

UMAP

**Modules in
Undergraduate
Mathematics
and Its
Applications**

**Published in
cooperation with**

**The Society for
Industrial and
Applied Mathematics,**

**The Mathematical
Association of America,**

**The National Council
of Teachers of
Mathematics,**

**The American
Mathematical
Association of
Two-Year Colleges,**

**The Institute for
Operations Research
and the Management
Sciences, and**

**The American
Statistical Association.**

The logo for COMAP, Inc., featuring the letters 'COMAP' in a stylized, cursive, handwritten font.

Module 767

The Mathematics of Scuba Diving

D. R. Westbrook



**Applications of Beginning Calculus to
Physiology**

INTERMODULAR DESCRIPTION SHEET:	UMAP Unit 767
TITLE:	The Mathematics of Scuba Diving
AUTHOR:	D.R. Westbrook Dept. of Mathematics and Statistics University of Calgary Calgary, Alberta, Canada T2N 1N4 westbroo@acs.ucalgary.ca
MATHEMATICAL FIELD:	Beginning calculus
APPLICATION FIELD:	Physiology
TARGET AUDIENCE:	Students in beginning calculus
ABSTRACT:	Exponential solutions of differential equations are used to construct decompression schedules for dives of various durations to various depths.
PREREQUISITES:	A knowledge of differential and integral calculus related to exponential functions.
RELATED UNITS:	Unit 676: <i>Compartment Models in Biology</i> , by Ron Barnes. <i>The UMAP Journal</i> 8 (2): 133–160. Reprinted in <i>UMAP Modules: Tools for Teaching 1987</i> , edited by Paul J. Campbell, 207–234. Arlington, MA: COMAP, 1988.

Tools for Teaching 1997, 191–219. ©Copyright 1997, 1998 by COMAP, Inc. All rights reserved.

Permission to make digital or hard copies of part or all of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice. Abstracting with credit is permitted, but copyrights for components of this work owned by others than COMAP must be honored. To copy otherwise, to republish, to post on servers, or to redistribute to lists requires prior permission from COMAP.

COMAP, Inc., Suite 210, 57 Bedford Street, Lexington, MA 02173
(800) 77-COMAP = (800) 772-6627, or (781) 862-7878; <http://www.comap.com>

The Mathematics of Scuba Diving

D.R. Westbrook
Dept. of Mathematics and Statistics
University of Calgary
Calgary, Alberta, Canada T2N 1N4
westbroo@acs.ucalgary.ca

Table of Contents

1. INTRODUCTION	1
2. A BRIEF HISTORY OF DIVING	1
3. HALDANE'S MODEL	2
4. SOLUTION OF THE DIFFERENTIAL EQUATION	4
5. THE HALF-TIME	6
6. SCUBA AND NO-STOP DIVES	7
7. DIVES WITH DECOMPRESSION STOPS	8
8. REPETITIVE DIVES	14
9. CHANGES IN PRESSURE DURING DESCENT AND ASCENT	16
10. CONCLUSION	19
11. SOLUTIONS TO THE EXERCISES	19
REFERENCES	23
ABOUT THE AUTHOR	25

MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS (UMAP) PROJECT

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications, to be used to supplement existing courses and from which complete courses may eventually be built.

The Project was guided by a National Advisory Board of mathematicians, scientists, and educators. UMAP was funded by a grant from the National Science Foundation and now is supported by the Consortium for Mathematics and Its Applications (COMAP), Inc., a nonprofit corporation engaged in research and development in mathematics education.

Paul J. Campbell
Solomon Garfunkel

Editor
Executive Director, COMAP

1. Introduction

Are you a scuba diver? Can you use the diving tables? Do you know the mathematical basis for the diving tables? Could you construct your own diving tables? The purpose of this module is to describe the physiological basis for the diving tables and the mathematics used for the calculations.

2. A Brief History of Diving

Diving is an ancient pastime. Diving for profit—the collectinn of sponges, shells, and pearls—and diving for food have been with us for some time, and probably so has diving for pleasure. Divers were used for military purposes by the Greeks and are still of strategic importance today.

Ancient diving was essentially free (or breath-hold) diving, although Alexander the Great was reported to have used a primitive diving bell around 330 B.C. A diving bell is essentially a weighted inverted receptacle that retains its air (or other gases) as it is lowered into the water, giving a source of oxygen at depth to which the diver may return as needed or even be connected by a flexible tube. The air in the bell deteriorates in quality as the dive progresses, and various methods have been devised to replenish it.

In 1691, Sir Edmund Halley (of comet fame) built and patented what may have been the first practical diving bell, with a volume of approximately 60 cubic feet. The air was replenished from barrels, and the fouled air was vented out by means of a valve. (A 6-foot-high cylinder of diameter $3\frac{1}{2}$ ft has volume $\simeq 56$ ft³.) Nearly 100 years passed before a successful forcing pump was developed to enable a supply of fresh air to be pumped to the bell from the surface. This technique later developed into personal diving suits supplied from the surface and then to self-contained underwater breathing apparatus (SCUBA).

As dives became deeper and longer, it became apparent that there were various physiological risks involved. One such risk is decompression sickness, or the “bends,” which was associated with a rapid return to the surface after a long or deep dive.

In addition to diving, the nineteenth century saw the introduction of “caissons,” large chambers equipped with an air lock and kept under high pressure, which enabled tunnellers and bridge builders to work underground or underwater without the chamber flooding. It soon became clear that special procedures were needed so that the workers, who may have been working in a high-pressure environment for several hours, did not suffer injuries or even death when they returned to normal atmospheric pressure. The need for a careful decompression sequence became obvious. In 1854, physicians B. Pol and T.J.J. Wattelle stated in a report, “The danger does not lie in entering a shaft containing compressed air; nor in remaining there a longer or shorter time; decompression alone is dangerous” [Hills 1977].

The decompression routines of this time were usually linear (i.e., a reduction in pressure at a fixed constant rate in atmospheres per minute) and were generally devised by experience that involved much pain and some deaths on the part of the experimental subjects. Of the approximately 600 men who worked on the St. Louis bridge, 119 suffered serious neurological decompression sickness, and 14 died. The name “the bends” apparently originated from the gait of these bridge workers, caused by pains in their joints. This resembled the “Grecian bend” of fashionable ladies of the time, who walked voluntarily in this manner.

In the early twentieth century, military needs led various navies to become interested in decompression sickness, and more careful research was begun. The most influential of this research was performed by the physiologist J.S. Haldane for the Royal Navy in 1906. Haldane’s diving tables (1908) were remarkably effective in almost eliminating decompression sickness as a diving hazard and were used for some time. As more experience was gained, it became clear that Haldane’s tables were somewhat conservative for short dives, so adjustments were made. Then, as longer deeper dives were undertaken, it was found that the tables were not conservative enough for such dives, and more refinements were made. Many further refinements have taken place in more recent times, but the tables are still essentially based on adaptations of Haldane’s original ideas.

In the following sections, we examine these basic ideas and the mathematics behind them. To construct adequate universal tables is arithmetically intensive, but we will use the ideas in simplified form to construct our own tables.

**The tables that we construct are not to be used in any dive!
Use the tables that your scuba instructors give you.**

3. Haldane’s Model

When Haldane began his experiments, it had been established that the major cause of decompression sickness was the release of bubbles of nitrogen, an inert gas in the air, into various tissues and into the arterial bloodstream. While a diver is underwater, she is breathing air under high pressure and, as a result, more nitrogen is forced into her blood. When she ascends, the air that she is breathing returns to a lower pressure, and the nitrogen dissolved in her blood forms bubbles. (Because oxygen in the air that is dissolved in the blood is metabolized, it does not cause a problem.) The effect can be seen when the lid of a pop bottle is unscrewed. The gas in the fluid is under pressure that is suddenly reduced when the lid is unscrewed, and bubbles rapidly form.

