Experiments in Lavatory Dynamics

William Mueller
Dept. of Mathematics
University of Arizona
Tucson, AZ 85721
mueller@math.arizona.edu

Introduction

Until very recently, most flush toilets were controlled by a mechanism whose workings were filled with differential poetry. In the rear tank of the toilet was a bulbous metal float, budding in hues of oxidation at the end of a gently curving metal stem. As the tank filled with water after flushing, the float would gently rise on the water's roiling surface. Through the stem, the rising float would gradually close off the inflow valve on a vertical ball cock connected to the water pipe.

Throughout this process, a high-pitched sound would emanate from the tank. The pitch would rise, and rise, and rise—interminably, it seemed—as if reaching for some unattainable ultimate tone. Bounded, however, by what could never be, the mechanism would finally cross over, in heroic gesture, to an indefinite silence. Contemplation of this tragic aria was the closest that many people would ever come to the sublime asymptotics of the bounded exponential function.

Sadly, these marvelous mechanisms have become almost obsolete. They have been replaced with plastic floats, climbing vertical slides in tedious linear ascents, clicking off matter-of-factly as they reach the top. It is not, however, too late to hear the exponential's sad song. Such mechanisms continue to function in the cold-water apartments, restaurants, and bars in the crumbling neon districts of many cities. I regularly send my beginning differential equations students out searching for them; for lessons in modeling, data analysis, and autonomous differential equations, certainly; but also for a glimpse of the "ancient heavenly connection to the starry dynamo in the machinery of night" [Ginsberg 1956].

My students are instructed as follows. Upon locating such a mechanism, and securing the restroom door behind them, they are to play the role of a team of experimental scientists. First of all, the porcelain cover on the rear tank must

The UMAP Journal 19 (1) (1998) 5–10.

be removed, revealing the magnificent workings within. A yardstick is then submersed into the void. The toilet is flushed, and as one of the team members calls out five-second intervals on their watch—"Now! Now! Now!"—another reads off the corresponding water levels on the yardstick—"5, 6.5, 7.5, 8"—while a third member (perhaps in a lab coat) tabulates the data on a clipboard. Additional team members, should there be any—particularly the large, "less meticulous" types—are usefully posted outside of the door. Anyone wishing to use the restroom, or expressing concern about the cries of "Now! Now! Now!" coming from within, can usually be turned away with a few stern words about a Health Department inspection.

After several trials have been averaged for accuracy, a typical plot of the data looks something like **Figure 1**. This toilet[1] had a float that engaged about 30 seconds after flushing.

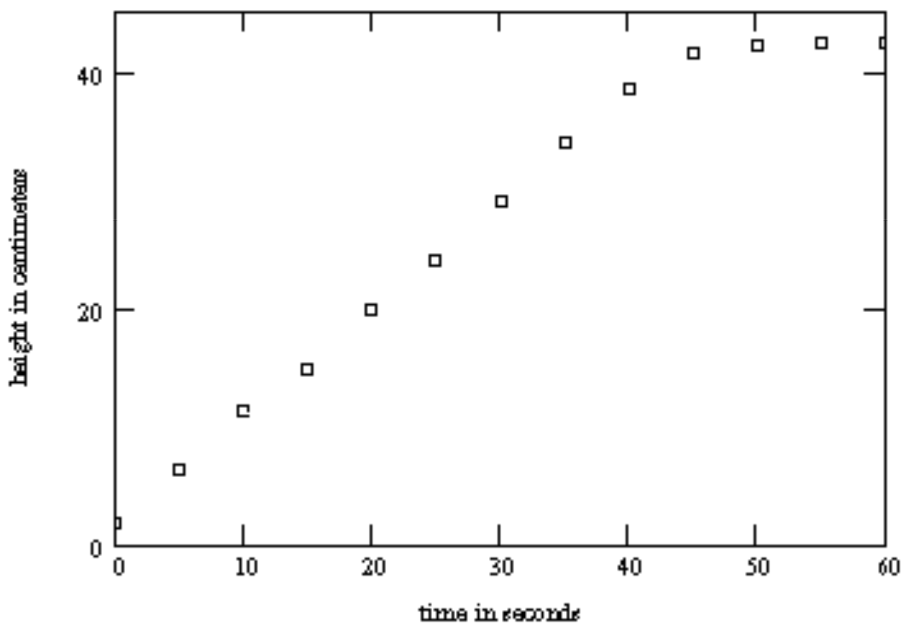

Figure 1. Toilet data.

