Mathematical Model for a Mission to Mars

Glenn Ledder
Dept. of Mathematics and Statistics
University of Nebraska–Lincoln
Lincoln, NE 68588–0323
gledder@math.unl.edu

Introduction

The United States space agency NASA is planning to send a robot vehicle to Mars. The basic mission will be similar to the moon landings of the 1960s and 1970s: The main spacecraft will orbit the planet, a Mars lander will be sent to the surface to collect samples, the lander (or possibly a smaller return vehicle carried by the lander) will return to the main spacecraft, and the main spacecraft will return to Earth.

We consider a mathematical model inspired by this mission. Suppose that we are asked to design a set of lander vehicles for collecting samples on Mars and other heavenly bodies, such as one of Jupiter's moons. For convenience, we refer to the heavenly body of interest as a "planet."

We consider a simple conceptual model that omits some specific details that are likely to have only a minor impact on the design. We assume that there are no forces other than gravity, that there are no fuel stages that must be shed during the launch, and that the launch is successful if and only if the launch protocol succeeds in giving the vehicle enough momentum to escape the planet's gravity. The goal of our investigation is to advise the engineers designing the return vehicle on some of the design parameters.

We assume that the reader has some background in elementary differential equations; the introduction to the subject that is now common in calculus books should be sufficient. Other needed mathematical ideas, as well as the physics necessary for the derivation of the models, are contained in this article.

The UMAP Journal 23 (4) (2002) 417–428.

Basic Celestial Mechanics

Any investigation of space flight must certainly begin with the two basic principles of celestial mechanics: Newton's Second Law of Motion and the Inverse Square Law of Gravitation.

Newton's Second Law of Motion

Newton's Second Law of Motion is generally considered to be the formula $F = ma$; however, this standard formula is actually a special case of the Second Law that is not always valid for problems of space flight.

What Newton actually said was that *the momentum acquired by an object is equal to the impulse given to it*. Suppose that we apply a constant force F to an object during a time interval Δt. The *impulse* given to the object is defined to be the quantity $F\Delta t$. The *momentum* of the object is defined by the quantity mv, where m is the object's mass and v is its velocity. Thus, the Second Law of Motion, for a constant applied force, is

$$\Delta(mv) = F\Delta t.$$

If the force is changing with time, then this equation applies only in the limit $\Delta t \to 0$. Thus, the formula becomes

$$d(mv) = F\,dt. \quad \textbf{(1)}$$

The familiar result $F = ma$ follows from the additional assumption that the mass is constant, an assumption that is not always true in space flight.

The Inverse Square Law of Gravitation

The Inverse Square Law of Gravitation is used to determine the force exerted on an object by a planet; this is the primary force responsible for momentum changes of objects in space. This law is generally written as

$$F_g = -\frac{GMm}{r^2},$$

where M is the mass of the planet, r is the distance between the centers of mass of the object and the planet, and G is the universal gravitation constant. In the context of space flight, the mass of the object is so much smaller than that of the planet that the planet can be considered to be unaffected by the gravitational force. It is reasonable to consider the planet as a frame of reference, so we replace r with z, the distance of the object from the center of the planet.

Now, suppose that the planet has radius R. Then the quantity

$$g = \frac{GM}{R^2}$$

is defined to be the surface gravitational constant for the planet. Replacing r by z and substituting gR^2 for GM, we obtain a convenient form for the inverse square law:

$$F = -mg\left(\frac{R}{z}\right)^2. \tag{2}$$

Note that this formula simplifies to the standard $F = -mg$ for the case where $z \approx R$.