Initially it was thought that there would be a critical drop in pressure above which sickness would occur; but Haldane’s experiments, which were performed on goats, led to a different conclusion. (Haldane had found that the sensitivity of goats to decompression sickness was acceptably close to that of

humans.) He found that no matter what the original pressure is, decompression sickness does not occur if the pressure is reduced by less than some fixed fraction. That is, there is a value M for which a pressure P_1 can be reduced to $P_2 = MP_1$ without the occurrence of "the bends." Haldane suggested a value M just slightly less than $1/2$. We will use $1/2.15 \approx .465$ in our calculations.

The subjects of these experiments were exposed to the higher pressure for long periods, so the dissolved gases were brought to saturation levels. In dives, this might not be the case. In addition, for long dives at an absolute pressure of more than twice atmospheric pressure, the subject could not be brought to atmospheric pressure without one or several intermediate stops. (An absolute pressure of two atmospheres occurs at a depth of about 10 m \approx 33 ft of water.)

To determine an appropriate set of stops, a model of how gases are dissolved in and released from body tissues is needed. First, it is known that the pressure of inert gas in the pulmonary circuit is almost instantaneously equalized with that in the lungs, which is the ambient external pressure. Thus, blood entering the arterial system has gas pressure equal to the ambient pressure. A model must now be made of the distribution of the gas to the various tissues in the body.

The simple model that we use in this Module is based on the following assumptions:

- The blood flows through a tissue at a constant volume rate ν ml/sec.
- If the gas pressure in the blood and tissue is p , then the concentration of the gas in the blood is s_1p g/ml and in the tissue is s_2p g/ml, where s_1, s_2 are constants with different values of s_2 for different tissues.

The model is a simple compartment model (see Barnes [1987]). Gas enters the pulmonary circuit from the lungs at pressure p_e , the ambient external pressure. We assume that the gas pressure in the blood as it enters a tissue compartment is p_e . The pressure in the tissue and the blood is quickly equalized to the local pressure p , and the blood leaves the compartment at pressure p .

A balance of mass for the gas must hold:

$$\begin{aligned} \text{The rate of increase of mass in the compartment} &= \\ \text{Rate at which mass flows in} &- \text{Rate at which mass flows out.} \end{aligned}$$

The mass of gas in the compartment at any time is $V_1s_1p + V_2s_2p$, where V_1 and V_2 are measured in ml and represent the respective volumes of blood and tissue in the compartment. The rate of increase of mass is then

$$\frac{d}{dt}[(V_1s_1 + V_2s_2)p] \quad \text{g/sec.}$$

Gas enters the compartment at a rate νs_1p_e g/sec and leaves at a rate of νs_1p g/sec. The balance of mass gives

$$[V_1s_1 + V_2s_2] \frac{dp}{dt} = \nu s_1(p_e - p) \quad \text{or} \quad \frac{dp}{dt} = k(p_e - p),$$

where $k = \nu s_1 / (V_1 s_1 + V_2 s_2)$ is a constant for the tissue. A simple diagram for this model is presented in **Figure 1**.

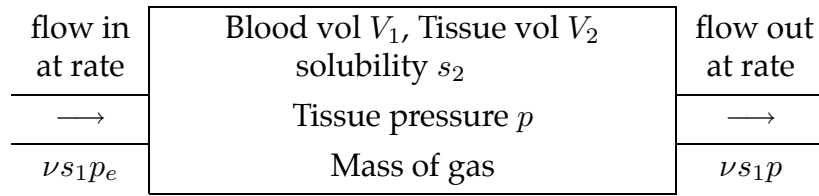


Figure 1. Diagram for the compartment model.

In Haldane's time, this model was thought to be appropriate for both compression ($p_e \geq p$) and decompression ($p \geq p_e$). It was known already that various tissues in the body required different values of s_2 , V_1 , V_2 , and ν , and the same blood does not flow through all tissues. In devising his tables, Haldane considered five different values for the constant k in the differential equation. His calculations were based on solutions of the differential equation and on the experimental result that the external absolute pressure could be reduced by the factor M at any time without an attack of the bends occurring.

In the work that follows, we assume for simplicity that air is all nitrogen. It can be shown that this in fact makes no significant difference to the results (see **Exercise 6**).

4. Solution of the Differential Equation

The differential equation

$$\frac{dp}{dt} = k(p_e - p), \quad (1)$$

where k and p_e are known constants, can be solved to find the pressure p at any time t , provided that the pressure p is known at one instant of time, usually taken to be $t = 0$ (we measure the elapsed time from the instant at which the pressure is known), i.e., $p(0) = p_0$, a known constant. If you know enough integral calculus, you can find the solution of the equation, as shown below, by the method of separation of variables. If you do not know integral calculus, the solution can be verified directly by substitution in (1).

To separate variables, we write (1) as

$$\frac{1}{p_e - p} \frac{dp}{dt} = k$$

and integrate (antidifferentiate) both sides with respect to t . This gives

$$\int \frac{1}{p_e - p} \frac{dp}{dt} dt = \int \frac{1}{p_e - p} dp = \int k dt.$$

Performing the integrations, we get

$$-\ln |p_e - p| = kt + c$$

where c is an arbitrary constant. Taking exponentials of both sides gives

$$|p_e - p| = e^{-(kt+c)} = e^{-kt} e^{-c} = Ae^{-kt},$$

where A is an arbitrary constant, $A = e^{-c}$. Since we also require $p(0) = p_0$, it follows that $|p_e - p_0| = A$, and we obtain the solution

$$p = p_e - (p_e - p_0)e^{-kt}. \tag{2}$$

Graphs of solutions for the case $p_0 = 1$ atm, $p_e = 3$ atm with (a) $k = 0.2$ min⁻¹, (b) $k = 0.1$ min⁻¹ are given in **Figure 2**. The curves represent the pressure p in the tissues of a diver at time t min after descending from the surface ($p = 1$ atm) to a depth of about 66 ft ($p = 3$ atm). Similarly, graphs for the case $p_0 = 3$ atm, $p_e = 1$ atm with (a) $k = 0.2$ min⁻¹, (b) $k = 0.1$ min⁻¹ are given in **Figure 3**. Here the curves represent the pressure t minutes after ascending to the surface from a point where the tissue pressure is 3 atm.

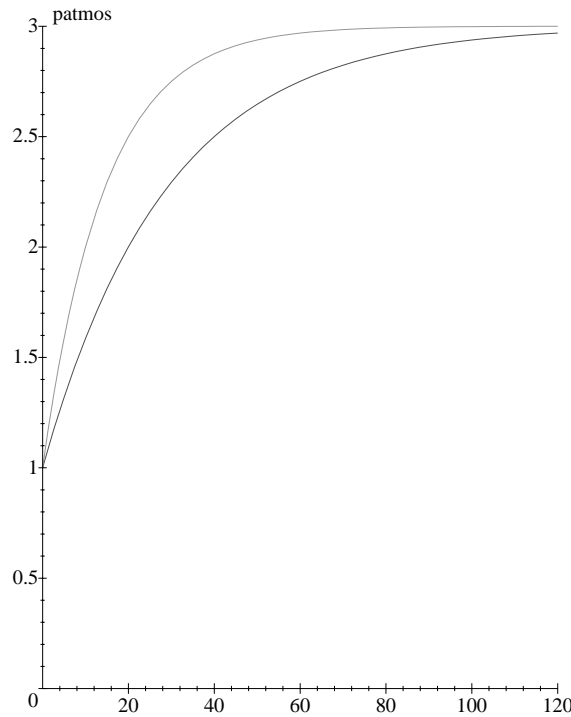


Figure 2. Solutions for the case $p_0 = 1$ atm, $p_e = 3$ atm. The lower curve is for $k = 0.2$ min⁻¹ and the upper curve is for $k = 0.1$ min⁻¹. The curves give the pressure p in the tissues of a diver at time t min after descending from the surface ($p = 1$ atm) to a depth of about 66 ft ($p = 3$ atm).