Upon returning to the well-lit classrooms on campus, I ask my students to switch roles and assume the attitude (pipes instead of lab coats?) of a team of theoretical scientists. Their goal is to construct a model of the toilet tank based entirely on reasonable assumptions about its behavior. They already have the advantage, of course, of having looked inside of the system that they will be modeling—an opportunity not afforded most theoretical scientists—and this makes the exercise a much more accessible introduction to modeling. The height of the water in the tank, $h(t)$, is usually identified as the essential state variable, and students write down something like this without much prompting:

[1]Women's restroom; Shelter Lounge; Tucson, AZ.

$$\frac{dh}{dt} = kQ, \tag{1}$$

where k is a constant and $Q(t)$ is the inflow rate in, e.g., gallons per minute. This is a good point to start asking troublesome questions. What about the shape of the tank: Does it matter if the sides are vertical or slanted ? What about variations in water pressure? How can the model account for these? Should it?

It takes students a bit longer to struggle with the idea that Q will have to be defined piecewise—for the times before and after the float is engaged. A typical model that emerges is:

$$Q(t) = \begin{cases} a, & \text{for } h \leq h_1; \\ b(h_2 - h), & \text{for } h_1 < h < h_2, \end{cases}$$

where h_1 is the height at which the float is engaged, h_2 is the cut-off height, and a and b are constants. Combining with **(1)** leads to the initial value problem:

$$\frac{dh}{dt} = \begin{cases} A, & \text{for } h \leq h_1; \\ B(h_2 - h), & \text{for } h_1 < h < h_2, \end{cases} \tag{2}$$

with $h(0) = h_0$. Students are now asked to investigate their "theoretical toilet." A slope field for the differential equation ((which students must understand as a first step in developing a geometric understanding of problems like this) is shown in **Figure 2**. Here $A = 1$, $B = .05$, and $h_2 = 45$. Students will have to experiment with these values to find a theoretical toilet that looks like an accurate model of their data.

I now nudge the students into a discussion of the apparently infinite family of theoretical toilets that their model seems to have predicted. For different initial water heights $h_0 > 0$, the model predicts, respectively,

- toilets that fill asymptotically toward the cut-off height, starting from empty at some negative time before flushing;
- toilets that never flush but maintain the cut-off height for all time; and
- toilets that drain to the cut-off height from some infinite capacity.

I like to quote the physicist Paul Dirac, who predicted the existence of positrons solely on the evidence of unexplained solutions to the Schrödinger equation. He wrote that "It is more important to have beauty in one's equations than to have them fit experiment" [Rothstein 1995, 151]. This usually brings about tongue-in-cheek predictions of "anti-toilets" in more beautiful unseen worlds. I am, of course, happy to have students thinking about beautiful unseen worlds which only mathematics can reveal; but the lesson is also quite practical: For the rest of the semester, I have little trouble getting them to