Motion Near a Planet

Consider a vehicle that moves away from a planet. The acceleration of the vehicle is determined by **(1)**, and we take the force to be strictly that of gravity **(2)**. Combining these laws results in a differential equation for the velocity.

$$\frac{dv}{dt} = -\frac{gR^2}{z(t)^2}. \tag{3}$$

Now, suppose we are given values v_0 for the velocity and z_0 for the height (measured from the center of the planet) at some time, which we can arbitrarily call $t = 0$. Given also that velocity is the derivative of position, we have a pair of initial value problems to describe the motion

$$\begin{aligned} \frac{dv}{dt} &= -\frac{gR^2}{z(t)^2}, \qquad && v(0) = v_0, \\ \frac{dz}{dt} &= v, && z(0) = z_0. \end{aligned}$$

These equations can be combined to yield a nonlinear second-order equation for the height, which we cannot solve. An alternative approach is to make use of the lack of explicit dependence of the equations on t. Dividing the two equations yields

$$\frac{dv}{dz} = -\frac{gR^2}{vz^2}.$$

This equation is first-order and separable. We can therefore think of the problem in separated form as the equation

$$v\,dv = -\frac{gR^2}{z}\,dz,$$

with the additional requirement that the solution curve must pass through the point (z_0, v_0).

The solution is then given by

$$\frac{v^2 - v_0^2}{2gR} = \frac{R}{z} - \frac{R}{z_0}. \tag{4}$$

This result is a family of height-velocity curves. Given planetary data R and g and initial data (z_0, v_0), the curves indicate the height and velocity for $t > 0$. Of course, the height-velocity curves are only valid while $z > R$.

The Escape Curve

Of particular interest is the notion of escape from a planet's gravitational field. The idea is that given the planet parameters g and R and the initial height z_0, there is a critical initial velocity v_c for which the subsequent velocity just vanishes as $z \to \infty$. We can calculate this velocity by considering the height–velocity curves in the form

$$\frac{v^2}{2gR} - \frac{R}{z} = \frac{v_0^2}{2gR} - \frac{R}{z_0}.$$

If the curve is to approach the point $(\infty, 0)$, then the initial conditions must be related by

$$\frac{v_c^2}{2gR} - \frac{R}{z_0} = 0.$$

Solving for v_c, we obtain the formula

$$v_c^2 z_0 = 2gR^2 \tag{5}$$

for the escape curve. In particular, the escape velocity v_e is defined to be the critical initial velocity for an object whose flight begins at the surface of the planet:

$$v_e = \sqrt{2gR}.$$

The Dimensionless Height–Velocity Curves

In the form **(4)**, the height–velocity curves are different for each planet. We can improve the result considerably by replacing the original variables v and z by appropriate dimensionless variables. We define these by

$$V = \frac{v}{v_e}, \qquad Z = \frac{z}{R}, \qquad V_0 = \frac{v_0}{v_e}, \qquad Z_0 = \frac{z_0}{R}, \quad \text{where} \quad v_e = \sqrt{2gR}. \tag{6}$$

(Note that these variables are ratios of the corresponding dimensional variable to a representative value of that variable. In space flight near a planet, we expect that velocities will be on the order of v_e and distances will be on the order of R.) In terms of these dimensionless variables, the height–velocity curves are given as

$$V^2 - V_0^2 = \frac{1}{Z} - \frac{1}{Z_0}. \tag{7}$$

This equation does not depend explicitly on the planetary data, so we get the same dimensionless height–velocity curves for any planet. In particular, the escape curve takes the simple form

$$ZV^2 = 1. \tag{8}$$

These curves are illustrated in **Figure 1**. Note that the escape curve is the one that passes through the point $(1, 1)$.

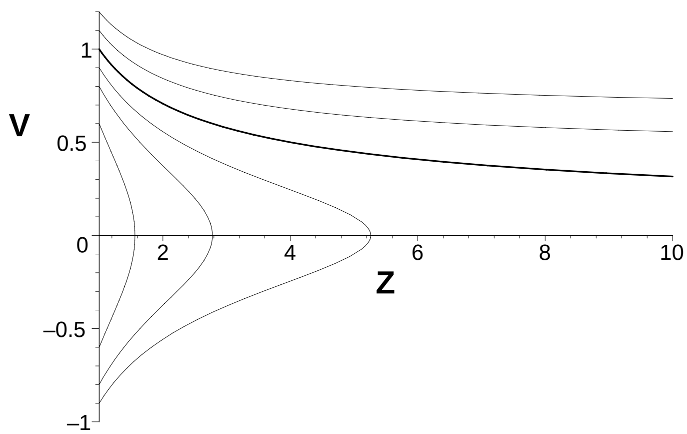

Figure 1. The dimensionless height-velocity curves.