The role of the constant k , which is measured in min⁻¹ if t is measured in min, is indicated in **Figures 2** and **3**. When p_0 and p_e are held constant, it takes

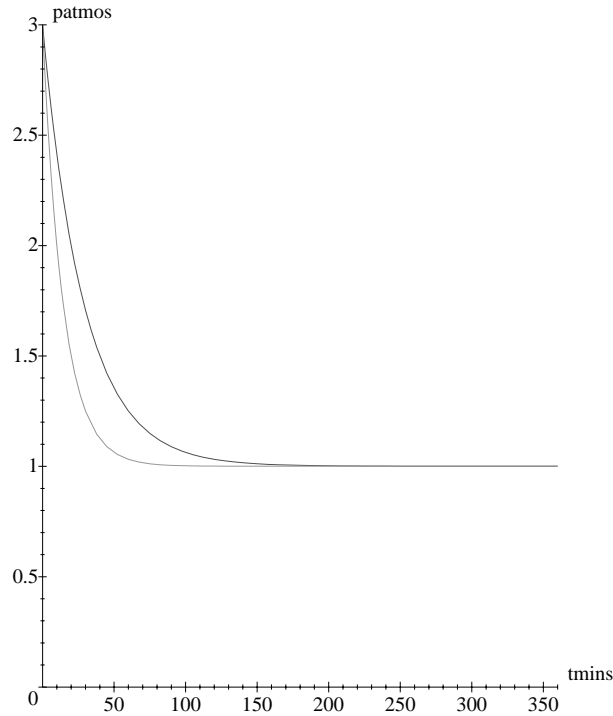


Figure 3. Solutions for the case $p_0 = 3 \text{ atm}$, $p_e = 1 \text{ atm}$. The lower curve is for $k = 0.2 \text{ min}^{-1}$ and the upper curve is for $k = 0.1 \text{ min}^{-1}$. The curves give the pressure t min after ascending to the surface from a point where the tissue pressure is 3 atm.

twice as long to attain a given pressure when $k = 0.1$ as it does when $k = 0.2$. We also see that for any positive k , p approaches the constant external pressure p_e as t becomes large ($t \rightarrow \infty$) no matter what the value of p_0 . In other words, the pressure equalizes over time, as expected.

5. The Half-Time

Because solutions of the exponential nature of (2) all have the same asymptote $p = p_e$ for all positive values of k , they are often characterized by their half-time, or half-life as it is called in the case of radioactive decay.

The half-time is the time required for the difference between p and the external pressure p_e to drop to exactly one half of its original value, that is, the time at which $(p - p_e) = (p_0 - p_e)/2$.

From (2), we see that if T is the half-time, then

$$p - p_e = (p_0 - p_e)e^{-kT} = \frac{1}{2}(p_0 - p_e)$$

and hence

$$e^{-kT} = \frac{1}{2} \Rightarrow e^{kT} = 2 \Rightarrow kT = \ln 2. \quad (3)$$

From this equation we see that $k = \ln 2/T$ no matter what the values of p_0 and p_e are, and that the half-time T for a tissue completely determines the value of k in (2). This makes the half-time extremely useful in characterizing the various tissues in the body.

The relationships between bottom times and decompression programmes differ for different half-times. The human body contains many different tissues, as Haldane knew, and a safe decompression programme must make sure that the bends do not occur in any of them. Haldane did not have exact values for half-times, so to compile his tables he used five different values (5, 10, 20, 40, and 75 min) in the belief that this would cover any reasonable spectrum of half-times. His tables were successful over the wide range of dives undertaken at that time and for some considerable time thereafter.

Noting that

$$e^{-kT} = \frac{1}{2}, \quad e^{-kt} = e^{-kT(\frac{t}{T})} = (e^{-kT})^{(t/T)} = \left(\frac{1}{2}\right)^{t/T},$$

we rewrite (2) as

$$p = p_e + (p_0 - p_e) \left(\frac{1}{2}\right)^{t/T}. \tag{4}$$

6. Scuba and No-Stop Dives

Most recreational divers usually dive to a given depth, remain at (or above) that depth for a certain time, and then ascend directly to the surface. This is the “no stop” or “no decompression” dive, as shown in **Table 1** below. The time allowed at the bottom depends on the depth of the dive. For example, the table says that you may stay (“stay” includes descent and ascent) at 70 ft for 50 min.

Table 1.
Diving table (from Hammes and Zimos [1988]).

Depth (ft)	40	50	60	70	80	90	100	110	120	130
Time (min)	200	100	60	50	40	30	25	20	15	10

A no-stop diving table can be produced from our model in the following manner.

We wish to model a situation in which a diver starts with an initial gas tissue pressure of 1 atm and wishes to stay at a depth d ft where the external pressure is $p_e = 1 + d/33$ (33 ft of water gives a pressure of 1 atm; the equation contains a 1 because there is already a pressure of 1 atm at the surface $d = 0$). We use (2) to tell us the tissue gas pressure after t min, which will be

$$p = 1 + \frac{d}{33} - \frac{d}{33} e^{-kt} \quad (k \text{ being known for the given tissue}).$$

Haldane's decompression experiment says that the diver may ascend directly to the surface where the pressure is 1 atm provided that the pressure p attained in the tissues is less than 2.15 atm. Thus the diver has a limiting dive time t_d given by

$$2.15 = 1 + \frac{d}{33} (1 - e^{-kt_d}),$$

$$\frac{d}{33} = \frac{1.15}{1 - e^{-kt_d}}.$$

This relation gives the time for the tissue as characterized by its value of k (equivalently, by its half-time $T = \ln 2/k$).

The allowable time t_d becomes longer as k becomes less, that is, as the half-time $T (= \ln 2/k)$ becomes greater. To be safe for all tissues, t_d is limited by the tissue with the shortest half-time, which is 5 min in Haldane's scheme. This would give the relation

$$d = \frac{38}{1 - \exp(-t_d \ln 2/5)}.$$

Tables are usually written with t_d as a function of depth d , which our model gives as

$$t_d = \frac{5 \ln \left(\frac{d}{d-38} \right)}{\ln 2}.$$

You will find that this relation gives qualitative agreement with published tables (see **Figure 4**); but the quantitative agreement is not very good, because of the conservative nature of Haldane's value of M and his tissue half-time of 5 min for short dives.

7. Dives with Decompression Stops

For dives that fall outside the no-stop dive range, a more complicated set of conditions must be satisfied. Again we follow Haldane's recipe.

The standard method to calculate a decompression routine is to consider a series of stops at depths that are multiples of 10 ft. The first stop must be such that the external pressure at that depth is not less than M times the pressure in each of the tissues that has been reached during the stay at the diving depth. The tissue pressures depend on the time spent at depth and on the tissue half-time. The greatest tissue pressure will be in the tissue with the shortest half-time. Consider the following three examples.

