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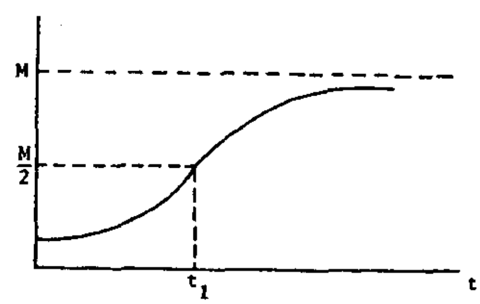
Module 444

Some Applications of Exponential and Logarithmic Functions

W. Thurmon Whitley



Applications of Calculus

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MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS (UMAP) PROJECT

The goal of UMAP was to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications to be used to supplement existing courses and from which complete courses may eventually be built.

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SOME APPLICATIONS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

by

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Intermodular Description Sheet: UMAP Unit 444

Title: SOME APPLICATIONS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

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Classification: APPL CALC/ARCH, MED, POP GROWTH

Target Audience: Second and third semester calculus courses.

Abstract: This unit contains five examples of exponential and logarithmic functions being used to study the behavior of real-world phenomena. The examples study the technique of dating very old objects by using the radioactive element of carbon-14, the infusion of glucose into a patient's body, and increase and decrease in the sizes of populations of organisms in the cases in which the organisms compete among themselves, and in which they compete with other species. Students reinforce differentiation and integration techniques from elementary calculus which involve either exponential or logarithmic functions, observe techniques from elementary calculus being used to study the behavior of real-world phenomena, are able to construct mathematical models of some simple real-world situations, and understand assumptions and refinements needed in the construction of mathematical models.

Prerequisites: Understand differentiation and integration of exponential and logarithmic functions, infinite limits, L'Hopital's Rule, and integration by partial fractions.

1. INTRODUCTION

In this unit, we discuss some applications of two closely related types of functions-exponential functions and logarithmic functions. These functions occur frequently in the study of phenomena in which the rate of growth or decay of some substance is closely related to the amount of the substance present at each instant of time.

Throughout this module, $\ln x$, where $x > 0$, will denote the natural logarithm of x, and e will denote the base of the natural logarithm. Recall that e is an irrational number whose decimal value is between 2.71828 and 2.71829.

2. RADIOCARBON DATING

2.1 How Radiocarbon Dating Works

One of the most important breakthroughs in modern archaeological study was the discovery in 1947 of the technique of radiocarbon dating to determine ages of once-living organisms. This technique was discovered by W. F. Libby, an American chemist, who won the Nobel Prize in Chemistry in 1960 for his work.

Here is how the technique works. All living tissue contains carbon, mostly in the form carbon-12, C^{12}. However, while the tissue is alive, it also absorbs carbon-14, C^{14}, from the atmosphere. (Carbon-14 is a radioactive isotope produced by cosmic rays.) While the tissue is alive, it contains 15.3 atoms of C^{14} disintegrating every minute for each gram of C^{12} in the tissue. When the tissue dies, it is no longer able to absorb the C^{14}, so the C^{14} in the tissue begins to decay.

Scientists assume, on the basis of substantial evidence, that the half-life of C^{14} is about 5730 years. (That is, at the end of any period of 5730 years, the amount of C^{14} remaining is half of what it was at the beginning of the period.)

Let us assume that we wish to determine the age of some very old object. A small sample (about one ounce) is carefully cleaned to remove any younger or older carbon-containing material such as tree roots or crude oil. The material is then burned to form carbon dioxide. This carbon dioxide, after purification, is either measured directly in a Geiger counter, or first converted to elementary carbon black and then measured in a Geiger counter, to determine the amount of C^{14} still present. Since the amount of C^{14} in living tissue is known, the age of the object can then be estimated using Equation (2.6) below.

For objects up to about 50,000 years old, this technique works reasonably well. Many of its findings have been confirmed by a recently developed dating technique called thermoluminescence dating. Other techniques have been developed in recent years to determine the ages of objects more than 50,000 years old. See the references for this section.

2.2 An Equation for the Rate of Decay

Now let us look at the mathematics of radiocarbon dating. Let y(t) denote the number of grams of C^{14} present in a once-living organism t years after its death. It is reasonable to assume that y is a continuous function of t. Further, based on experimental observation, scientists feel that it is reasonable to assume that

$$\frac{dy}{dt} = ky, \quad k < 0, \tag{2.1}$$

where k is constant. That is, the rate of decay is proportional to the amount present at time t.

2.3 The Amount of C^{14} Present at Time t

Equation (2.1) has a unique solution for y, the number of grams of C^{14} present in a sample, in terms of t, namely

$$y(t) = Ce^{kt}, \tag{2.2}$$

where C is a constant. (In Exercise 1 you will be asked to verify that this function is indeed a solution of Equation (2.1). You will see how to derive this solution in Section 3.2.) By substituting 0 for t in Equation (2.2), we see that $C = y(0)$, the amount of C^{14} present when the organism dies.

Since C^{14} has a half-life of 5730 years, we obtain

$$\frac{1}{2}\, y(0) = y(0)e^{k(5730)}$$

$$\frac{1}{2} = e^{5730k} \tag{2.3}$$

$$\ln \frac{1}{2} = 5730k$$

$$k = -\frac{\ln 2}{5730}.$$

Hence, Equation (2.2) becomes

$$y(t) = y(0)e^{-(\ln 2/5730)t}. \tag{2.4}$$

Alternatively, using the fact that $2 = e^{\ln 2}$, we could write Equation (2.2) as

$$y(t) = y(0)\ 2^{-t/5730}. \tag{2.5}$$

See Figure 2-1 for the graph of y(t).

