

UMAP

Modules in
Undergraduate
Mathematics
and Its
Applications

Published in
cooperation with
the Society
for Industrial
and Applied
Mathematics, the
Mathematical
Association of
America, the
National Council
of Teachers of
Mathematics,
the American
Mathematical
Association of
Two-Year Colleges,
The Institute
of Management
Sciences, and the
American Statistical



Module 308

The Richardson Arms Race Model

Dina A. Zinnes
John Gillespie
G.S. Tahim

"Permit me to discuss a generalized public speech, fictitious but typical of the year 1937. The Defense Minister of Jedesland, when introducing his estimates, said:

'The intentions of our country are entirely pacific. We have given ample evidence of this by the treaties which we have recently concluded with our neighbors. Yet, when we consider the state of unrest in the world at large and the menaces by which we are surrounded, we should be failing in our duty as a government if we did not take adequate steps to increase the defenses of our beloved land.'

-- Richardson, L.F. 1960. *Statistics of Deadly Quarrels*. Pittsburgh: Boxwood Press, p. 14.

Applications of First Order
Differential Equations
to International Relations

MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS (UMAP) PROJECT

The goal of UMAP was to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications to be used to supplement existing courses and from which complete courses may eventually be built.

The Project was guided by a National Advisory Board of mathematicians, scientists, and educators. UMAP was funded by a grant from the National Science Foundation and is now supported by the Consortium for Mathematics and Its Applications (COMAP), Inc., a non-profit corporation engaged in research and development in mathematics education.

COMAP STAFF

Paul J. Campbell	Editor
Solomon A. Garfunkel	Executive Director, COMAP
Laurie Aragón	Development Director
Roland D. Cheyney	Project Manager
Philip A. McGaw	Production Manager
Laurie M. Holbrook	Copy Editor
Dale Horn	Design Assistant
Robin Altomonte	Distribution

This material was prepared with the support of National Science Foundation Grant No. SED76-19615 A02 and No. SPE 8304192. Recommendations expressed are those of the author and do not necessarily reflect the views of the NSF or of COMAP.

THE RICHARDSON ARMS RACE MODEL

by

Dina A. Zinnes, John V. Gillespie
and

G.S. Tahim

Center for International Policy Studies
Indiana University
Bloomington, Indiana 47401

TABLE OF CONTENTS

1. INTRODUCTION	1
2. THE ARMS RACE MODEL	3
3. RICHARDSON'S COUPLED EQUATIONS AS A SINGLE EQUATION	8
4. RICHARDSON'S ARMS RACE MODEL IN MATRIX NOTATION	10
5. STABILITY PROPERTIES OF THE RICHARDSON ARMS RACE MODEL	21
6. A SENSITIVITY ANALYSIS OF THE RICHARDSON ARMS RACE MODEL	29
7. CONCLUSION	32
8. REFERENCES	33
9. ANSWERS TO EXERCISES	34

Intermodular Description Sheet: UMAP Unit 308

Title: THE RICHARDSON ARMS RACE MODEL

Authors: Dina A. Zinnes, John V. Gillespie, and G.S. Tahim
Center for International Policy Studies
Indiana University
Bloomington, IN 47401

Review Stage/Date: IV 11/20/80

Classification: APPL DIFF EQUA/INTER REL

Prerequisite Skills:

1. Know how to integrate and differentiate.
2. Be familiar with a first-order differential equation and its solution.
3. Know how to calculate the limit of a function.

Output Skills:

1. Demonstrate how interacting entities like nations can be represented by second order difference equations.
2. Demonstrate alternate ways of representing such sets of equations.
3. Illustrate the usefulness of analyzing the equilibrium, stability, and sensitivity properties of an interactive model.

Related Units:

The Geometry of the Arms Race (Unit 311)

This unit was presented in preliminary form at the Shambaugh Conference on Mathematics in Political Science Instruction held December, 1977 at the University of Iowa. The Shambaugh Fund was established in memory of Benjamin F. Shambaugh who was the first and for forty years served as the chairman of the Department of Political Science at the University of Iowa. The funds bequeathed in his memory have permitted the department to sponsor a series of lectures and conferences on research and instructional topics. The Project would like to thank the participants in the Shambaugh Conference for their reviews, and all others who assisted in the production of this unit.