Example 1. Consider a one-hour dive at a depth of 66 ft, where the pressure is approximately 3 atm. To save some calculations, we assume

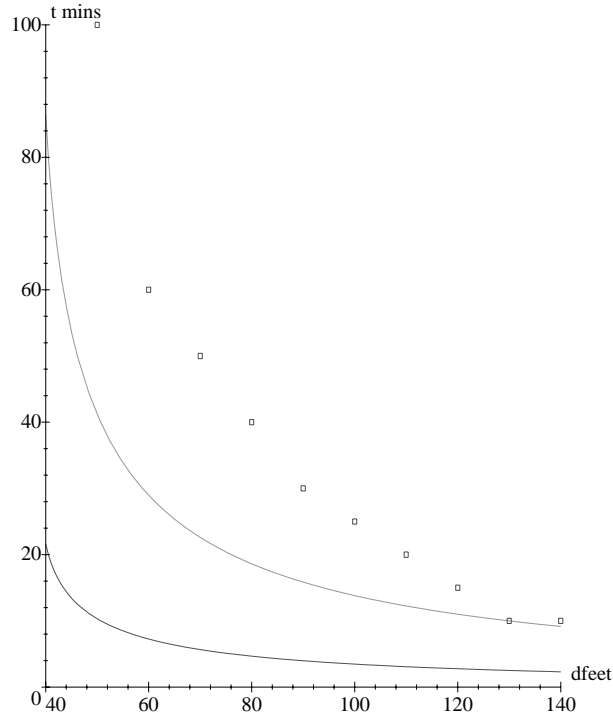


Figure 4. No-stop dive. Graph of $t_d = 5 \ln \left(\frac{d}{d-38} \right) / \ln 2$ and $t_d = 20 \ln \left(\frac{d}{d-38} \right) / \ln 2$, compared with points from the diving table of **Table 1**.

that there are three tissues (as opposed to Haldane’s five) with half-times 10, 20, and 40 min. From (4), the pressure of a tissue at an external pressure p_e is

$$p = p_e + (p_0 - p_e) \left(\frac{1}{2} \right)^{t/T},$$

where p_0 is the initial tissue pressure, T the tissue half-time, and t is the length of time at depth (in minutes). The pressure p_0 at the beginning of the dive is 1 atm. After one hour at 66 ft (3 atm for 60 min, $p_e = 3$ atm), tissue pressures are

$$\begin{aligned} T = 10\text{-min tissue} : \quad p &= 3 - 2 \left(\frac{1}{2} \right)^6 && \approx 2.97 \\ T = 20\text{-min tissue} : \quad p &= 3 - 2 \left(\frac{1}{2} \right)^3 && \approx 2.75 \\ T = 40\text{-min tissue} : \quad p &= 3 - 2 \left(\frac{1}{2} \right)^{3/2} && \approx 2.29. \end{aligned}$$

It is safe to ascend to an external pressure of $2.97/2.15 = 1.35$, or about 12.5 ft. To keep the ascent steps in multiples of 10 ft, the first ascent is made to 20 ft (1.60 atm).

At this point, the diver makes a stop. We have to decide how long this stop should be. To do this, we must decide the depth for the next stop. We choose 10 ft or 1.30 atm. The diver must remain at 20 ft until

all tissue pressures have declined to a value that will be safe when the diver ascends to 1.30 atm—that is, until all tissue pressures are reduced to $2.15 \times 1.30 = 2.795$ atm. The three tissue pressures at the beginning of the 20-ft stop are 2.97, 2.75, 2.29. The pressures of the 20-min and 40-min tissues are already low enough to ascend to 10 ft. For the 10-min tissue, t min will result in pressures

$$\text{10-min tissue:} \quad p = 1.6 + 1.37 \left(\frac{1}{2}\right)^{t/10}.$$

(Again we are using (4), with $p_e = 1.6$ and $p_0 = 2.97$ for $T = 10$.) The diver must remain at the 20-ft level until all tissue pressures are below the pressure 2.795 atm that is safe at the 10-ft stop (1.3 atm). For the 10-min tissue, this means t must be greater than the solution of

$$2.975 = 1.6 + 1.37 \left(\frac{1}{2}\right)^{t/10} \quad \text{or} \quad t = 10 \ln \left(\frac{1.37}{1.195}\right) / \ln 2 \approx 1.971.$$

Suppose that we make a 2-min stop at 20 ft. We must calculate the tissue pressures after 2 min at 20 ft:

$$\begin{aligned} \text{10-min:} \quad p &= 1.6 + 1.37 \left(\frac{1}{2}\right)^{.2} = 2.79 \\ \text{20-min:} \quad p &= 1.6 + 1.15 \left(\frac{1}{2}\right)^{.1} = 2.67 \\ \text{40-min:} \quad p &= 1.6 + .69 \left(\frac{1}{2}\right)^{.05} = 2.27. \end{aligned}$$

These are the initial pressures at the 10-ft (1.3-atm) stop. The next ascent will be to the surface (1 atm), where the safe pressure will be 2.15. The stop at 10 ft (1.3 atm) must be long enough that all three pressures will drop below 2.15. For a stop of t min, the pressures will be

$$\begin{aligned} \text{10-min:} \quad p &= 1.3 + 1.49 \left(\frac{1}{2}\right)^{t/10} \\ \text{20-min:} \quad p &= 1.3 + 1.37 \left(\frac{1}{2}\right)^{t/20} \\ \text{40-min:} \quad p &= 1.3 + .97 \left(\frac{1}{2}\right)^{t/40}, \end{aligned}$$

and t must be large enough that all three are less than 2.15. For the 10-min tissue, this requires 7.62 min, for the 20-min tissue 13.77 min, and for the 40-min tissue 8.09 min. The stop at 10 ft must be greater than 13.77 min—say 14 min. An appropriate decompression procedure for a one-hour dive at 66 ft would feature stops of

$$\begin{array}{ll} 2 \text{ min} & \text{at 20 ft} \\ 14 \text{ min} & \text{at 10 ft.} \end{array}$$

The ascent would also be lengthened by the time to ascend the 66 ft, about 1.5 min.

Example 2. We take an ascent as recommended in Haldane’s tables [Hempleman 1982, 330]. For a dive of 130 min at 90 ft, Haldane’s tables recommend stops of

5 min at 30 ft
 25 min at 20 ft
 30 min at 10 ft.

In this calculation, we will use all five of Haldane’s half-times of 5, 10, 20, 40, and 75 min.

First, we calculate the saturation levels for a dive of 130 min at 90 ft ≈ 3.73 atm. Then we calculate the pressures at the end of the period spent at each stopping point. Finally, we note the safe pressure to ascend to the next stop (see **Table 2**).

Table 2.
 Analysis of ascent recommended by Haldane for a 130-min dive at 90 ft.

Tissue half-time (min)	Pressure			
	90 ft = 3.73 atm	30 ft = 1.9 atm	20 ft = 1.6 atm	10 ft = 1.3 atm
5	3.73	2.82	1.64	1.30
10	3.73	3.19	1.88	1.37
20	3.70	3.41	2.36	1.67
40	3.44	3.31	2.71	2.14
75	2.91	2.86	2.60	2.285
Safe pressure at next stop	4.08	3.44	2.8	2.15

We see that at every stage except one, a safe pressure is attained in each tissue to allow the diver to ascend to the next stop. The exception is the last ascent to the surface for the 75-min tissue. Haldane allowed 2 min to move to and from the stops; if this time were included, the final pressures would be slightly reduced. This example, however, shows a problem with Haldane’s tables for long dives.

A much more recent U.S. Navy Table T-10 (reproduced in Hammes and Zimos [1988]) gives for this dive stopping times of

5 min at 30 ft
 36 min at 20 ft
 74 min at 10 ft.

This decompression procedure allows for even larger half-times than 75 min.

Figure 5 shows graphs of the tissue pressures for half-times of 5, 10, 20, 40, and 75 min, using the decompression scheme from Haldane’s tables. The piecewise “step” graph at the right indicates the safe pressure at the stops.