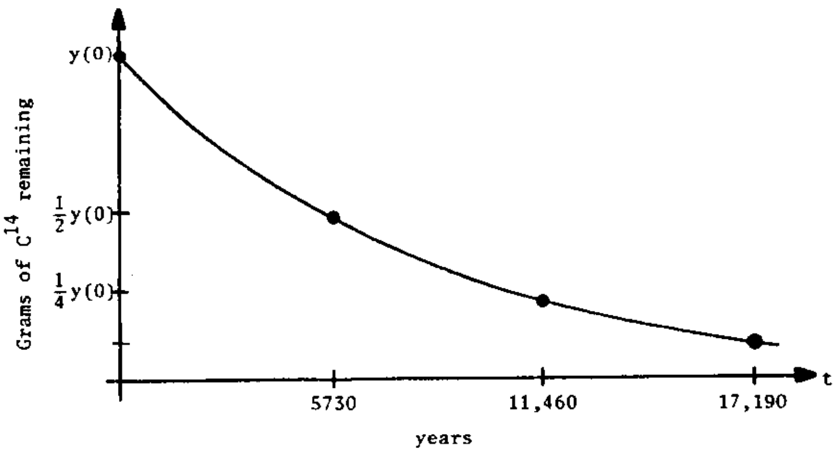

Figure 2-1. The carbon-14 decay function, $y(t) = y(0)2^{-t/5730}$.

2.4 How Old is the Object?

Let us carry Equation (2.4) (or Equation (2.5) if you prefer) one step further. Since we are interested in values of t for specific values of y, we Equation (2.5) for t, we obtain

$$t = -\left(\frac{5730}{\ln 2}\right)\ln\left(\frac{y(t)}{y(0)}\right) . \tag{2.6}$$

Equation (2.6) can be used to calculate the age of the object. (See Exercises 2 and 4.)

Exercises

1. (a) Verify that the function y(t) in Equation (2.2) is a solution of Equation (2.1) by direct substitution.
   (b) Solve Equation (2.4) (or Equation (2.5)) for t, and thus derive Equation (2.6).

2. (a) A fossilized bone of a man found in Western Pennsylvania contained approximately 17 per cent of its original C^{14}. Estimate the year the man died. (Assume that the discovery was made in the year 2000 A.D.)
   (b) Do part (a) under the assumption that the bone contained 18 per cent of its original C^{14}.
   (c) Do part (a) under the assumption that the bone contained 16 per cent of its original C^{14}. Do you see the effects on the age estimate caused by a relatively small error in the estimate of the amount of C^{14} present?

3. Estimate the percentage of C^{14} present in the body of an organism 2000 years after its death.

4. A bone uncovered in Kenya was found to contain only 10 per cent of its original C^{14}. Approximately how long ago did death occur?

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3. INTRAVENOUS FEEDING OF GLUCOSE

3.1 An Outline of the Problem

An important medical process is the infusion of glucose into the bloodstream of an ill person. Let us assume a physician decides to give glucose to a patient at a rate of c grams per hour. At the same time the glucose is being infused into the bloodstream, the body is converting the glucose and removing it from the bloodstream at a rate approximately proportional to the amount present at each instant of time. The physician needs to know much glucose is actually in the bloodstream at any given time. The physician also needs to know how long is required to raise the glucose level in the body to a given level.

3.2 A Differential Equation Related to the Glucose Problem

The "glucose problem" involves a differential equation, that is, an equation involving derivatives of a function. We digress briefly to derive the solution of this equation.

The differential equation we wish to consider is

$$\frac{df}{dt} + a\ f(t) = b \tag{3.1}$$

where a and b are constants, $a \neq 0$, and f is a differentiable function of t. (Equation (2.1) is a special case of (3.1), with $b = 0$, $a = -k$, and $f = y$.)

Multiplying both sides of Equation (3.1) by e^{at}, we obtain

$$e^{at}\frac{df}{dt} + ae^{at}\ f(t) = be^{at}. \tag{3.2}$$

Now from the product rule for derivatives, we see that the left side of Equation (3.2) is the derivative

$$\frac{d}{dt}\ (f(t)e^{at}).$$

Thus, Equation (3.2) becomes

$$\frac{d}{dt}\ (f(t)e^{at}) = be^{at}. \tag{3.3}$$

Integrating both sides of Equation (3.3) with respect to t, we obtain

$$f(t)e^{at} = \frac{b}{a} e^{at} + k, \text{ k constant.} \tag{3.4}$$

If we set t = 0 in Equation (3.4), we see that

$$k = f(0) - \frac{b}{a}.$$

Substituting this expression for k into Equation (3.4) and multiplying both sides of the resulting equation by e^{-at}, we obtain

$$f(t) = \frac{b}{a} + (f(0) - \frac{b}{a})\ e^{-at}. \tag{3.5}$$

Equation (3.5) is the general solution to Equation (3.1).

Exercise

5. Show, by direct substitution that the function given by Equation (3.5) satisfies Equation (3.1).

3.3 The Amount of Glucose Present at Time t

Now let us return to the "glucose problem" of Section 3.1. The glucose is being infused into the bloodstream at a constant rate of c grams per hour. Further, let us assume that the glucose is simultaneously being converted and removed from the bloodstream at a rate of r grams per hour per gram of glucose.