This unit was field-tested and/or student reviewed in preliminary form by: Carole Uhlener of Northwestern University; Deborah Frank Lockhart of Michigan Technical University; Donald T. Snow of Brigham Young University; and Jonathan Choate of The Groton School, and has been revised on the basis of data received from these sites.

1. INTRODUCTION

Why do wars occur? A question as broad as this necessarily has a variety of possible answers. But one answer that many agree upon is that the existence of weapons--military arsenals--increases the likelihood of violent conflict. Without weapons, it is argued, men would be forced to settle their differences by other means. This is the basic assumption behind the long history of negotiations that have sought to limit the military capabilities of major powers.

It was also the assumption that led Lewis Fry Richardson to begin his study of arms races. Richardson was a Quaker by conviction and deeply troubled by the two major wars. His scientific training (physicist) led him to believe that wars were phenomena that could be studied, explained, and thus ultimately controlled. Towards this end he collected considerable data on wars and performed numerous statistical tests (Statistics of Deadly Quarrels, 1960a). But his most famous undertaking was the construction of a model to represent an arms race (Arms and Insecurity, 1960b).

Richardson believed that arms races were often, but not always, preludes to war. Hence, if he could obtain a reasonable representation of how and why nations increase or decrease their arms expenditures for defense, a study of the dynamics of this process might expose the conditions under which wars would be likely. If hostile nations were pumping greater and greater amounts of money into their defense budgets at an accelerating rate then perhaps a small spark (like the assassination of an Archduke) would start a major conflagration. On the other

hand, if two hostile nations were decreasing their defense expenditures, small incidents might be less likely to lead to violence.

This unit contains five parts. In the next section we will show how Richardson used the mathematics of coupled linear difference equations to represent an arms race between two nations. Following this presentation we will demonstrate that the arms race model can be written in two alternate and sometimes more convenient forms. Then, having constructed the model, we will explore three of its properties. Like Richardson, we will be interested in considering the equilibrium point of the system. This analysis will not only highlight one of the properties of the model but will also illustrate the usefulness of one of the alternate representations of the model, namely its matrix formulation. The second analysis will also follow the work of Richardson. Here we will be interested in the "stability" properties of the model. Runaway arms races, arms races which are continually accelerating, are in effect "unstable" reaction systems. If, as Richardson thought, these might be settings within which wars could occur, then it is important to find the set of conditions under which the arms race model can be said to be "stable" or "unstable." Our analysis of the stability properties of the arms race model will differ from that given by Richardson, but the results are the same.

Our last set of analyses consider the importance of the parameters of the model. When Richardson developed

and analyzed his arms race model, "sensitivity" analysis, as it is now called, was relatively new and unknown. Thus this last analysis has no parallel in the work of Richardson but demonstrates yet another property of the model that can be analyzed. A sensitivity analysis shows which parameters of the model, when changed slightly, will have the greatest impact on the dynamics of the process. If an arms race can be shown to be stable and it is known that slight adjustments in a particular parameter can have a major impact on the dynamics, then to guarantee stability, decision makers should not tamper with this parameter.

2. THE ARMS RACE MODEL

Since it would be difficult to improve on Richardson's presentation of his arms race model, we will let him provide the steps in the modeling process. Richardson's formulation, however, was in terms of linear differential equations; our presentation takes the form of linear difference equations. The mathematics of the two formulations are similar so this slight alteration in no way affects Richardson's argument or analysis. The use of difference equations, however, does require that we specify

$x(n)$ = the armament of nation X at $t = n$

where t is time, and that the change in armament level for X from $t = n-1$ to $t = n$ be represented by:

$$\Delta x(n) = x(n) - x(n-1). \quad (2.1)$$

Similarly, for another nation Y we note:

$$\Delta y(n) = y(n) - y(n-1). \quad (2.2)$$

We will now follow Richardson's discussion (1960b, p. 14):

"Permit me to discuss a generalized public speech, fictitious but typical of the year 1937. The Defense Minister of Jedsland, when introducing his estimates, said:

"'The intentions of our country are entirely pacific. We have given ample evidence of this by the treaties which we have recently concluded with our neighbors. Yet, when we consider the state of unrest in the world at large and the menaces by which we are surrounded, we should be failing in our duty as a government if we did not take adequate steps to increase the defenses of our beloved land.'