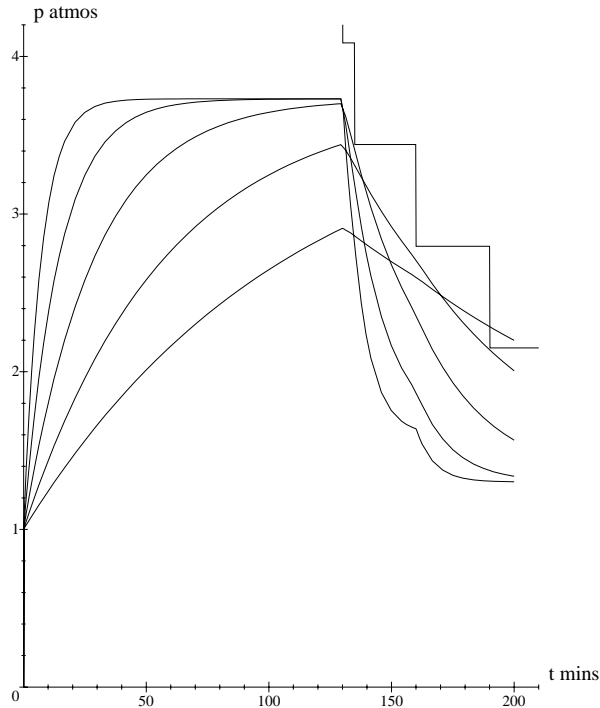


Figure 5. A 130-min dive to 90 ft followed by ascent with decompression stops as recommended by Haldane's tables. At left, from top to bottom, are tissue pressures at 90 ft for half-times of 5, 10, 20, 40, and 75 min. At right, from top to bottom, are the tissue pressures during ascent. The piecewise "step" graph at far right indicates the safe pressures at the stops.

Example 3. We consider a dive to 80 ft = 3.43 atm for one hour. Haldane's tables give stops of

9 min at 20 ft
18 min at 10 ft.

Again we give the pressures as the diver *leaves* each level to proceed to the next (see **Table 3**).

In this case, a safe tissue pressure has been reached at all levels for all tissues before proceeding. This decompression procedure, however, is now considered to be rather conservative. The U.S. Navy table suggests 17 min at 10 ft as the only stop for this dive.

A procedure of this kind can be calculated with as many tissues as appropriate. (You might like to write a computer programme to carry out the steps.)

Table 3.

Analysis of ascent recommended by Haldane for a 60-min dive at 80 ft.

Tissue	Pressure		
	80 ft = 3.43 atm	20 ft = 1.6 atm	10 ft = 1.3 atm
5	3.43	2.13	1.37
10	3.39	2.56	1.66
20	3.17	2.75	2.08
40	2.57	2.43	2.13
75	2.03	2.00	1.89
Safe pressure at next stop	3.44	2.80	2.15

Exercises

In all exercises, assume that $M = 1/2.15$.

1. Find a decompression procedure for a dive of 40 min at 3.5 atm (80–85 ft) with stops at 1.7 atm (23 ft) and 1.3 atm (10 ft). (Consider only 10- and 20-min tissues.)
2. Find a decompression procedure for a 2-hr dive at 4.0 atm (100 ft) with stops at 1.9 atm (30 ft), 1.6 atm (20 ft), and 1.3 atm (10 ft). (Consider 10-, 20-, and 40-min tissues.)
3. Show that a slightly faster ascent for the dive of **Exercise 2** could be made if three stops of equal duration T_1 are made, the first at 1.9 atm (30 ft) and the second and third at depths to be determined. (As a first step, consider only the 40-min tissue; then verify that the steps are appropriate for the 10-min and 20-min tissues.)
4. Show that for a single tissue half-time T and an n -stop decompression schedule, the shortest total ascent time is achieved by using equal times at each step and determining the depths of each step according to the time. (The actual time at each step is determined by the number of steps.)
5. Show that for a single tissue, it is possible to have a continuous ascent in which the tissue pressure at time t is exactly 2.15 times the external pressure that the diver is experiencing at that time. Find the diver's depth at time t (pressure = $1 + d/33$ atm, where d is in feet). Using such a scheme, find how long it would take to ascend from a long dive at 4 atm. (Assume a single tissue of half-time 40 min and an instantaneous ascent from 4 atm to $1.86 = 4/2.15$ atm.)
6. If the nitrogen (partial) pressure in a tissue is 80% of the pressure, and the safe nitrogen pressure for a no-stop dive is 2.15 times that of the nitrogen partial pressure in the atmosphere (0.8 atm), show that the equation relating time and depth for no-stop dives is unaltered.

7. Check for safety the following recommendations from Haldane's tables for a dive of 45 min at 85 ft (3.58 atm). Stop 2 min at 30 ft, 7 min at 20 ft, 15 min at 10 ft. (U.S. Navy Table T-10 [Hammes and Zimos 1988] gives one stop of 17 min at 10 ft for this dive.)

8. Repetitive Dives

A major portion of the scuba diving tables is devoted to repetitive diving. The problem with repetitive diving is the fact that after one "no decompression" dive, the tissue pressure may be 2.15 times atmospheric pressure. An immediate dive back to a depth greater than 37 ft (external pressure greater than 2.15 atm) would raise the tissue pressure to above the limit that would allow a safe ascent to the surface. A break at the surface between dives lessens the pressure when the second dive is commenced, but it takes about twelve hours to restore all tissue pressures to 1 atm. The tissue pressure remaining after the first dive is known as the *residual nitrogen pressure* (RNP). We consider only a 20-min tissue in making our calculations, to keep things simple.

Example 4. Dive (1): 15 min at 80 ft. Dive (2) is to be to a depth of 100 ft after a one-hour break at the surface. We calculate the safe time for a "no decompression" second dive (20-min tissue only).

Tissue pressure p after 15 min at 80 ft ($p_e \approx 3.4$ atm):

$$p = 3.4 - 2.4 \left(\frac{1}{2} \right)^{3/4} = 1.97.$$

Since this is less than 2.15, it is safe to ascend to the surface.

Tissue pressure p after one hour at the surface ($p_e = 1$ atm):

$$p = 1 + .97 \left(\frac{1}{2} \right)^3 = 1.12.$$

Descent to 100 ft (4 atm):

$$p = 4 - 2.88 \left(\frac{1}{2} \right)^{t/20}.$$

The diver may remain until $p = 2.15$, that is, until

$$t = 20 \ln(2.88/1.85) / \ln 2 = 12.77 \text{ min.}$$

Figure 6 shows the pressure as a function of time for this example.

Actual scuba tables cover the large numbers of different calculations by classifying the residual nitrogen pressures into groups A, B, C, etc. The group

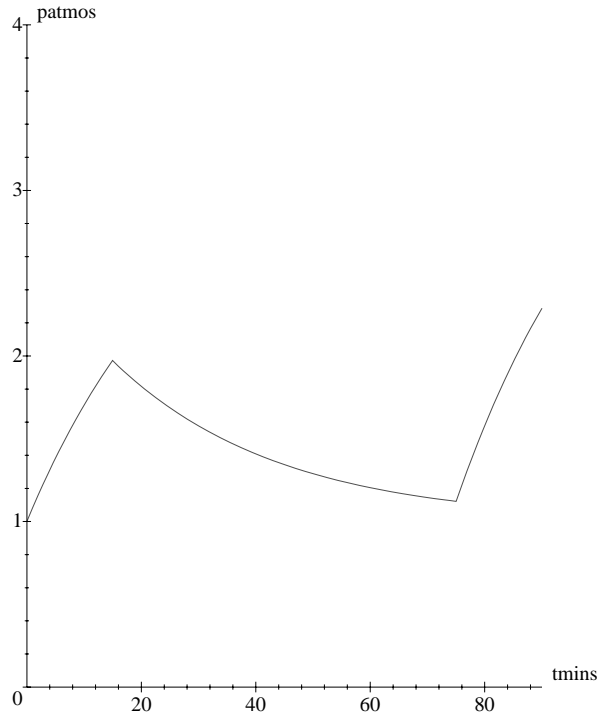


Figure 6. Repetitive dive.

is found after the first dive. The effect of remaining at the surface for a given time period is to change the group; the new group determines the safe time for the next dive. We give an example.

