see if your answer makes sense. You could even make additional recommendations and/or comments if you feel the inclination to do so.]

References

American Society of Civil Engineers. n.d. ASCE History and Heritage of Civil Engineering: Whipple bowstring truss bridge. `http://www.asce.org/history/brdg_whipple.html` .

Historic Bridge Foundation. n.d. Texas bridges in the Hill Country: Faust Street Bridge. `http://www.historicbridgefoundation.com/ipages/texas/hillcountry/comal/faust.html` .

Janberg, Nicolas. 2007. Squire Whipple. `http://en.structurae.de/persons/data/index.cfm?ID=d000317` .

Whipple, Blaine. 1998. America's railroads and skyscrapers indebted to civil engineer Squire Whipple. `http://www.whipple.org/blaine/squire.html`.

Whipple Website. n.d. Whipple truss bridge and plaque, Boonville, N.Y.`http://www.whipple.org/photos/whipplebridge.html` .

6. Sample Solution

Below are the corresponding equations summing forces for each joint:

Joint	Direction	Equation		
A	x	$A_x + F_{AB} + \frac{8}{10} F_{AE}$	$=$	0
	y	$A_y - \frac{6}{10} F_{AE}$	$=$	0
B	x	$- F_{AB} + F_{BC} + \frac{8}{10} F_{BH}$	$=$	0
	y	$- F_{BE} - \frac{6}{10} F_{BH}$	$=$	0
C	x	$- F_{BC} + F_{CD}$	$=$	0
	y	$- F_{CH}$	$=$	5720
D	x	$- F_{CD} - \frac{8}{10} F_{HD}$	$=$	0
	y	$D_y - \frac{6}{10} F_{HD}$	$=$	0
E	x	$\frac{-8}{10} F_{AE} + F_{EH}$	$=$	0
	y	$\frac{6}{10} F_{AE} + F_{BE}$	$=$	0
H	x	$\frac{-8}{10} F_{BH} - F_{EH} + \frac{8}{10} F_{HD}$	$=$	0
	y	$\frac{6}{10} F_{BH} + F_{CH} + \frac{6}{10} F_{HD}$		0

Matrix operations can now be used to solve the above system of equations. **Table S1** contains the coefficients of the 12 variables in each of the 12 equations.

Let the matrix A consist of the coefficients in **Table S1** and let the vector represent the variables. Using technology we can now easily solve the equation $A\vec{x} = \vec{b}$ for the vector $\vec{b}$ of the 12 unknowns, to find the following solution:

Table S1.

Coefficients of variables in the linear system.

	A_x	A_y	D_y	F_{AB}	F_{AE}	F_{BE}	F_{BC}	F_{BH}	F_{EH}	F_{CH}	F_{CD}	F_{HD}
joint A (x-dir)	1	0	0	1	8/10	0	0	0	0	0	0	0
joint A (y-dir)	0	1	0	0	- 6/10	0	0	0	0	0	0	0
joint B (x-dir)	0	0	0	-1	0	0	1	8/10	0	0	0	0
joint B (y-dir)	0	0	0	0	0	-1	0	- 6/10	0	0	0	0
joint C (x-dir)	0	0	0	0	0	0	-1	0	0	0	1	0
joint C (y-dir)	0	0	0	0	0	0	0	0	0	-1	0	0
joint D (x-dir)	0	0	0	0	0	0	0	0	0	0	-1	- 8/10
joint D (y-dir)	0	0	1	0	0	0	0	0	0	0	0	- 6/10
joint E (x-dir)	0	0	0	0	- 8/10	0	0	0	1	0	0	0
joint E (y-dir)	0	0	0	0	6/10	1	0	0	0	0	0	0
joint H (x-dir)	0	0	0	0	0	0	0	- 8/10	-1	0	0	8/10
joint H (y-dir)	0	0	0	0	0	0	0	6/10	0	1	0	6/10

$$
\begin{aligned}
A_x &= 0\\
A_y &= 4760\\
D_y &= 5240\\
F_{AB} &= -6347\\
F_{AE} &= 7933\\
F_{BE} &= -4760\\
F_{BC} &= -6987\\
F_{BH} &= 800\\
F_{EH} &= 6347\\
F_{CH} &= -5720\\
F_{CD} &= -6987\\
F_{HD} &= 8733
\end{aligned}
$$