The Liftoff Model

So far, we have been following the course of a standard example that appears in some ordinary differential equations texts. See, for example, Boyce and DiPrima [1997, 76–78], who conclude their example with a note that in reality the escape velocity is not achieved instantly but rather over a period of several minutes. It is this extra detail that will occupy our attention now.

We consider the situation of a vehicle that begins on the surface of a planet with zero velocity. The flight of the vehicle consists of a *liftoff phase*, during which a propulsion system causes the velocity to increase, followed by a *free-fall phase*, during which the velocity decreases because of the pull of gravity. We have already studied the free-fall phase; the subsequent motion will escape from the planet if the plot of the motion in the ZV-plane rises above the escape curve. If $ZV^2 \geq 1$ at the end of the liftoff phase, then the launch will be successful; if not, the gravitational force will pull the vehicle back to the surface.

The differential equation describing the liftoff process is based on the differential form of Newton's Second Law of Motion **(1)**. The use of the law is complicated by the fact that the rocket is not a closed system. Fuel is burnt and the exhaust is blown out of the rocket; hence, we have a closed system only if we include both the rocket and the fuel. The left side of the equation must include the momentum change of both the vehicle and the fuel, while the right side must include both the gravitational force and any force due to combustion.

Momentum Change

Let M be the mass of the vehicle and payload and let P be the mass of fuel initially carried by the vehicle. We assume that the fuel is burned at a rate α, which could perhaps be a function of time or state (z, v). The fuel is either a solid or a pressurized fluid, and the products of the burning of the fuel are gases that are blown out of the vehicle at high speed. We assume that the velocity of the exhaust, relative to the vehicle, is β_1.

In a small amount of time dt, both the mass and the velocity of the vehicle change. The change in momentum for the vehicle is then given by

$$d(mv) = m\,dv + v\,dm,$$

where m is the time-dependent mass consisting of the vehicle and payload and what remains of the fuel. Given that the vehicle and payload are of fixed mass and the fuel burns at rate α, we have $dm = -\alpha\,dt$. Taken together, the differential momentum change of the vehicle is

$$d(mv)\big|_{\text{vehicle}} = m\,dv - \alpha v\,dt.$$

The calculation for the momentum change of the exhaust has a subtlety. Each molecule of exhaust removes momentum from the vehicle only at the moment it is emitted; thus, any subsequent changes in the molecule's velocity are irrelevant. Hence, we have

$$d(mv) = v\,dm.$$

The correct velocity to use in this calculation is the velocity of the exhaust relative to the surface of the heavenly body, which is the sum of the velocity of the vehicle relative to the heavenly body (v) and that of the exhaust relative to the vehicle ($-\beta_1$). Thus,

$$d(mv)\big|_{\text{exhaust}} = \alpha(v - \beta_1)\,dt.$$

Combining the momentum changes of the vehicle and exhaust gives us the total change in momentum:

$$d(mv) = m\,dv - \alpha\beta_1\,dt. \tag{9}$$

Forces

The vehicle is certainly subject to a gravitational force. Additionally, we assume that the burning of the fuel creates energy that in turn creates a force propelling the vehicle upward. The mechanism by which the fuel burns is unimportant; we assume that the force produced by burning is proportional to the rate at which fuel is burnt, with proportionality constant β_2. Thus, the total force is

$$F = \alpha\beta_2 - \frac{mgR^2}{z^2}. \tag{10}$$

The Differential Equation of Motion

Combining **(9, 10)** with **(1)** yields the equation

$$m\,dv = \alpha(\beta_1 + \beta_2)\,dt - \frac{mgR^2}{z^2}\,dt.$$

The differential equation of motion is obtained by dividing through by $m\,dt$:

$$\frac{dv}{dt} = \frac{\alpha\beta}{m} - \frac{gR^2}{z^2},$$

where we have defined $\beta = \beta_1 + \beta_2$.

Note that there are actually two mechanisms that contribute to the acceleration of the vehicle: the propulsive force caused by the energy of combustion, and also the propulsion obtained by the downward momentum change of the exhaust. Although these two mechanisms sound different, their mathematical details are equivalent, and only the sum is needed. In effect, the energy created by the combustion of the fuel simply increases the exhaust velocity that would be obtained if there were no combustion.