Example 5.

- Dive 1: 100 ft for 15 min
- Dive 2: 80 ft
- Time at surface between dives: 1 hr

We consult **Table 4**. First look at the row for a dive to 100 ft. Note that the no-stop time is 25 min. Our dive is for 15 min, so we go across the row until we reach 15. We then move down the corresponding column and find the repetitive group label “E”.

The stay at the surface is for 60 min. We continue along the column until we come to the two numbers that bracket 60 min:

$$\begin{aligned} &0 : 55 \\ &1 : 57. \end{aligned}$$

We now proceed left across this row until we find a new repetitive group label “D”.

For the second dive, at depth 80 ft, we use the label D. We continue across the row until we reach the column corresponding to 80 ft (at right).

The entry contains the numbers 18 (RNT) and 22 (TR). This means that because of the previous dive, it is as if we had already been at this depth for 18 min, and our time remaining is 22 min. We must be back at the surface within 22 min.

Exercises

Use (2) in the following exercises.

8. Consider the same sequence of dives as in **Example 4** but include a 40-min tissue. Does this make a difference for the second stop time?
9. Find the safe time for a second dive to 80 ft one hour after a first dive to 100 ft for 10 min. Consider tissue half-times of 20 min and 40 min.

9. Changes in Pressure During Descent and Ascent

To this point, we have assumed that the passage from one level to another is instantaneous. This is not possible; moreover, rapid motion is not recommended. A steady ascent or descent rate of about 60 ft/min is not unreasonable, and we will now examine the effect on tissue pressure of ascending at such a rate.

Our basic equation

$$\frac{dp}{dt} = k(p_e - p)$$

(where p_e is the external pressure) still holds, but p_e is no longer constant. For a descent at a constant rate of 60 ft/min, we have $p_e = 1 + 60t/33$ atm, and the differential equation becomes

$$\frac{dp}{dt} = k \left(1 + \frac{60t}{33} - p \right) \quad \text{or} \quad \frac{dp}{dt} + kp = k \left(1 + \frac{60t}{33} \right). \quad (5)$$

This is no longer a separable equation but a first-order linear equation, and it must be solved in a different manner. Here we describe one possible method.

First we try to guess a solution. After examining the equation, we feel that $p = A + Bt$, where A and B must be selected, seems a possible guess. If we substitute this into (5), we see that we get a solution if we can choose A , B so that

$$B + k(A + Bt) = k \left(1 + \frac{60t}{33} \right).$$

Table 4.
Diving table. ©NAUI 1987. Reproduced by permission.

NAUI[®]

DIVE TABLES

TABLE 1 - END-OF-DIVE LETTER GROUP

ABT ACTUAL BOTTOM TIME
+RNT RESIDUAL NITROGEN TIME
TBT TOTAL BOTTOM TIME
 (USE THIS FIGURE TO DETERMINE END-OF-DIVE LETTER GROUP)

START DEPTH M FEET	00 NO-DECOMPRESSION LIMIT (NDL)					00 BOTTOM TIME REQUIRING DECOMPRESSION 00 NO. MINUTES REQUIRED AT 10' STOP (3m)								
	12 40 →	5	15	25	30	40	50	70	80	100	110	130	150	170
15 50 →		10	15	25	30	40	50	60	70	80	90	100	140	160
18 60 →		10	15	20	25	30	40	50	55	60	70	80	100	120
21 70 →		5	10	15	20	30	35	40	45	50	60	70	80	90
24 80 →		5	10	15	20	25	30	35	40	50	60	70	80	90
27 90 →		5	10	12	15	20	25	30	35	40	50	60	70	80
30 100 →		5	7	10	15	20	22	25	30	35	40	50	60	70
33 110 →			5	10	13	15	20	25	30	35	40	50	60	70
36 120 →			5	10	12	15	20	25	30	35	40	50	60	70
39 130 →			5	8	10	15	20	25	30	35	40	50	60	70
42 140 →			5	7	10	15	20	25	30	35	40	50	60	70

M FT.	NEW GROUP													
	A	B	C	D	E	F	G	H	I	J	K	L	M	N
7 193	6	5	4	4	3	3	3	3	2	2	2	2	2	2
17 183	13	11	9	8	7	7	6	6	6	5	5	5	5	5
25 175	21	17	15	13	11	10	10	9	8	7	7	7	7	7
37 163	29	24	20	18	16	14	13	12	11	10	10	10	10	10
49 151	38	30	26	23	20	18	16	15	13	12	12	12	12	12
61 138	47	36	31	28	24	22	20	18	16	15	15	15	15	15
73 127	56	44	37	32	29	26	24	21	19	18	18	18	18	18
87 113	66	52	43	38	33	30	27	25	22	20	20	20	20	20
101 90	76	61	50	43	38	34	31	28	25					
116 84	87	70	57	48	43	38								
138 62	99	79	64	54	47									
161 30	111	88	72	61	53									
187 13	124	97	80	68	58									
213	142	107	87											

TABLE 3 - REPETITIVE DIVE TIMETABLE

00 LIGHT FACE NUMBERS ARE RESIDUAL NITROGEN TIMES (RNT)
 00 BOLD FACE NUMBERS ARE ADJUSTED NO-DECOMPRESSION LIMITS (ANDL) • ACTUAL BOTTOM TIME SHOULD NOT EXCEED THIS NUMBER

TABLE 2 - SURFACE INTERVAL TIME (SIT) TABLE

TIME RANGES IN HOURS MINUTES ©1987 NAUI

The choice $B = 60/33$, with $kA + B = k$, hence $A = 1 - 60/33k$ gives a solution

$$p = 1 + \frac{60}{33} \left(t - \frac{1}{k} \right).$$

We call this a *particular integral*. If we then write

$$u = p - \left[1 + \frac{60}{33} \left(t - \frac{1}{k} \right) \right] = p - 1 - \frac{60}{33}t + \frac{60}{33k},$$

where p is any solution of the equation, it follows that

$$\frac{du}{dt} + ku = \frac{dp}{dt} + kp - \frac{60}{33} - k \left(1 + \frac{60t}{33} \right) + \frac{60}{33} = 0,$$

since p is a solution of (5).

If $du/dt + ku = 0$, then we can again use separation of variables to get

$$\int \frac{1}{u} \frac{du}{dt} dt = \int k dt$$

which implies that $-\ln|u| = kt + C$, or $u = Ae^{-kt}$, where A is an arbitrary constant. In this approach, u is usually called the *complementary function*. Thus, if p is any solution of (5), it can be written as

$$p = 1 + \frac{60}{33} \left(t - \frac{1}{k} \right) + u = 1 + \frac{60}{33} \left(t - \frac{1}{k} \right) + Ae^{-kt};$$

that is, any solution is the sum of a particular integral and a complementary function. The technique may be used on any first-order linear equation. To satisfy an initial condition $p(0) = p_0$, we get

$$1 - \frac{60}{33k} + A = p_0 \quad \text{or} \quad A = p_0 - 1 + \frac{60}{33k},$$

$$p = 1 + \frac{60t}{33} - \frac{60}{33k} + \left(p_0 - 1 + \frac{60}{33k} \right) e^{-kt} \quad (6)$$

$$= 1 + \frac{60t}{33} - \frac{60}{33k} + \left(p_0 - 1 + \frac{60}{33k} \right) \left(\frac{1}{2} \right)^{(t/T)}. \quad (7)$$

A similar solution could be obtained for an ascent from a given depth.