If the force acting on a member is negative, then our initial assumption of the member being in tension was incorrect, and the member is in compression. The internal and external forces acting on the truss are as follows:

$$
\begin{aligned}
A_x &= 0 & &\\
A_y &= 4760 & &\text{lb tension}\\
D_y &= 5240 & &\text{lb tension}\\
F_{AB} &= 6347 & &\text{lb compression}\\
F_{AE} &= 7933 & &\text{lb tension}\\
F_{BE} &= 4760 & &\text{lb compression}\\
F_{BC} &= 6987 & &\text{lb compression}\\
F_{BH} &= 800 & &\text{lb tension}\\
F_{EH} &= 6347 & &\text{lb tension}\\
F_{CH} &= 5720 & &\text{lb compression}\\
F_{CD} &= 6987 & &\text{lb compression}\\
F_{HD} &= 8733 & &\text{lb tension}
\end{aligned}
$$

Since the forces acting on all members in tension are less than 9000 lb, none of the members fail in tension. However, two of the members, BC and CD, fail in compression, since the forces F_{BC} and F_{CD} are greater than 6500 lb. Thus the bridge fails; and unfortunately, the burgers do not make it to the tailgate.

Figures 5 and **6** depict two different scenarios for a bridge of the same structure but with the weight and the dimensions of the bridge changed.

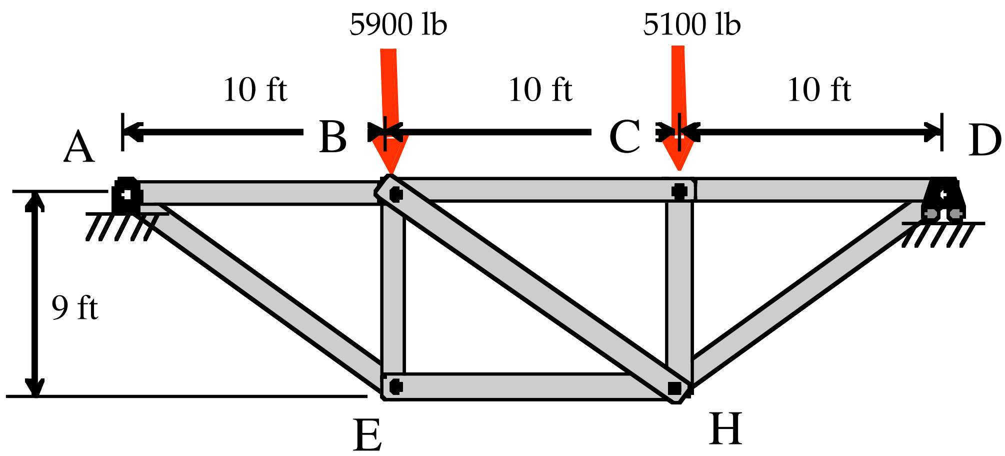


Figure 5. Schematic of a second truss.

The internal and external forces acting on the truss of **Figure 5** are as follows:

$$
\begin{aligned}
A_x &= 0 \\
A_y &= 5633 \text{ lb tension} \\
D_y &= 5367 \text{ lb tension} \\
F_{AB} &= 6222 \text{ lb compression} \\
F_{AE} &= 8408 \text{ lb tension} \\
F_{BE} &= 5633 \text{ lb compression} \\
F_{BC} &= 5927 \text{ lb compression} \\
F_{BH} &= 398 \text{ lb compression} \\
F_{EH} &= 6222 \text{ lb tension} \\
F_{CH} &= 5100 \text{ lb compression} \\
F_{CD} &= 5927 \text{ lb compression} \\
F_{HD} &= 8010 \text{ lb tension}
\end{aligned}
$$