". . .the simplest representation of what that generalized defense minister said is this:"

$$\Delta x(n) = \delta_1 y(n-1) \quad (2.3)$$

"where [n] . . . is time, x represents his own defenses, y represents the menaces by which he is surrounded, and [δ_1] is a positive constant, which will be named a 'defense coefficient.' Let us for simplicity assume that what he euphemistically called 'surroundings' is, in fact, a single nation. Its defense minister asserts similarly that"

$$\Delta y(n) = \delta_2 x(n-1). \quad (2.4)$$

But, continues Richardson (p. 15):

"Surely the cost of armaments exercises some restraint. Leading statesmen have expressed this opinion. Thus Mr. Winston Churchill, (1923, p. 29) records that on November 3, 1909, while he was President of the Board of Trade, he began a minute to the Cabinet with these words:

"Believing that there are practically no checks upon German naval expansion except those imposed by the increasing difficulties of getting money, I have had the enclosed report prepared with a view to showing how far those limitations are becoming effective. It is clear that they are becoming terribly effective.

". . . So let the equations be improved. . ."

$$\Delta x(n) = \delta_1 y(n-1) - \alpha_1 x(n-1) \quad (2.5)$$

$$\Delta y(n) = \delta_2 x(n-1) - \alpha_2 y(n-1) \quad (2.6)$$

"where . . . [α_1 and α_2] are positive constants representing the fatigue and expense of keeping defenses. . ."

But Richardson is still not satisfied and he quotes Mr.

L. S. Amery, M.P. in the House of Commons on July 20, 1936,

in a discussion of World War I causes (p. 16):

". . . The armaments were only the symptoms of the conflict of ambitions and ideals, of those nationalist forces, which created the war. The War was brought about because Serbia, Italy, and Rumania passionately desired the incorporation in their States of territories which at that time belonged to the Austrian Empire and which the Austrian Government were not prepared to abandon without a struggle. France was prepared if the opportunity ever came to make an effort to recover Alsace-Lorraine. It was in those facts, in those insoluble conflicts of ambitions and not in the armaments themselves that the cause of the War lay."

And so Richardson proposes (p. 16):

"Mr. Amery's objections should, I think, be met. . . by inserting additional terms, namely g and h, to represent grievances and ambitions, provisionally regarded as constants, so that the equations become"

$$\Delta x(n) = \delta_1 y(n-1) - \alpha_1 x(n-1) + g \quad (2.7)$$

$$\Delta y(n) = \delta_2 x(n-1) - \alpha_2 y(n-1) + h. \quad (2.8)$$

As we have seen, in Richardson's construction of the model the parameters δ_1 , α_1 , g and h had very special meanings which suggested that these constants be positive. However, it has since been argued (Zinnes, Gillespie and Rubison, 1976) that negative parameters can have equally relevant interpretations and that both mathematically and substantively it makes more sense to consider a general model in which parameters are not constrained. We therefore rewrite (2.7) and (2.8) in a more standard form:

$$\Delta x(n) = \alpha_1 x(n-1) + \delta_1 y(n-1) + g \quad (2.9)$$

$$\Delta y(n) = \alpha_2 y(n-1) + \delta_2 x(n-1) + h \quad (2.10)$$

and using (2.1) and (2.2) obtain

$$x(n) = (1 + \alpha_1)x(n-1) + \delta_1 y(n-1) + g \quad (2.11)$$

$$y(n) = \delta_2 x(n-1) + (1 + \alpha_2)y(n-1) + h. \quad (2.12)$$

If we define

$$1 + \alpha_1 \equiv \beta_1, \quad 1 + \alpha_2 \equiv \beta_2$$

we can write

$$x(n) = \beta_1 x(n-1) + \delta_1 y(n-1) + g \quad (2.13)$$

$$y(n) = \delta_2 x(n-1) + \beta_2 y(n-1) + h.$$

The pair of equations in (2.13) are coupled first order difference equations, representing a discrete interpretation of Richardson's arms race model for two nations.