This latter statement means that for each gram of glucose in the bloodstream, r grams of glucose will be removed from the bloodstream in one hour. Thus, for example, if there are 10 grams of glucose in the bloodstream at some point, 10r grams will be removed during the next hour.

Now let G(t) denote the number of grams of glucose present in the bloodstream at time t. Since the rate at which G(t) is changing is the rate at which glucose is being added minus the rate at which it is being removed, we have the equation

$$\frac{dG}{dt} = c - r\ G(t),$$

or equivalently,

$$\frac{dG}{dt} + r\ G(t) = c. \tag{3.6}$$

Observe that since r and c are constants, (3.6) is in the form of Equation (3.1). Then from Equation (3.5), with the substitutions f = G, a = r, and b = c, we find that the amount of glucose present in the bloodstream at time t is

$$G(t) = \frac{c}{r} + (G(0) - \frac{c}{r})\ e^{-rt}. \tag{3.7}$$

Note that to determine G(t) specifically for a given value of t, the physician must know not only the infusion rate c but also the conversion rate r and the amount G(0) of glucose initially in the bloodstream. These latter two constants are determined by clinical measurements. (See the answer to Exercise 8 for a graph of Equation (3.7).)

3.4 An Equilibrium Point

Equation (3.7) can give us an additional piece of important information. If G(t) is given by Equation (3.7), we see that

$$\lim_{t \to \infty} G(t) = \frac{c}{r}. \tag{3.8}$$

The number c/r is called the <u>equilibirum point</u>. If the infusion were continued for a long period of time, the number of grams of glucose in the bloodstream would approach c/r.

<u>Exercises</u>

6. Assume the physician orders an infusion of 10 grams of glucose per hour for a certain patient. Laboratory technicians determine that the patient has 2 grams of glucose in his bloodstream just prior to the start of the infusion, and that the patient's body will remove the glucose from the bloodstream at a rate of 3 grams per hour per gram of glucose.
   (a) How much glucose will be in the patient's bloodstream t hours after the infusion is started?
   (b) How much glucose will be in the patient's bloodstream after 2 hours?
   (c) How long will it take for the glucose level in the bloodstream of the patient to reach 3 grams?
   (d) Find the equilibrium amount of glucose in the bloodstream.
7. Suppose the bloodstream of a patient has 2 grams of glucose. Her physician wishes to bring this amount up to 3.5 grams in 3 hours. It is determined that the patient's system removes glucose from her bloodstream at a rate of 4 grams per hour per gram of glucose. How fast should the physician order the glucose to be infused into the patient's body?
8. (a) Using the first derivative, show that if G(t) is given by Equation (3.7), $G(0) < c/r$, and $r > 0$, then G(t) is an increasing function. (This will show that the glucose level is always increasing toward the equilibrium level of c/r, rather than sometimes being above c/r and sometimes below it.)
   (b) Under the assumptions of 7(a), show that $G''(t) < 0$ for all t.
   (c) Sketch the graph of Equation (3.7) for $t > 0$, assuming that $r > 0$ and $G(0) < c/r$.

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4. POPULATION GROWTH - A COMPETITION MODEL

4.1 The Basic Problem

In this section, we will study populations in which members of the group compete among themselves for food, water, etc. This competition will, in all likelihood, retard the growth rate of the group.

4.2 The Fundamental Assumptions of the Model

Let $A(t)$ denote the size of our population at time t, with initial population size $A(0) = N$. If our population were to grow without restriction, a reasonable model of its growth would be

$$\frac{dA}{dt} = r\, A, \; r > 0.$$

That is, the rate of growth would be proportional to the size of the population.

However, let us assume that competition between individuals tends to slow the growth rate. Suppose that two members of the group compete for resources (food, water, etc.) until one of the two succeeds in taking over the resources of the other. The loser then competes with one other member of the group to obtain a new supply of food and water. Such a situation might occur when a population controls a large territory, with each individual in the group controlling a smaller subterritory. One example might be a swarm of insects in a tomato patch, one bug to a leaf. At any given time there are $A(A-1)/2$ possible pairs of individuals which may be engaged in such competition. It then seems reasonable to assume that the rate of population growth is retarded by an amount proportional to $A(A-1)/2$.

4.3 A "Growth Rate" Equation

From our assumptions in Section 4.2, we have

$$\begin{aligned} \frac{dA}{dt} &= r\, A - cA(A-1) \quad , \; r > 0, \; c > 0 \\ &= k\, A - cA^2 \qquad , \; k = r + c \qquad (4.1) \\ &= A\, (k - cA). \end{aligned}$$

For simplicity, let $M = k/c$. Then Equation (4.1) becomes

$$\frac{dA}{dt} = c\, A(M-A) \qquad (4.2)$$

or equivalently,

(4.3) $$\frac{1}{A(M-A)} \frac{dA}{dt} = c.$$

4.4 The Size of the Population at Time t

Using partial fractions, we see that

$$\frac{1}{A(M-A)} = \frac{1}{M} \left(\frac{1}{A} + \frac{1}{M-A}\right).$$

Hence, integrating Equation (4.3) with respect to t, under the assumption that $M - A(t) > 0$ for $t > 0$, yields

(4.4) $$\frac{1}{M} [\ln A - \ln(M-A)] = ct + c_1, \quad c_1 \text{ constant,}$$

or equivalently,

(4.5) $$\frac{A}{M-A} = c_2 e^{cMt}, \quad c_2 = e^{Mc_1}.$$

Note that in obtaining Equation (4.4), we assumed that $M - A(t) > 0$. This is equivalent to assuming that $k - c A > 0$. In light of Equation (4.1), if the population is to be increasing in size, this latter assumption is not unreasonable. However, as we shall see below, all we actually need is that either $M - A(t) > 0$ or $M - A(t) < 0$. (See Exercise 10.)