The Full Model

The differential equation of motion is supplemented by the differential equations for height and mass. Including the initial conditions, we have

$$\frac{dv}{dt} = \frac{\alpha\beta}{m} - \frac{gR^2}{z^2}, \qquad v(0) = 0, \tag{11}$$

$$\frac{dz}{dt} = v, \qquad z(0) = R, \tag{12}$$

$$\frac{dm}{dt} = -\alpha, \qquad m(0) = M + P. \tag{13}$$

These equations need to be supplemented by an equation that prescribes α in terms of t, v, and/or z.

Note also that the liftoff phase ends when the fuel is gone. Thus, the equation

$$m(t_f) = M$$

determines the time at the end of the liftoff phase. The launch is then successful if and only if

$$Z(t_f)V^2(t_f) \geq 1,$$

where we assume the same dimensionless variables for the full model that were used for the free-fall model.

The Return Vehicle Design Problem

The full model **(11–13)** has 6 parameters, including 4 design parameters: α, β, M, and P. (As noted above, the parameter α could actually be a function.) There are also 2 parameters, g and R, that are characteristics of the planet. The *return vehicle design problem* is the problem of determining, for any particular planet, the suitable region in the 4-dimensional $\alpha\beta MP$-space for which the escape curve is reached by the time the fuel supply is exhausted.

Figure 2 is a schematic diagram of the model for the return vehicle design problem, which is conveniently viewed as a "black box" in which the parameters are entered and the calculations inside the box yield a Boolean result with possible outcomes "success" and "failure." In this sense, the model can be thought of as a function of its parameters. The parameters serve as constants or independent variables, depending on the depth of one's viewpoint. They are constants in the model for vehicle flight, which is on the inside of the box, but they are independent variables in the model for the return vehicle design problem, which is outside of the box.

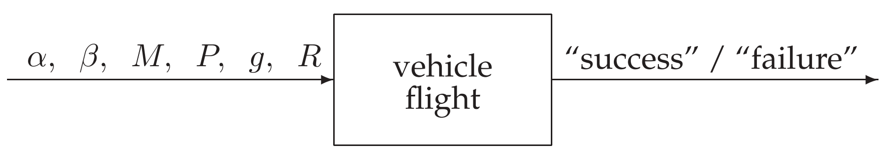

Figure 2. A schematic diagram of the return vehicle design problem.

Simplification

We can simplify the problem with little loss of generality by assuming that the fuel is burned at a constant rate during the liftoff phase. In reality, a given amount of fuel will do more good if it is used as early in the launch as possible. With this simplification, the initial value problem for the mass can be solved immediately, with the result

$$m = M + P - \alpha t. \quad \textbf{(14)}$$

The fuel is exhausted when $m = M$, or $t_f = P/\alpha$. The full model then reduces to

$$\frac{dv}{dt} = \frac{\alpha\beta}{M + P - \alpha t} - \frac{gR^2}{z^2}, \qquad v(0) = 0, \quad 0 \le t \le \frac{P}{\alpha}, \tag{15}$$

$$\frac{dz}{dt} = v, \qquad z(0) = R, \quad 0 \le t \le \frac{P}{\alpha}. \tag{16}$$

Having 4 design parameters makes for an unwieldy investigation. We consider a simplified model in which there are only 3 design parameters.

Assume that we want to carry the maximum amount of fuel. This maximum amount can be determined by considering the initial moment of the vehicle liftoff. We can determine the initial acceleration by substituting the initial conditions into the differential equation of motion:

$$\frac{dv}{dt}(0) = \frac{\alpha\beta}{M + P} - g.$$

To achieve liftoff, the initial propulsive force must be at least sufficient to overcome the gravitational force. It is therefore necessary to require

$$(M + P)g \le \alpha\beta.$$

The maximum amount of fuel that can be carried is then given by

$$P = \alpha\beta g^{-1} - M, \tag{17}$$

and we assume that this is the amount actually chosen. The problem is then

$$\frac{dv}{dt} = \frac{\beta g}{\beta - gt} - \frac{gR^2}{z^2}, \qquad v(0) = 0, \qquad 0 \le t \le \frac{\beta}{g} - \frac{M}{\alpha},$$

$$\frac{dz}{dt} = v, \qquad z(0) = R, \qquad 0 \le t \le \frac{\beta}{g} - \frac{M}{\alpha}.$$