In the following sections we explore two alternate but equivalent representations of the arms race model given by (2.13). In section 3 we show how the coupled equations of (2.13) can be decoupled so that the armament of X (and similarly of Y) can be rewritten solely in terms of X's (or Y's) past behavior. In so rewriting these equations we obtain second order difference equations. Furthermore, also it will be shown that in the absence of "grievances" X's armament process, rewritten solely in terms of its own past behavior, will be identical to Y's armament process, similarly rewritten only in terms of Y's past behavior.

In section 4 we provide yet a third representation of the arms race model using matrix algebra. Richardson's initial model considered only two interacting nations. However, in later chapters of his volume he suggests a more complex formulation involving n nations. To represent and subsequently analyze n interacting nations using the form given in (2.13) is exceedingly cumbersome. Hence to consider a more general model it is necessary to find a compact representation. Such a compact form can be obtained using matrix algebra. Although the analyses to be presented here will not involve n nations, it is useful to show how a two nation model can be represented in matrix form and how such a representation can aid the analysis. Thus following a discussion of matrix algebra, provided for those unfamiliar with basic definitions and operations, we will consider the equilibrium properties of the model.

3. RICHARDSON'S COUPLED EQUATIONS AS A SINGLE EQUATION

The equations (2.13) represent the behavior of the two nations such that the armament of the nation X depends upon the armament of the nation Y. These can be rewritten in a form such that the arms of a nation depend only on the arms of the same nation at previous times. In other words, we want to rewrite (2.13) which are coupled equations in an uncoupled form. To do so, we will rewrite the second equation of (2.13) by considering $y(n-1)$ and thus changing n to $n-1$, and $n-1$ to $n-2$:

$$y(n-1) = \delta_2 x(n-2) + \beta_2 y(n-2) + h. \quad (3.1)$$

Substitute (3.1) into the first equation of (2.13):

$$\begin{aligned} x(n) &= \beta_1 x(n-1) + \delta_1 [\delta_2 x(n-2) + \beta_2 y(n-2) + h] + g \\ &= \beta_1 x(n-1) + \delta_1 \delta_2 x(n-2) + \delta_1 \beta_2 y(n-2) + \\ &\quad \delta_1 h + g. \end{aligned} \quad (3.2)$$

Rewriting the first equation of (2.13) in terms of $y(n-1)$ we have

$$y(n-1) = \frac{1}{\delta_1} [x(n) - \beta_1 x(n-1) - g]. \quad (3.3)$$

If we now let n be $(n-1)$ and $(n-1)$ be $(n-2)$ we can rewrite (3.3) as

$$y(n-2) = \frac{1}{\delta_1} [x(n-1) - \beta_1 x(n-2) - g]$$

which can then be substituted into (3.2). With this substitution, some simplification and rearrangement we obtain:

$$x(n) = [\beta_1 + \beta_2]x(n-1) + [\delta_1\delta_2 - \beta_1\beta_2]x(n-2) + [\delta_1h + g - g\beta_2]. \quad (3.4)$$

In an analogous fashion we obtain:

$$y(n) = [\beta_1 + \beta_2]y(n-1) + [\delta_1\delta_2 - \beta_1\beta_2]y(n-2) + [\delta_2g + h - h\beta_1]. \quad (3.5)$$

Examining equations (3.4) and (3.5) we observe the following significant points. First, we have obtained an expression for x (and similarly for y) that reflects how X will arm as a function of X 's previous armament behavior. Y 's behavior is no longer reflected in X 's equation (and vice versa for Y). Second, equation (3.4) in contrast to the first equation of (2.13), represents X 's behavior at time n in terms of the previous two time points, $(n-1)$ and $(n-2)$. Thus equation (3.4) (and similarly 3.5) is a second order difference equation. Third, if we compare equations (3.4) and (3.5) we observe that if $g = h = 0$ the equations are completely parallel. Thus, in the absence of grievance and ambition the armament processes of the two nations are the same.