Now solving Equation (4.5) for A, we obtain

(4.6) $$A(t) = \frac{M c_2}{c_2 + e^{-cMt}} .$$

Using the fact that $A(0) = N$, we have

(4.7) $$A(t) = \frac{M N}{N + (M-N)e^{-cMt}} .$$

Recall that in obtaining Equation (4.4) we assumed that $M - A(t) > 0$. However, it can be shown by direct substitution, that, so long as $N + (M-N)e^{-cMt}$ is not zero, Equation (4.7) is a solution to Equation (4.1). Hence, the requirement that $M - A(t) > 0$ is not necessary.

Note that if $M > N$, then $N + (M-n)e^{-cMt}$ will always be positive.

4.5 Bounds on the Population Size

Let us see what we can learn about our population from Equation (4.7).

First if we assume that $M > N$, manipulation with some inequalities shows that $N \leq A(t) < M$ for all $t > 0$. Now recall that $M = (r + c)/c = (r/c) + 1$, and that r is a "growth" constant and c is a "competition" constant. Then we see that if r is sufficiently bigger than c (so that

$(r/c) + 1 > N)$, the population size never gets below the initial size N, and never gets above M.

Further, observe that

$$\lim_{t\to\infty} A(t) = M.$$

Thus, M is the least upper bound for the population size.

4.6 The Maximum Rate of Increase of Population Size

If we compute the second derivative $A''(t)$ of $A(t)$ from Equation (4.7), we see that $A(t)$ has an inflection point

$$t_1 = -\frac{1}{cM}\ln\left(\frac{N}{M-N}\right).$$

Further, we see that the graph of $A(t)$ is concave up for $0 \le t < t_1$ and concave down for $t > t_1$. (See Figure 4.1).

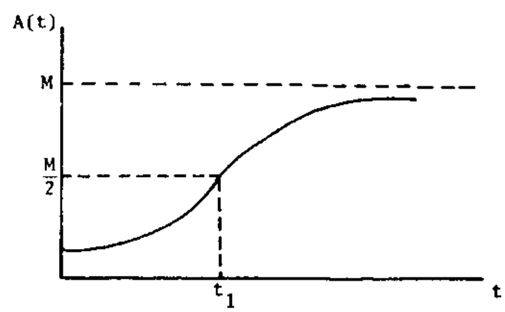

Figure 4-1. Graph of A(t), the population size at time t.

Thus, dA/dt, the rate of increase of the population size attains a maximum at t_1. By direct calculation, we see that $A(t_1) = M/2$. Hence, the rate of growth increases until the population reaches half its maximum size; from that point on, the rate of growth decreases. (See Figure 4.1)

4.7 Some History and Applications of the Model

Equations of the form of Equation (4.1) were first introduced about 1840 by the Belgian sociologist P. F. Verhurst. Verhurst used such equations, now called logistic or saturation curves, in studies dealing with the increase in human populations.

Logistic equations were rediscovered in the 1920's by American biologists R. Pearl and L. J. Read. They are now used to study such diverse problems as population growth of fruit flies in biology, and learning rates in psychology. (See, for example, Problem 3 on the Model Examination.)

Exercises

9. In Equation (4.6) find an expression for c_2 in terms of M and N. (Hint: Use the fact that A(0) = N.)

10. Derive an equation for A(t) under the assumption that M - A(t) < 0 for all t. Use Equation (4.3). (Hint: The equation

$$\frac{1}{A(M-A)}\frac{dA}{dt} = c \text{ is equivalent to the equation } \frac{1}{A(A-M)}\frac{dA}{dt} = -c.)$$

11. Suppose M = 200, N = 50, $c = \frac{1}{2}$.

    (a) Find t such that A(t) = 100.
    (b) Find t such that A(t) = 150.
    (c) Find t such that A(t) = 190.

5. WOLVES VERSUS RABBITS

5.1 A Battle For Survival

Now we will study a model of population growth in which one species is the principal source of food for another.

Suppose some wolves and some rabbits live in a certain forest. Of course, the wolves eat the rabbits. In fact, we shall assume that the rabbits are the principal food source for the wolves. However, if the wolves eat too many rabbits, so that few rabbits are left, the food source for the wolves will be greatly diminished. The wolves will begin to die off or leave the forest. As the wolf population decreases, the rabbit population will begin to increase, since not as many rabbits are being eaten. With the increase of the rabbit population, more food will become available for the wolves, so the wolf population will again begin to grow.

In this section, we want to look at a model for this "wolf-rabbit" competition. Our model was first described about 1925, independently, by A. J. Lotka, an American biophysicist, and Vito Volterra, an Italian mathematician. (See references for this section.) In his work, Lotka was studying the effects of certain parasites feeding off, and thus killing, most insects. Volterra was studying the Italian fishing industry just prior to and during World War I. The "wolves" in Volterra's study were sharks, and the "rabbits" were edible fish. The model we will disucss very closely approximated the observed situations in both Lotka's "parasite" study and Volterra's "fishing" study. Since the original work of Lotka and Volterra, their model and its refinements have been used to study predator-prey relationships between species in a wide variety of situations. See, for example, Huffaker, Leslie and Gower, and Pielou in the references for this section.