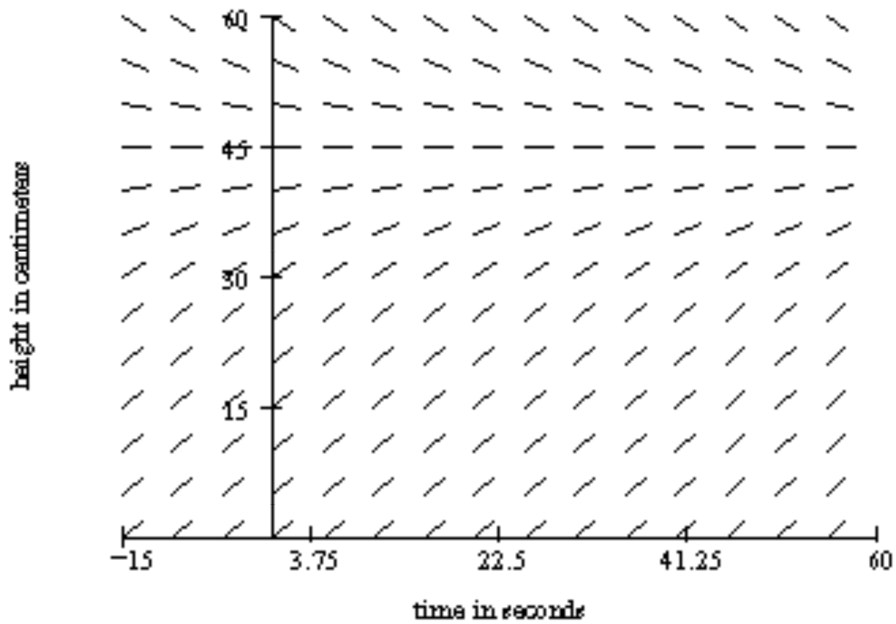

Figure 2. The theoretical toilet.

actively consider the variety of qualitatively different solutions to a differential equation.

The two pieces of the differential equation given in **(2)** are quite separable, and so provide an elementary opportunity for students to compare the relative merits of the holy (as students see it) analytic solution with the numerical and graphical representations that they have obtained earlier. The particular solution of the IVP in **(2)** is:

$$h(t) = \begin{cases} At + h, & \text{for } h \le h_1; \\ h_2 - Ce^{-Bt}, & \text{for } h_1 < h < h_2, \end{cases}$$

where $C = (h_2 - h_1)e^B$ and $h(t_1) = h_1$. The model predicts a linear rise in h, followed by bounded exponential growth after the float is engaged.

Values for A, B, and the cut-off height h_2 can be obtained from the data, and this provides some elementary lessons in data analysis. A linear regression on the data for $h < h_1$ gives a value for the slope A. Another regression, this time on a linearized plot of $\ln(h_2 - h)$ vs. t for the data with $h > h_1$, gives a value for the slope $-B$ (see **Figure 3**).

Picking a value for the cut-off height h_2 is slightly tricky: Using the actual cut-off from the data leads to the unacceptable $\ln(0)$; and the regression is quite sensitive to small perturbations of h_2 above this value. In **Figure 3**, a value of h_2 that is 0.1 cm above the actual cut-off has been chosen. Once all of these values are obtained, a plot of the analytic solution through the data gives a good confirmation of the analysis. Of course, it must be noted here that the theoretical toilet never actually shuts off.