Exercises:

$$(1) \quad x(n) = 2x(n-1) + 7y(n-1)$$

$$y(n) = 4x(n-1) + 5y(n-1)$$

Using the above two coupled equations obtain a single equation involving only one variable x or y .

$$(2) \quad x(n) = 7y(n-1)$$

$$y(n) = 4x(n-1)$$

Decouple the above two equations to obtain a single equation in one variable. Comment on the difference between this single equation and that obtained in the first exercise.

4. RICHARDSON'S ARMS RACE MODEL IN MATRIX NOTATION

Richardson's arms race model can be written in yet a third form, using matrix algebra. Since not all students may be familiar with matrix algebra, we will provide the necessary definitions and explanations as we proceed.

A matrix is a rectangular array of numbers. The horizontal lines are called rows and vertical lines are called columns. Matrices are usually denoted by block letters, e.g. A, B, N, . . .

$$A \equiv \begin{bmatrix} a_{ij} \end{bmatrix}_{m \times n}$$

represents a matrix with m rows and n columns, $(m \times n)$ is the size or order of the matrix; $\begin{bmatrix} a_{ij} \end{bmatrix}_{2 \times 3}$ represents a matrix

with two rows and three columns and its (i,j) th entry is a_{ij} , i.e.

$$A \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}. \quad (4.1)$$

A matrix with one column (or one row) is called a vector (some authors call only a matrix with one column a vector but we shall use the more general convention); x is a vector and x can be:

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, [1, 2, 3], \begin{bmatrix} 1 \\ 0 \\ -2 \\ 4 \end{bmatrix}, \dots \quad (4.2)$$

The addition and subtraction of matrices is defined only for matrices of like sizes, i.e., an $(n \times p)$ matrix can only be added to (or subtracted from) another $(n \times p)$ matrix, e.g.

$$\begin{bmatrix} a_{11} & a_{12} & -a_{13} \\ a_{21} & -a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & -a_{13} + b_{13} \\ a_{21} + b_{21} & -a_{22} + b_{22} & a_{23} + b_{23} \end{bmatrix} \quad (4.3)$$

$$\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}.$$

Only conformable matrices can be multiplied.

Def.: A and B are called conformable if the number of columns of A and the number of rows of B are the same.

Rule: A: (n x p) - a matrix with n rows and p columns

B: (p x m) - a matrix with p rows and m columns

A and B can be multiplied as (n x p) and (p x m)
 ↑ ↑
 are conformable.

This is unlike the case with real numbers (or complex numbers) where any two can be multiplied. The size of the product matrix is: (n x m).

We shall illustrate the procedure for multiplication by the following examples:

$$\begin{array}{c}
 [1 \quad 0 \quad -1] \cdot \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} = \text{a matrix with 1 row and 1 column} \\
 \uparrow \qquad \qquad \qquad \uparrow \\
 (1 \times 3) \qquad \qquad \qquad (3 \times 1) \\
 \uparrow \\
 = (1) \cdot (1) + (0) \cdot (0) + (-1) \cdot (-4) \\
 = 1 + 0 + 4 \\
 = 5.
 \end{array}$$

$$\begin{array}{c}
 \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -7 \\ 9 \end{bmatrix} = \text{a matrix with 2 rows and 1 column} \\
 \uparrow \qquad \qquad \qquad \uparrow \\
 (2 \times 3) \qquad \qquad \qquad (3 \times 1) \\
 \uparrow \\
 = \begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix}
 \end{array}$$

where:

$$c_{11} = [2 \quad 1 \quad 0] \cdot \begin{bmatrix} 4 \\ -7 \\ 9 \end{bmatrix} = (2) \cdot (4) + (1) \cdot (-7) + (0) \cdot (9).$$

$$c_{21} = [1 \quad 2 \quad 4] \cdot \begin{bmatrix} 4 \\ -7 \\ 9 \end{bmatrix} = (1) \cdot (4) + (2) \cdot (-7) + (4) \cdot (9) \\
 = 4 - 14 + 36 = 26.$$

$$\begin{array}{c}
 \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \\
 (2 \times 2) \\
 \uparrow
 \end{array}
 \cdot
 \begin{array}{c}
 \begin{bmatrix} 1 & 0 & 4 \\ 1 & -1 & 0 \end{bmatrix} \\
 (2 \times 3) \\
 \uparrow
 \end{array}
 = \text{a matrix with 2 rows and 3 columns} \\
 = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