5.2 The Underlying Assumptions and the Basic Equations

Let x(t) denote the number of rabbits present at time t, and let y(t) denote the number of wolves present at time t. Our first task will be to write dx/dt and dy/dt as functions of time t.

First, consider the rabbits. If the rabbits were permitted to grow without restraint, i.e. without being eaten by wolves, the size of the population would grow very rapidly. Then a reasonable model of the rabbit growth rate would be

$$\frac{dx}{dt} = rx,$$

where r is a positive constant. This equation is the usual Malthusian exponential growth equation. In fact, however, the rabbits are being restrained (eaten), so that their growth rate is diminished. We assume that this "diminishing" is proportional to the number of possible pairs of wolves and rabbits since the two species normally interact in pairs. (One wolf eats one rabbit.) Thus, a reasonable assumption for the rate of growth of the rabbit population is

$$\frac{dx}{dt} = rx - axy \tag{5.1}$$

where r and a are positive constants.

Using similar reasoning, we obtain as a model for the rate of growth of the wolf population

$$\frac{dy}{dt} = -sy + bxy \tag{5.2}$$

where b and s are positive constants.

Note that b is assumed to be positive since, with lots of rabbits available, the wolf population will increase. If few rabbits are available, the wolf population will decrease. Thus, we also assume that s is positive.

Equations (5.1) and (5.2) are called the Lotka-Volterra equations.

5.3 An Equation Relating the Two Population Sizes

Since the number of wolves will depend upon the number of rabbits available for food, we assume that y is a function of x. Then from the chain rule for derivatives, we have

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

Equivalently,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(bx - s)y}{(r - ay)x} . \tag{5.3}$$

Rewriting Equation (5.3), we obtain

$$(\frac{r}{y} - a) \frac{dy}{dx} = (b - \frac{s}{x}). \tag{5.4}$$

Integrating Equation (5.4) with respect to x, we obtain

$$r \ln y - ay = bx - s \ln x + k, \text{ k constant,} \tag{5.5}$$

or,

$$r \ln y + s \ln x - ay - bx = k. \tag{5.6}$$

Using properties of the natural logarithm function and the exponential function, we obtain an equivalent form of Equation (5.6) to be

$$\frac{y^r}{e^{ay}} \frac{x^s}{e^{bx}} = c, \text{ c constant.} \tag{5.7}$$

Equation (5.7), which we shall call the "wolf-rabbit" equation, is the basic equation we shall use to investigate the "wolves versus rabbits" survival situation described in Section 5.1.

Exercise

12. Derive Equation (5.7) from Equation (5.6).

5.4 Solutions to the Wolf-Rabbit Equation

Now let us assume that y is a fixed number. We want to determine how many corresponding values of x there are. That is, for a given number y of wolves in the forest, could there be more than one size x for the corresponding rabbit population?

Equation (5.7) can be rewritten as

$$\frac{x^s}{e^{bx}} = c \frac{e^{ay}}{y^r}.$$

Now let $f(x) = x^s/e^{bx}$. Using the first derivative test (see Exercise 13), we see that f(x) has a unique relative maximum when x = s/b and no relative minima. Further, f(0) = 0, and by use of L'Hôpital's Rule, we see that

$$\lim_{x \to \infty} f(x) = 0.$$

Thus, $f(x)$ assumes each positive value less than $f(s/b)$ exactly twice, and it assumes the values $f(s/b)$ and 0 exactly once each. The graph of $f(x)$ is sketched in Figure 5-1.

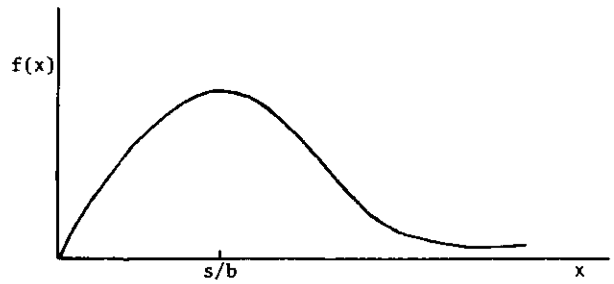

Figure 5-1. The graph of $f(x) = x^s/e^{bx}$.

From our analysis above, we can conclude that if the wolf population size y, is such that

$$\frac{c\ e^{ay}}{y^r} < f(\tfrac{s}{b}) = \frac{\left(\frac{s}{b}\right)^s}{e^s},$$

then there are exactly two corresponding sizes for the rabbit population; if

$$\frac{c\ e^{ay}}{y^r} = f(\tfrac{s}{b}),$$

then there is exactly one size for the rabbit population, namely $x = s/b$; if

$$\frac{c\ e^{ay}}{y^r} > f(\tfrac{s}{b}),$$

no solution for x is possible.

A similar analysis holds if we assume the rabbit population is fixed and try to determine the number of corresponding values for y.

5.5 Maximum and Minimum Values of the Population Sizes

Next, let us see if there are bounds on the population sizes of the two species.

From Equation (5.1), we see that if x has maximum or minimum values, such values must occur when $y = r/a$. Let us suppose that exactly two values of x, say x_1 and x_2, exist such that $(x_1, r/a)$ and $(x_2, r/a)$ satisfy Equation (5.7). (Recall our results on the number of solutions to

Equation (5.7) from Section 5.4. Also see Exercise 13(a).) Computing d^2x/dt^2 from Equation (5.1), we see that a member of the set $\{x_1, x_2\}$ less than s/b yields a minimum for x, while a member of the set $\{x_1, x_2\}$ greater than s/b yields a maximum for x. (See Exercise 13.)