The model can be modified to reflect the discontinuous jump to the cut-off

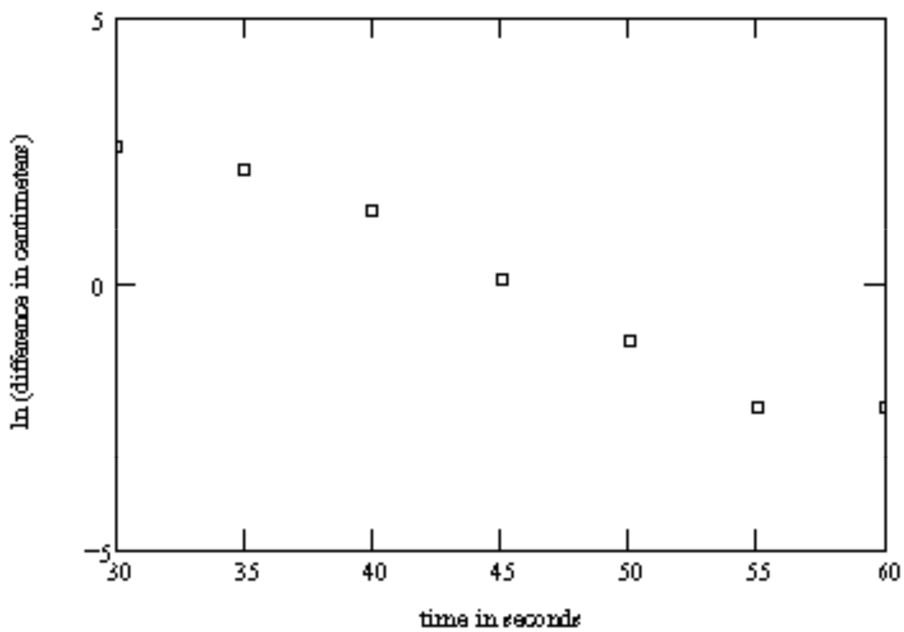

Figure 3. Re-expressed toilet data.

height, simply by adding a third height interval to **(1)**. It is equally instructive, however, to have students discuss the "practical" solution many people apply to an infinitely running toilet: bending the float stem. The effects of this stratagem are summarized in **Figure 4**.

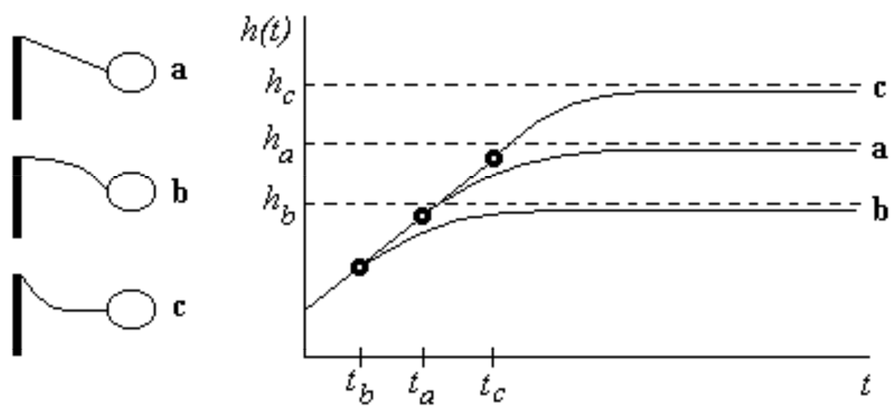

Figure 4. Bending the float stem.

We see that bending the float stem does nothing to quiet the toilet's high-pitched explorations of the asymptote—it runs on just the same. Rather, the effect is to shift the solution, both horizontally and vertically. The only practical benefit comes from the water that would be saved by the lowered cut-off height in *b*.

By means of such a simple experiment and modeling exercise, students are able to carry away memorable lessons that can be usefully referred to throughout the semester. (There are also the less measurable lessons that come from making the "ancient heavenly connection" to the world outside of the classroom. One group of students, for example, received a sumptuous Japanese meal, free of charge—so proud were the proprietors of the restaurant to have "such mathematics" being carried out in their restroom. This cultural respect for mathematics was less evident at a local Denny's, where another group of students was asked to leave.) I have had students tell me, long after completing this project, that they *cannot* go to the bathroom without thinking about differential equations. They are usually glaring at me, and I am usually smiling.

Acknowledgment

This article is reprinted and adapted, with permission of the author and of the editor, from "Hydraulics in an Unlikely Place," *C•ODE•E, Newsletter for the Consortium for Ordinary Differential Equations Experiments* (Spring–Summer 1996): 8–10.

References

Ginsberg, Allen. 1956. *Howl.* New York: City Lights Books.

Rothstein, Edward. 1995. *Emblems of Mind.* New York: Times Books.

About the Author

William Mueller received his bachelor's degree in mathematics from MIT, his master's degree in mathematics from the University of Michigan, and his Ph.D. in mathematics from Duke University. He has a long history of interest in curriculum development and the use of technology in the mathematics classroom. He is currently working on a number of projects to bring interactive mathematics lessons to the Web.