Example 6. Find the pressure in a 20-min tissue on arrival at a depth of 100 ft (4 atm) after a descent from the surface at a rate of 60 ft/min.

The time to descend 100 ft at 60 ft/min is $100/60 = 5/3$ min.

The initial pressure is $p_0 = 1$, and $k = \ln 2/T = .03466$.

Therefore,

$$p = 1 + \frac{60}{33} \cdot \frac{5}{3} - \frac{60}{33(.03466)} + \frac{60}{33(0.3466)} \left(\frac{1}{2}\right)^{1/12} = 1.086.$$

To do a complete dive, we would have to include these changes of pressure in the complete diving schedule. We will not do this, although it is merely tedious rather than difficult.

We note finally that if the descent had been considered instantaneously, the pressure after 5/3 min at a depth of 100 ft would be 1.17 atm.

Exercise

- Find the tissue pressure for a 20-min tissue at the end of an ascent from 100 ft to 10 ft at a speed of 60 ft/min, assuming that the pressure at the beginning of the ascent was 4 atm. Compare it with the pressure at 10 ft after an instantaneous ascent.

10. Conclusion

In this Module, we have discussed a simple technique for derivation of diving tables, which is based on a model proposed by Haldane. Although modern diving tables cannot be devised by means of such simple techniques, most of them have been developed by refinements to the simple model and methods proposed by Haldane, as tempered by experience (see, for example, Bornmann [1970]).

11. Solutions to the Exercises

All solutions use either (4) or its inverse:

$$p = p_e + (p_0 - p_e) \left(\frac{1}{2}\right)^{t/T} \quad \text{or} \quad t = T \ln \left(\frac{p_0 - p_e}{p - p_e}\right) / \ln 2.$$

- During the dive, $p_e = 3.5$, $t = 40$ min, $p_0 = 1$.
For $T = 10$, $p = 3.344$; for $T = 20$, $p = 2.875$. It is safe to ascend to 20 ft = 1.6 atm, because $2.15 \times 1.6 = 3.44$. The stop at 1.6 should be long enough that an ascent to 1.3 will be safe. This requires that p be reduced to $2.15 \times 1.3 = 2.795$.

For $T = 10$, this requires $t = 10 \ln \left(\frac{3.344 - 1.6}{2.795 - 1.6}\right) / \ln 2 \simeq 5.454$ min
($p_0 = 3.344$, $p_e = 1.6$).

For $T = 20$, this requires $t = 20 \ln \left(\frac{2.875 - 1.6}{2.795 - 1.6} \right) / \ln 2 \simeq 1.870$ min ($p_0 = 2.875, p_e = 1.6$).

Thus, a stop of 5.454 min is required. After 5.454 min, the pressure in the $T = 20$ tissue is 2.655, and that in the $T = 10$ tissue is 2.795.

The stop at 1.3 (10 ft) should be long enough that an ascent to the surface (1 atm) is safe. This requires that p be reduced to 2.15.

For $T = 10$, this requires $t = 10 \ln \left(\frac{2.795 - 1.3}{2.15 - 1.3} \right) / \ln 2 \simeq 8.146$ min.

For $T = 20$, this requires $t = 20 \ln \left(\frac{2.655 - 1.3}{2.15 - 1.3} \right) / \ln 2 \simeq 13.455$ min.

A safe schedule is then a 5.454-min stop at 1.6 (20 ft) and a 13.455-min stop at 10 ft. The total stopping time is 18.909 min.

2. By similar means as in **Exercise 1**, the pressures at the end of the dive where $p_e = 4, p_0 = 1, t = 120$ are: for $T = 10, p = 4$; for $T = 20, p = 3.953$; for $T = 40, p = 3.625$.

Stop 1 at 1.9 atm: (This is safe since since $1.9 \times 2.15 = 4.085$.) Times to reduce pressure to $1.6 \times 2.15 = 3.44$ are: for $T = 10, 4.47$ min; for $T = 20, 8.296$ min; for $T = 40, 6.547$ min.

A stop of 8.296 min is required. After this stop, the $T = 10$ tissue will have a pressure below that of the $T = 20$ tissue, and this will remain true for the rest of the dive. We need not consider the $T = 10$ tissue further.

After 8.296 min at 1.9, $T = 20$ has pressure 3.44 and $T = 40$ has pressure 3.39.

Stop 2 at 1.6 atm: Times to reduce pressure to $2.15 \times 1.3 = 2.795$ are: for $T = 20, 12.5$ min; for $T = 40, 23.4$ min. From this point on we need only consider the $T = 40$ tissue. After stop 2, its pressure is 2.795.

Stop 3 at 1.3 atm: Time to reduce pressure to 2.15 is 32.584 min for the $T = 40$ tissue.

The total time for all stops is 64.3 min.

3. The same dive as in **Exercise 2**. We consider the 40-min tissue only and make three stops of equal time. The first stop is at 1.9, but the depth of the remaining stops must be calculated from the condition of equal times.

After the dive, the pressure in the $T = 40$ tissue is 3.625. Ascent to 1.9 is certainly safe.

Suppose that the second and third stops are at pressures p_2, p_3 . Then the diver must stay at 1.9 until $p = 2.15p_2$, must stay at p_2 until $p = 2.15p_3$, and must stay at p_3 until $p = 2.15$. From the inverse of (4), the equalization times are

$$\begin{aligned} t_1 &= \frac{40}{\ln 2} \ln \left(\frac{3.625 - 1.9}{2.15p_2 - 1.9} \right) = \frac{40}{\ln 2} \ln \left(\frac{2.15p_2 - p_2}{2.15p_3 - p_2} \right) \\ &= \frac{40}{\ln 2} \ln \left(\frac{2.15p_3 - p_3}{2.15 - p_3} \right). \end{aligned}$$

This gives

$$\frac{1.725}{2.15p_2 - 1.9} = \frac{1.15}{2.15\frac{p_3}{p_2} - 1} = \frac{1.15}{2.15\left(\frac{1}{p_3}\right) - 1}.$$

The last two equations give $p_3/p_2 = 1/p_3$, or $p_2 = p_3^2$. The first two then give $2.4725p_3^3 - 0.46p_3 - 3.70875 = 0$. The only real positive solution is $p_3 = 1.199$. Thus, $p_2 = p_3^2 = 1.438$ and $t_1 = 21.438$. The total stop time is $3t_1 = 64.314$, a very small improvement. We can verify that after the first stop, the $T = 20$ pressure is 2.877 and the $T = 10$ pressure is 2.375, both below the 3.092 ($= 2.15 \times 1.438$) of the $T = 40$.