Similarly, we can use Equation (5.2) to show that maximum and minimum values for y occur only if x = s/b. If the constants are such that there are two values of y, say y_1 and y_2, such that (s/b, y_1) and (s/b, y_2) satisfy Equation (5.7) (see Exercise 14), then the maximum for y will be the member of the set $\{y_1, y_2\}$ that is greater than r/a, and the minimum will be the value of y which is less than r/a. Again, see Exercise 13.

5.6 Graphing the Wolf-Rabbit Equation

We have shown that for most values of y, there are either exactly two or zero corresponding values of x satisfying Equation (5.7). Similarly, we have shown that for most values of x, there are either exactly two or zero corresponding values of y satisfying Equation (5.7). These results suggest that the graph of the Wolf-Rabbit equation may be somewhat like an ellipse or an oval.

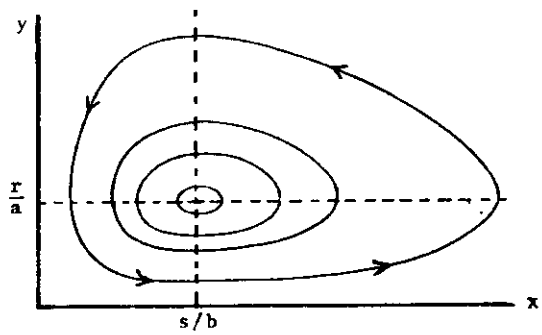

Figure 5-2. Graphs of the Wolf-Rabbit Equation.

We give several possibilities for the graph of Equation (5.7) in Figure 5-2. The outer paths correspond to smaller values of c.

One further point is worth noting here. From Equation (5.1), we see that $dx/dt > 0$ if $y < r/a$ and $dx/dt < 0$ if $y > r/a$. Thus, x is increasing if $y < r/a$ and x is decreasing if $y > r/a$. This tells us that the motion around each path in Figure 5-2 is counterclockwise.

5.7 The Average Population Sizes

If $z = h(u)$ is a continuous function of u on a closed interval $a \leq u \leq b$, the average value of h(u), $\bar{z}$, is defined to be

$$\bar{z} = \frac{1}{b - a} \int_a^b h(u)\ du.$$

Using this definition, we are able to calculate the average number of wolves and the average number of rabbits present at any given time.

Let T denote the time necessary to complete one cycle of Figure 5-2. Then since the functions x(t) and y(t) are cyclic, we have

$$\bar{y} = \frac{1}{T} \int_0^T y(t)\ dt, \quad \bar{x} = \frac{1}{T} \int_0^T x(t)\ dt.$$

From Equation (5.2) we have

$$\frac{1}{y} \frac{dy}{dt} = bx - s$$

so that

$$x = \frac{1}{b} (s + \frac{1}{y} \frac{dy}{dt}).$$

Therefore,

$$\bar{x} = \frac{1}{T} \int_0^T \frac{1}{b} (s + \frac{1}{y} \frac{dy}{dt})\ dt$$

$$= \frac{1}{bT} (st + \ln y(t) \Big|_0^T$$

$$= \frac{1}{bT} sT$$

$$= \frac{s}{b} .$$

Note that in the preceding equation, we need the fact that $y(T) = y(0) > 0$. Thus, the average number of rabbits present at any given time is s/b. (Be sure to carry out the details).

Similarly, using Equation (5.1), we find that the average number of wolves present at any given time is r/a. (See Exercise 15.)

5.B Some Limitations of the Model

You should be aware that several factors have not been taken into account in this model. First, we have assumed that the only cause of diminution of the wolf population is a lack of rabbits for food. We have not taken into account such factors as hunters, forest fires, other food sources. Can you think of some other factors we have neglected? (See Exercise 17.)

Similarly, we have assumed that the only cause of diminution of the rabbit population is predation by wolves. We have ignored such factors as hunters, other predators, and food supply. What other factors could affect the size of the rabbit population? (See Exercise 17.)

In spite of the omissions just mentioned, studies have shown that our model gives good approximations to the interaction between certain species in many "predator-prey" situations. (See the references for this section.)

Exercises

13. Let $f(x) = \frac{x^s}{e^{bx}}$, $s > 0$, $b > 0$, $x > 0$. Using the first derivative test, show that $f(x)$ has a unique relative maximum when $x = s/b$, and that $f(x)$ has no relative minima.

14. (a) From Equation (5.1) compute $\frac{d^2x}{dt^2}$.

    (b) Let $(x_1, r/a)$ be a solution of Equation (5.7). (See the discussion in Section 5.5). By substitution in d^2x/dt^2, show that if $x_1 < s/b$, then $x(t)$ is a minimum, and if $x_1 > s/b$, then $x(t)$ is a maximum. (Hint: From Equation (5.2), $dy/dt = -sy + bxy$.)

    (c) From Equation (5.2), compute d^2y/dt^2.

    (d) Let $(s/b, y_1)$ be a solution of Equation (5.7). (See the discussion in Section 5.5.) By substitution in d^2y/dt^2, show that if $y_1 > r/a$, then $y(t)$ is a maximum, and if $y_1 < r/a$, then $y(t)$ is a minimum. (Hint: From Equation (5.1), $dx/dt = rx - axy$.)

15. (a) Using methods similar to those in Section 5.7 show that $\bar{y} = r/a$.

    (b) If $a = 2$, $b = 3$, $r = 120$, and $s = 450$, find the average number of wolves present, and the average number of rabbits present at any given time.