The internal and external forces acting on the truss of **Figure 6** are as follows:

$$
\begin{aligned}
A_x &= 0 \\
A_y &= 5642 \text{ lb tension} \\
D_y &= 5358 \text{ lb tension} \\
F_{AB} &= 6232 \text{ lb compression} \\
F_{AE} &= 8422 \text{ lb tension} \\
F_{BE} &= 5642 \text{ lb compression} \\
F_{BC} &= 5917 \text{ lb compression} \\
F_{BH} &= 409 \text{ lb compression} \\
F_{EH} &= 6232 \text{ lb tension} \\
F_{CH} &= 5100 \text{ lb compression} \\
F_{CD} &= 5917 \text{ lb compression} \\
F_{HD} &= 7996 \text{ lb tension}
\end{aligned}
$$

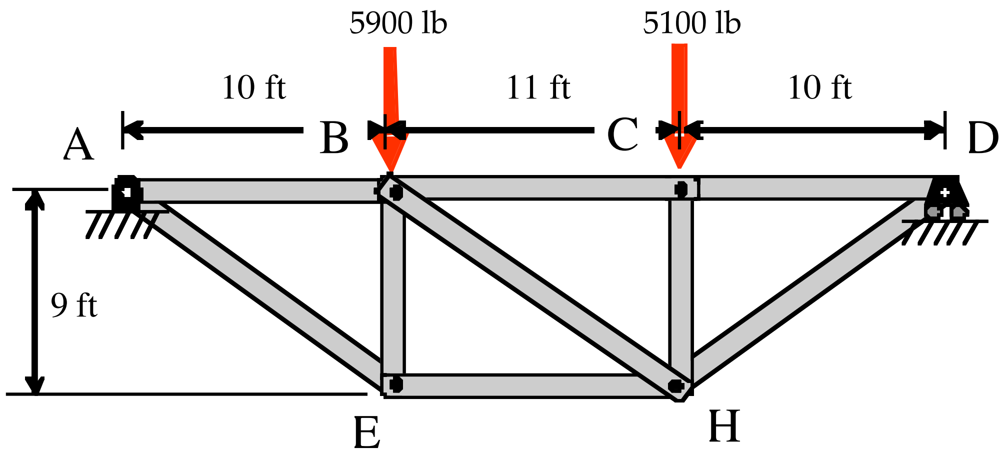
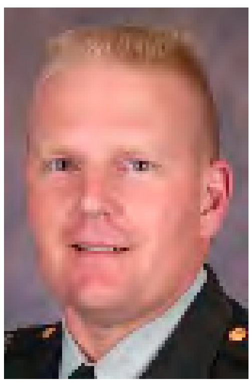
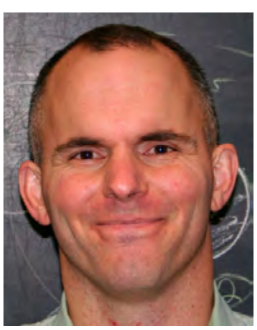


Figure 6. Schematic of a third truss.

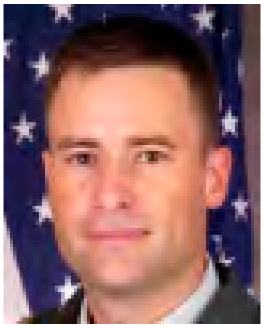
About the Authors



James Dzwonchyk is an Instructor in the Department of Mathematical Sciences at the United States Military Academy. He received his B.S. in Engineering Management from the United States Military Academy in 1993 and an M.S. in Operations Research from the Naval Postgraduate School in 2002.



Gerald Kobylski is an Assistant Professor in the Department of Mathematical Sciences at the United States Military Academy. He received his B.S. in Engineering from the United States Military Academy in 1988. He received an M.B.A. from Western New England College in 1991, an M.S. in Operations Research from the Georgia Institute of Technology in 1997, an M.A. in National Security and Strategic Studies from the Naval War College in 2002, and a Ph.D. from the Stevens Institute of Technology in 2005.



Thomas Messervey is an Assistant Professor in the Department of Mathematical Sciences at the United States Military Academy. He received his B.S. in Civil Engineering from the United States Military Academy in 1994. He received an M.S. in Civil Engineering from Stanford University in 2003 and is licensed as a Professional Engineer in the State of Missouri.