4. We assume that the tissue pressure at the beginning of the ascent is p_0 , which is known. The three stops will be at pressures p_1, p_2, p_3 , where $p_1 = p_0/2.15$ and the pressures at the ends of the stops will be $2.15p_2, 2.15p_3, 2.15$. The times at each stop will then be

$$t_1 = \frac{T}{\ln 2} \ln \left(\frac{p_0 - p_1}{2.15p_2 - p_1} \right) = \frac{T}{\ln 2} \ln \left(\frac{1.15}{2.15\frac{p_2}{p_1} - 1} \right),$$

$$t_2 = \frac{T}{\ln 2} \ln \left(\frac{2.15p_2 - p_2}{2.15p_3 - p_2} \right) = \frac{T}{\ln 2} \ln \left(\frac{1.15}{2.15\frac{p_3}{p_2} - 1} \right),$$

$$t_3 = \frac{T}{\ln 2} \ln \left(\frac{2.15p_3 - p_3}{2.15 - p_3} \right) = \frac{T}{\ln 2} \ln \left(\frac{1.15}{2.15\left(\frac{1}{p_3}\right) - 1} \right).$$

We wish to minimize $t_1 + t_2 + t_3$ by choosing p_2 and p_3 . This is equivalent to maximizing

$$F(p_2, p_3) = \ln \left(\frac{p_2}{p_1} - M \right) + \ln \left(\frac{p_3}{p_2} - M \right) + \ln \left(\frac{1}{p_3} - M \right),$$

where $M = 1/2.15$ and p_1 is known. Using calculus, we find

$$\frac{\partial F}{\partial p_2} = \frac{1}{\left(\frac{p_2}{p_1} - M\right)} \frac{1}{p_1} + \frac{1}{\left(\frac{p_3}{p_2} - M\right)} \left(\frac{-p_3}{p_2^2}\right) = 0,$$

$$\frac{\partial F}{\partial p_3} = \frac{1}{\left(\frac{p_3}{p_2} - M\right)} \frac{1}{p_2} + \frac{1}{\left(\frac{1}{p_3} - M\right)} \left(\frac{-1}{p_3^2}\right) = 0.$$

This gives

$$\frac{p_2}{p_2 - Mp_1} = \frac{p_3}{p_3 - Mp_2} = \frac{1}{1 - Mp_3},$$

and hence $p_2^2 = p_1p_3$, $p_2 = p_3^2$, and finally

$$p_3 = p_1^{1/3}, \quad p_2 = p_1^{2/3}.$$

This also gives

$$t_1 = t_2 = t_3 = \frac{T}{\ln 2} \ln \left(\frac{1.15}{2.15/p_1^{1/3} - 1} \right).$$

(For $p_0 = 3.625$, we have $p_1 = 1.686$, $p_2 = 1.417$, $p_3 = 1.190$, $t_1 = 20.482$ min, and the total time $3t_1 = 61.447$ min.)

5. For a safe continuous ascent, the external pressure should be the tissue pressure divided by 2.15. The differential equation for $p(t)$ then becomes

$$\frac{dp}{dt} = k(p_e - p) = k \left(\frac{p}{2.15} - p \right) = -k \frac{1.15}{2.15} p, \quad k = \ln 2/T$$

$$\frac{dp}{dt} = -.535kp.$$

The solution of this equation is $p = p(0)e^{-.535kt}$, where $p(0)$ is the pressure at time $t = 0$. The diver's depth at time t is related to $p_e(t)$ ($= p(t)/2.15$) by

$$1 + \frac{d}{33} = p_e(t) = p(0)e^{-.535kt}/2.15.$$

For a long dive at 4 atm and $T = 40$, we have

$$\begin{aligned} d &= 33 \times 1.86 \left(\frac{1}{2} \right)^{.535t/40} - 33 \\ &= 33 \left[1.86 \left(\frac{1}{2} \right)^{.0134t} - 1 \right]. \end{aligned}$$

The time to ascend to the surface is the value of t at which $d = 0$, that is,

$$t = \frac{1}{.0134} \frac{\ln 1.86}{\ln 2} \approx 66.81.$$

6. If the partial pressure of nitrogen is $0.8p$, where p is the tissue pressure, then the maximum safe pressure for the nitrogen is 0.8×2.15 , so that the condition $p < 2.15$ is retained. Moreover, if the external gas pressure is p_e , the external nitrogen pressure is $0.8p_e$, and the equation for absorption of nitrogen will be

$$\frac{d}{dt} (.8p) = k(.8p_e - .8p)$$

with initial nitrogen pressure $.8p_0$. Thus, the differential equation for the pressure is the same and the criterion for safe ascent is the same.

7. **Table 5** gives the pressures at the ends of the stops for the half-times 5, 10, 20, 40, and 75 min.

Table 5.
Pressures at the ends of the stops for the dive of **Exercise 7**.

	5	10	20	40	75	Safe pressure at next stop
45 min at 3.58	3.57	3.46	3.04	2.40	1.88	4.08
2 min at 1.9	3.17	3.26	2.96	2.38		3.44
7 min at 1.6		2.62	2.67	2.29		2.795
15 min at 1.3			2.11	2.06		2.15

From the table, we see that a safe pressure has been reached to ascend to the next stop in all cases. The blanks in the 75 column have not been calculated, since they will all be less than 1.88. In the 5 column, the blanks will be less than the corresponding entries in the 10 column, and the final entry in the 10 column will be less than that in the 20 column.

8. First dive at 3.4 atm for 15 min. Pressures will be for $T = 20$, 1.97; for $T = 40$, $p = 1.55$.
 After 60 min at the surface, $p_e = 1$. For $T = 20$, $p = 1.12$; for $T = 40$,
 Descent to 4 atm. Diver may remain until tissue pressure is 2.15. For $T = 20$, this requires 12.77 min; for $T = 40$, 24.12 min.
 The diver must still return to the surface after 12.77 min.
9. First dive 4 atm for 10 min. Pressures: for $T = 20$, $p = 1.88$; for $T = 40$, $p = 1.48$.
 After 60 min at 1 atm: for $T = 20$, $p = 1.11$; for $T = 40$, $p = 1.17$.
 Second dive to 3.4 atm until $p = 2.15$. For $T = 20$, 17.47 min; for $T = 40$, 33.40 min.
 The diver must ascend after 17.47 min.

10. We use (7). The time for ascent is $90/60 = 3/2$ min, so we have

$$p = 1 + \frac{60}{33} \cdot \frac{3}{2} - \frac{60}{33(\ln 2/20)} + \left(3 + \frac{60}{33(\ln 2/20)} \right) \left(\frac{1}{2} \right)^{3/40} = 3.92.$$

A stop of 1.5 min at 10 ft (1.3 atm) reduces the pressure to 3.86 atm.

References

- Bachrach, Arthur. 1982. A short history of man in the sea. In Bennett and Elliott [1982], 1–14.
- Barnes, Ron. 1987. Compartment models in biology. UMAP Modules in Undergraduate Mathematics and Its Applications: Module 676. *The UMAP Journal* 8 (2): 133–160. Reprinted in *UMAP Modules: Tools for Teaching 1987*, edited by Paul J. Campbell, 207–234. Arlington, MA: COMAP, 1988.
- Bennett, Peter B., and David H. Elliott, ed. 1982. *The Physiology and Medicine of Diving*. 3rd ed. London: Bailliere Tindall.
- Bornmann, Robert C. 1970. U.S. Navy experiences with decompression from deep helium oxygen saturation excursion diving. In *Human Performance and Scuba Diving*, Proceedings of the Symposium on Underwater Physiology at Scripps Institute of Oceanography, La Jolla, Calif. Chicago, IL: Athletic Institute.
- Hammes, Richard B., and Anthony G. Zimos. 1988. *Safe Scuba*. Long Beach, CA: The National Association of SCUBA Diving Schools. Chapter 9.
- Hempleman, Henry V. 1982. History of evolution of decompression procedures. In Bennett and Elliott [1982], 319–351.
- Hills, B.A. 1977. *Decompression Sickness*. New York: John Wiley & Sons.
- Moon, Richard E., Richard D. Vann, and Peter B. Bennett. The physiology of decompression illness. *Scientific American* 273 (2) (August 1995): 70–77.
- Vann, Richard D. 1982. Decompression theory and applications. In Bennett and Elliott [1982], 352–382.
- Walder, Dennis. 1982. The compressed air environment. In Bennett and Elliott [1982], 15–30.

About the Author

Prof. Westbrook received a B.Sc. in mathematics (1958) and a Ph.D. in applied mathematics (1961), both from University College, University of London. Currently he is an associate professor in the Dept. of Mathematics and Statistics at the University of Calgary, where he has been since 1966. His principal research interests are in mathematical modeling, perturbation theory, and the numerical solution of partial differential equations.