16. (a) Assume that values for a, b, r, and s are given. Show that in order for there to be two values of x, say x_1 and x_2, for which $(x_1, r/a)$ and $(x_2, r/a)$ satisfy Equation (5.7), it must be true that

$$c < \frac{(\frac{r}{a})^r (\frac{s}{b})^s}{e^{r+s}} . \qquad (5.8)$$

    (Hint: Recall Section 5.4.)

    (b) Compute the right side of the inequality in (5.8) for $r = 4$, $a = 2$, $s = 3$, and $b = 1$, to two decimal places. (You will probably need a calculator for this exercise.) This will give the least upper bound for values of c.

17. (a) List some factors, other than those given in Section 5.8, which could affect the size of the wolf population.

(b) List some factors, other than those given in Section 5.8, which could affect the size of the rabbit population.

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6. MODEL EXAMINATION

1. Find the percentage of the original amount of C^{14} in the remains of an organism 50,000 years after the death of the organism.

2. Newton's Law of Cooling states that the rate at which a body cools is proportional to the difference between its temperature and the temperature of the surrounding medium. Suppose an object is placed in air whose temperature is $30^{\circ}C$. Let y(t) be the temperature of the object at time t, (t measured in hours) and let k be the constant of proportionality.

   (a) Write a differential equation for the rate of change of the temperature of the object.

   (b) If the initial temperature of the object is $120^{\circ}C$, and its temperature one hour later is $60^{\circ}C$, find an expression for y(t) as a function of t. (Hint: The technique developed in Section 3.2 may help to solve your equation in part (a).)

   (c) How long does it take for the object to cool to a temperature of $40^{\circ}C$?

3. A learning psychologist is trying to teach a group of monkeys to do a certain trick. He works with each monkey over a period of time. The monkey is rewarded each time it performs the trick correctly. It is hoped that this increases the chances that the monkey will perform the trick correctly at future times.

   Let p(t) denote the probability that the monkey will perform the trick correctly t minutes after the start of the experiment. The psychologist deduces that a reasonable expression for the rate of change of p(t) is

$$\frac{dp}{dt} = kp\ [a(1-p) - bp]$$

   where k, a, and b are constants, $a > 0$, $b > 0$, and $[a(1-p) - b\,p] > 0$.

   (a) Derive an expression for p(t) in terms of k, a, b, and p(0). (Hint: First group "like terms" in the brackets portion of the above equation. Then use techniques similar to those used in Section 4 to solve the equation for p(t).

(b) Find $\lim_{t \to \infty} p(t)$.

Remark. The model discussed in the problem above was developed by Robert Bush and Frederick Mosteller, two pioneers in the field of learning psychology. (See the references for Section 6.)

7. ANSWERS TO EXERCISES

1. (a) $y'(t) = ky(0)e^{kt} = ky(t)$.
   (b) Take the natural logarithm of both sides of the equation.

2. (a) About 12648 B.C. (about 14648 years ago.)
   (b) About 12176 B.C. (About 14176 years ago.)
   (c) About 13149 B.C. (About 15149 years ago.)

3. 78.5%

4. About 19035 years ago.

5. First of all, we differentiate

$$f(t) = \frac{b}{a} + \left(f(0) - \frac{b}{a}\right)e^{-at}$$

to obtain

$$\frac{df}{dt} = 0 - a\left(f(0) - \frac{b}{a}\right)e^{-at}$$

$$= -af(0)\,e^{-at} + be^{at}$$

when we add this latter expression to

$$a\,f(t) = b + a\,f(0)\,e^{-at} - be^{-at}$$

we get Equation (3.1):

$$\frac{df}{dt} + a\,f(t) = b.$$

6. (a) $G(t) = 10/3 - (4/3)\,e^{-3t}$.
   (b) 3.33 grams.
   (c) 0.46 hours or between 27 and 28 minutes.
   (d) 3 1/3 grams.

7. About 14 grams per hour.

8. (a) $G'(t) = -(G(0) - c/r)\,re^{-rt} > 0$.
   (b) $G''(t) = (G(0) - c/r)\,r^2\,e^{-rt} < 0$.
   (c) G(t).

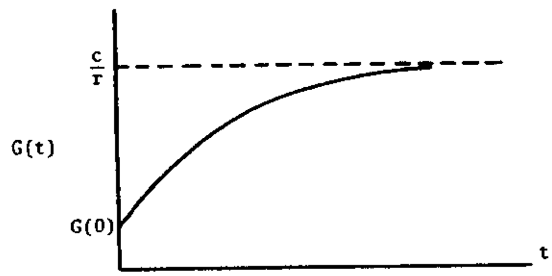

Figure 7-1. Graph of $G(t) = c/r + (G(0)-(c/r))e^{rt}$.

9. $c_2 = \frac{N}{M-N}$.

10. $A(t) = \frac{M\,N}{N+(M-N)\,e^{-cmt}}$.

11. (a) 0.011.
    (b) 0.022.
    (c) 0.040.

12. $r \ln y + s \ln x - ay - bx = k$

$$\ln y^r + \ln x^s = ay + bx + k$$

$$\ln y^r x^s = ay + bx + k$$

$$y^r x^s = e^{ay+bx+k}$$

$$= e^{ay} \cdot e^{bx} \cdot e^k$$

$$\left(\frac{y^r}{e^{ay}}\right)\left(\frac{x^s}{e^{bx}}\right) = e^k.$$

In Equation (5.7) the constant e^k is denoted by the letter c.