In the product matrix, the entry c_{ij} ; $i = 1, 2$, $j = 1, 2, 3$; is obtained by multiplying the i^{th} row of the first matrix with the j^{th} column of the second matrix as indicated below:

$$\begin{aligned}
 c_{11} &= [1 \ 2] \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (1) \cdot (1) + (2) \cdot (1) = 3 \\
 c_{12} &= [1 \ 2] \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} = (1) \cdot (0) + (2) \cdot (-1) = -2 \\
 c_{13} &= [1 \ 2] \cdot \begin{bmatrix} 4 \\ 0 \end{bmatrix} = (1) \cdot (4) + (2) \cdot (0) = 4 \\
 c_{21} &= [0 \ 4] \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (0) \cdot (1) + (4) \cdot (1) = 4 \\
 c_{22} &= [0 \ 4] \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} = (0) \cdot (0) + (4) \cdot (-1) = -4 \\
 c_{23} &= [0 \ 4] \cdot \begin{bmatrix} 4 \\ 0 \end{bmatrix} = (0) \cdot (4) + (4) \cdot (0) = 0.
 \end{aligned}$$

Division is not defined for matrices.

Given the above definitions it is now possible to rewrite the two coupled difference equations of (2.13) in matrix notation by defining the following:

$$\begin{aligned}
 X(n) &\equiv \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}, \quad X(n-1) \equiv \begin{bmatrix} x(n-1) \\ y(n-1) \end{bmatrix}, \\
 A &\equiv \begin{bmatrix} \beta_1 & \delta_1 \\ \delta_2 & \beta_2 \end{bmatrix}, \quad V \equiv \begin{bmatrix} g \\ h \end{bmatrix}
 \end{aligned} \tag{4.4}$$

The rule for matrix multiplication allows us to write:

$$X(n) = A X(n-1) + V, \quad n = 1, 2, 3, \dots \quad (4.5)$$

Consider for example the following set of parameter values for a specific realization of the Richardson model:

$$x(n) = 2x(n-1) + 3y(n-1) + g \quad (4.6)$$

$$y(n) = 5x(n-1) + 7y(n-1) + h.$$

As was done in (4.4), (4.6) can be rewritten in matrix notation:

$$X(n) = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} X(n-1) + V. \quad (4.7)$$

Exercises:

- (1) Rewrite the two equations in matrix notation:

$$x(n) = 7x(n-1)$$

$$y(n) = 2x(n-1) + 4y(n-1).$$

- (2) Rewrite the equations in matrix notation:

$$x(n) = 5y(n-1) + g$$

$$y(n) = 2x(n-1) + h.$$

- (3) What difference do you detect between Exercises

(1) and (2)?

Matrix notation allows us to represent systems of considerably greater complexity and size in a more compact form. Expansion of the set of equations only increases the size of the A matrix and the vectors X and V. This can be seen in the following example. Consider three nations in

an arms race. Assume the armament behavior is still described by the Richardson process, i.e.,

$$\begin{aligned} x(n) &= \alpha_1 x(n-1) + \alpha_2 y(n-1) + \alpha_3 z(n-1) + g_1 \\ y(n) &= \beta_1 x(n-1) + \beta_2 y(n-1) + \beta_3 z(n-1) + g_2 \\ z(n) &= \gamma_1 x(n-1) + \gamma_2 y(n-1) + \gamma_3 z(n-1) + g_3. \end{aligned} \tag{4.8}$$

By defining the following quantities:

$$X(n) \equiv \begin{bmatrix} x(n) \\ y(n) \\ z(n) \end{bmatrix}, \quad V \equiv \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}, \quad A \equiv \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \tag{4.9}$$

the equation (4.8) can be compactly rewritten as:

$$X(n) = A X(n-1) + V, \quad n = 1, 2, 3, \dots \tag{4.10}$$

Note that the equation (4.10) resembles (4.5), only the quantities, A , $X(n)$ and V are defined differently.