Scaling

We saw in the discussion of the escape velocity curves that models can be greatly simplified by nondimensionalization. It is particularly important to nondimensionalize using dimensional reference quantities that represent the order of magnitude of the variables (scales). The scales for height and velocity are obviously R and v_e, as before. For time, we could choose the duration of the liftoff phase; however, this is a rather complicated quantity. The simpler quantity β/g works just as well. This is what the duration of the liftoff would be in the special case where the vehicle is small compared to the mass of fuel. We therefore define the dimensionless variables

$$V = \frac{v}{v_e}, \qquad Z = \frac{z}{R}, \qquad \tau = \frac{gt}{\beta}.$$

Applying these changes to the velocity equation yields the result

$$\frac{v_e}{\beta}\frac{dV}{d\tau} = \frac{1}{1-\tau} - \frac{1}{Z^2}, \qquad V(0)=0, \qquad 0 \le \tau \le 1 - \frac{Mg}{\alpha\beta}.$$

Note that certain dimensionless groupings, such as v_e/β, appear in the result. It is particularly interesting that the design parameters M and α are not independent, so we have only two dimensionless design parameters, which we define as

$$a = \frac{v_e}{\beta}, \qquad b = \frac{Mg}{\alpha\beta}, \tag{18}$$

to consider. The dimensionless model now takes its final form:

$$a\frac{dV}{dT} = \frac{1}{1-T} - \frac{1}{Z^2}, \qquad V(0)=0, \qquad 0 \le \tau \le 1-b, \tag{19}$$

$$a\frac{dZ}{dT} = 2V, \qquad Z(0)=1, \qquad 0 \le \tau \le 1-b. \tag{20}$$

The Return Vehicle Design Results

Given a pair of values (a,b), with $b<1$, we now have a simple method to determine whether or not the launch is successful. We simply solve the system **(19–20)** numerically up to time $1-b$. If the graph of the solution rises above the escape velocity curve in the ZV-plane (in other words, if at some point $ZV^2 \ge 1$), then the launch is a success. In either case, the launch history after the time $1-b$ simply follows the appropriate height–velocity curve.

Figure 3 shows the launch histories for a successful launch and an unsuccessful launch. Note that this model is still too simple to correctly describe the details of the launch. There is a sharp cusp in the curves at the point where the fuel is exhausted. This corresponds to an instantaneous velocity change (a discontinuity in the acceleration) that we could expect to damage the vehicle. A more sophisticated model could be written in which the burn rate gradually changes from α to 0 over a time interval of a few seconds. However, this change unnecessarily complicates an investigation whose purpose is to determine the success or failure of a launch.

The system **(19–20)** is even more convenient than we could have anticipated. Of the two remaining design parameters, the parameter b does not actually appear in the differential equations or initial conditions. This means that, for any fixed a, all values of b can be tested by solving the system numerically for the given value of a.

Specifically, for a given value of a, we compute the solution of the differential equations up to the time T_0 for which $ZV^2 = 1$. **Figure 4** shows the graph of the launch up to time T_0 for the case $a=1$.

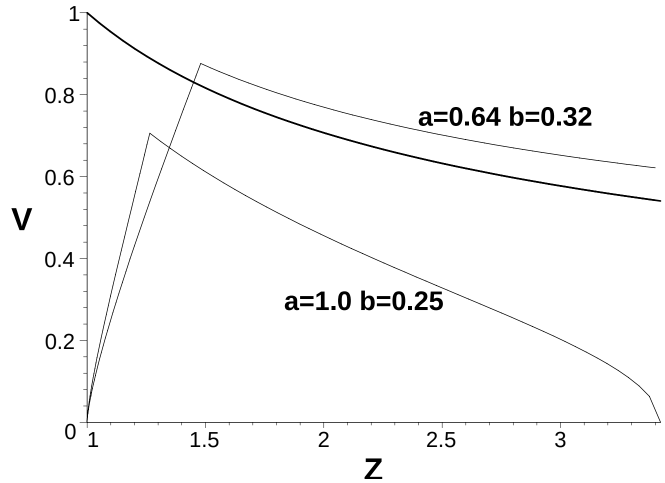

Figure 3. A successful launch (top curve) and an unsuccessful launch (bottom curve).