13. $f'(x) = \frac{\frac{1}{b} x^{s-1}(\frac{s}{b} - x)}{e^{bx}}$

$f'(x) > 0$ for $x < s/b$ and $f'(x) < 0$ for $x > s/b$.

14. (a) $\frac{d^2x}{dt^2} = r\frac{dx}{dt} - a\frac{dx}{dt}y - ax\frac{dy}{dt}$

(b) From Equation (5.1), if $y = r/a$, then $dx/dt = 0$. Then if $x = x_1$ and $y = r/a$, we have

$$\frac{d^2x}{dt^2} = -ax_1\frac{dy}{dt}$$

$$= -ax_1(-sy + bx_1y)$$

$= -ax_1y\ (-s + bx_1)$

$= -rx_1\ (-s + bx_1).$

Then

$$\frac{d^2x}{dt^2} > 0 \quad \text{if } x_1 < \frac{s}{b} \quad \text{and} \quad \frac{d^2x}{dt^2} < 0 \quad \text{if } x_1 > s/b.$$

(c) $\dfrac{d^2y}{dt^2} = -s\dfrac{dy}{dt} + b\dfrac{dx}{dt}y + bx\dfrac{dy}{dt}.$

(d) From Equation (5.2), if $x = s/b$, then $dy/dt = 0$. Then if $x = s/b$, $y = y_1$, we have

$$\frac{d^2y}{dt^2} = by_1\frac{dx}{dt}$$

$= by_1\ (rx - axy_1)$

$= by_1x\ (r - ay_1)$

$= sy_1\ (r - ay_1).$

Therefore, $\dfrac{d^2y}{dt^2} > 0$ if $y_1 < \dfrac{r}{a}$ and $\dfrac{d^2y}{dt^2} < 0$ if $y_1 > \dfrac{r}{a}.$

15. (a) $\overline{y} = \dfrac{1}{T}\displaystyle\int_0^T y(t)\,dt = \dfrac{1}{T}\int_0^T \frac{s}{s}\left(r - \frac{1}{x}\frac{dx}{dt}\right)dt = \frac{r}{a}$

(b) Average number of wolves: 60
Average number of rabbits: 150

16. (a) Since $\dfrac{c\ e^{ay}}{y^r} < \dfrac{\left(\frac{s}{b}\right)^s}{e^s}$, we must have

$$c < \frac{y^r\left(\frac{s}{b}\right)^s}{e^{ay}\ e^s},$$

If $y = r/a$, we have

$$c < \frac{\left(\frac{r}{a}\right)^r\left(\frac{s}{b}\right)^s}{e^{r+s}}$$

(b) 0.39.

17. (a) Severe weather such as draught, and snowstorms; disease.
(b) Severe weather; disease.

8. ANSWERS TO MODEL EXAMINATION

1. $\frac{y(50000)}{y(0)} = .0024$, i.e. about 0.24 per cent.

2. (a) $dy/dt = k(y(t) - 30)$, $y(t)$ the temperature at time t, k constant.

   (b) $y(t) = 90\, e^{kt} + 30$

   $y(t) = 90\,(1/3)^t + 30$, or $y(t) = 90\,(3^{-t}) + 30$ or

   $y(t) = 90\, e^{-t\ln 3} + 30$.

   (c) 2 hours.

3. (a) $p(t) = \frac{aKe^{akt}}{1+Ke^{akt}(a+b)}$, $K = \frac{p(0)}{a-(a+b)p(0)}$

   (b) $\lim_{t\to\infty} p(t) = \frac{a}{a+b}$

9. REFERENCES AND SOURCES FOR FURTHER READING

Section 2

Canby, Thomas Y., "The Search for the First Americans," National Geographic 156, No. 3, 1979, pp 330-363. See especially page 351.

Glyn, Daniel, "Megalithic Monuments," Scientific American 243, No. 1, 1980, pp. 78-90. See especially page 87.

Johanon, Donald C., "Ethiopia Yields First 'Family' of Early Man," National Geographic 150, No. 6, 1976, pp. 790-811. See especially page 801.

Section 3

Grossman, Stanley I., and Turner, James. Mathematics for the Biological Sciences. New York: Macmillan, 1974, p. 308.

Section 4

Maki, Daniel P., and Thompson, Maynard. Mathematical Models and Applications. Englewood Cliffs, N. J.: Prentice-Hall, 1973, pp. 313-318.

Section 5

Huffaker, C.B., "Experimental Studies on Predation: Dispersion Factors and Predator-Prey Oscillations." Reprinted in Readings in Population and Community Biology, W.E. Hazen, ed. Philadelphia: Saunders, 1970.

Kemeny, John G., and Snell, Laurie J. Mathematical Models in the Social Sciences, Cambridge, MA: The MIT Press, pp. 24-32.

Leslie, P.H. and Gower, J.C., "The Properties of Stochastic Model for the Predator-Prey Type of Interaction Between Two Species," Biometrika 46, 1960, pp. 219-234.
Lotka, Alfred J., Elements of Mathematical Biology, New York: Dover, 1965, pp. 83-92.
Pielou, E.C., An Introduction to Mathematical Ecology, New York: Wiley, 1969, pp. 67-68.
Volterra, Vito. "Lecons sur la Theorie Mathematique de la Lutta pour la vie" ("Lessons on the Mathematical Theory of the Fight for Survival"), Cahier Scientifique, Vol VII, 1931.

Section 6

Bush, R.R. and Mosteller, F., "A Mathematical Model for Simple Learning," Pschological Review 58, 1951, pp. 313-323.

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