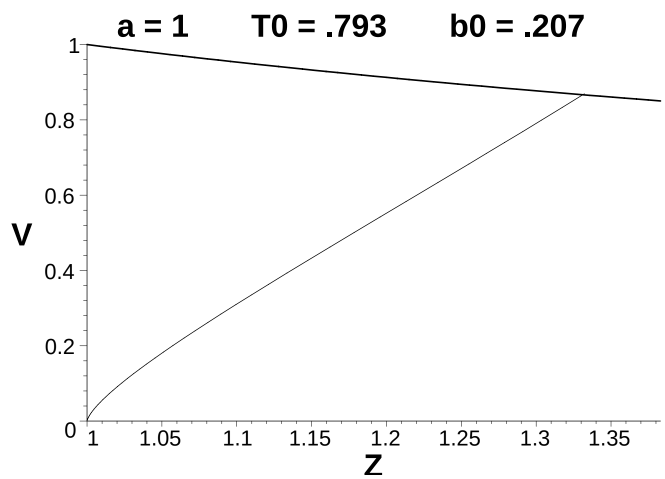

Figure 4. The launch curve for $a = 1$ up to the escape curve.

Note that a launch is successful if the value of b is small enough for the time T_0 to occur before the fuel runs out. Let

$$b_0 = 1 - T_0. \tag{21}$$

The criterion for success is then $b < b_0$. We can repeat the calculations to find the value of b_0 corresponding to any value of a. By connecting such points in the ab-plane, we obtain a curve that separates the parameter space into regions of success and failure. This curve, which appears in **Figure 5**, shows the vehicle design engineers what combinations of values of the dimensionless design parameters a and b leads to success.

Of course the actual vehicle design must still take into account the parameters R and g that are planet-dependent, because these quantities appear in the definitions of a and b. Given specific values for a and b, the vehicle design curve provides a quick check on whether a given set of values for the design parameters β, M, and α is sufficient.

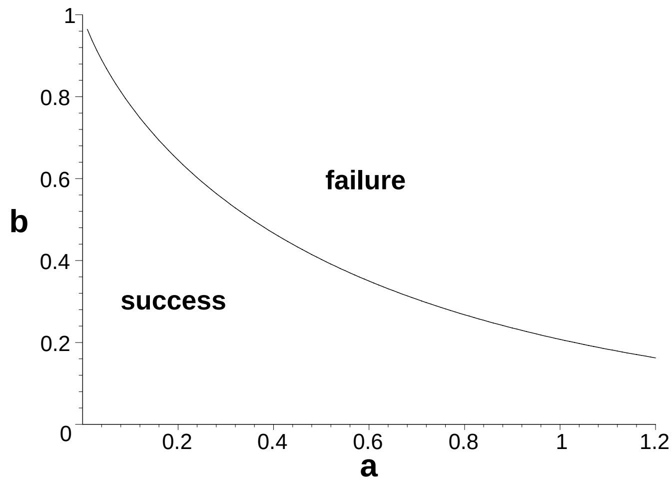

Figure 5. The vehicle design curve.

In practice, the engineers would choose a point in **Figure 5** that is not too close to the curve, in order to provide a factor of safety (the mathematical model approximates reality rather than describing it, so the model should not be used to make fine distinctions). This principle must also be balanced by a desire to avoid overdesigning the vehicle, which would make it more expensive than necessary.

References

Boyce, William E., and Richard C. DiPrima. 1997. *Elementary Differential Equations*. 6th ed. New York: Wiley.

About the Author

Glenn Ledder earned his B.S. in ceramic engineering from Iowa State University and his M.S. and Ph.D. in applied mathematics from Rensselaer Polytechnic Institute. His research area is mathematical modeling, particularly in hydrogeology and biology. He is currently an associate professor in the Department of Mathematics and Statistics at the University of Nebraska–Lincoln.