But a matrix representation not only allows us to write a system of equations in a more compact form. Once a system of equations is expressed in matrix notation the properties of that system can be more easily assessed using operations and theorems in matrix algebra. To illustrate this we will show how the equilibrium point of a system of difference equations can be found using matrix algebra. However, to do this we must define several additional operations and state a relevant theorem in matrix algebra.

To every $(n \times n)$ matrix, i.e., square matrix, A , is associated a number, called its determinant and denoted by

det. A. The determinant is also represented by two vertical lines, i.e. $\det. A \equiv |A|$. For a (2x2) matrix the determinant is defined as indicated below:

$$A \equiv \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \det. A \equiv \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \quad (4.11)$$

For example: $\det. \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} = 5$; $\det. \begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix} = 10 - 4 = 6$.

For a bigger order square matrix, the determinant is computed by the successive application of the expansion by rows (or columns) and using the definition (4.11) as shown below.

$$A \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \quad (4.12)$$

Before we evaluate the determinant of the above matrix let us define a useful term:

$A \equiv [a_{ij}]$ is an (n x n) matrix. Corresponding to the $(i, j)^{\text{th}}$ entry of the matrix A, the cofactor, denoted by Δ_{ij} , is defined as the determinant obtained after deleting the i^{th} row and the j^{th} column of A, multiplied by $(-1)^{i+j}$.

Consider the following example:

$$A \equiv \begin{bmatrix} 1 & 2 & 0 \\ 4 & 5 & 7 \\ 3 & 9 & 8 \end{bmatrix}$$

The cofactor associated with the (2,2) entry (i.e., 5),
 obtained by deleting the second row and second column is:

$$\Delta_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 0 \\ 3 & 8 \end{vmatrix} = 8$$

$$\Delta_{13} = (-1)^{1+3} \begin{vmatrix} 4 & 5 \\ 3 & 9 \end{vmatrix} = 36 - 15 = 21$$

Now we can write an expression for the determinant of (4.12):

$$\begin{aligned} \det. A &= a_{11}\Delta_{11} + a_{12}\Delta_{12} + a_{13}\Delta_{13} \quad (\text{expansion by first row}) \\ &= a_{31}\Delta_{31} + a_{32}\Delta_{32} + a_{33}\Delta_{33} \quad (\text{expansion by third row}) \\ &= a_{11}\Delta_{11} + a_{21}\Delta_{21} + a_{31}\Delta_{31} \quad (\text{expansion by first column}) \\ &= a_{12}\Delta_{21} + a_{22}\Delta_{22} + a_{32}\Delta_{32} \quad (\text{expansion by second column}) \end{aligned}$$

Exercise: Write the expressions for $\det. A$ as expansions
 by second row and third column.

As another example of finding the determinant of a (3x3) matrix,
 consider

$$\begin{aligned} \det. \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix} &= (1) \begin{vmatrix} 0 & -1 \\ 1 & 3 \end{vmatrix} - (2) \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + (0) \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \\ &= 1 - 2 \times 5 = -9. \end{aligned}$$

A very useful operation in matrix algebra is called
 the transpose. If $A = [a_{ij}]$ is a given matrix, then the
 $n \times m$

transpose of the matrix A , denoted by A^T is $\equiv [a_{ji}]$.
 $m \times n$

Essentially, transposing implies changing a row into a column.

Examples:

$$(1) \quad A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix}_{2 \times 3}, \quad A^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 3 \end{bmatrix}_{3 \times 2}$$

$$(ii) A = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}_{3 \times 1}, \quad A^T = \begin{bmatrix} 1 & 4 & 0 \end{bmatrix}_{1 \times 3}.$$

Let us now define the identity matrix, which is the matrix equivalent of the real identity, i.e., the real number 1. I_n is an $(n \times n)$ identity matrix if all of its main diagonal entries are the real number 1 and every other entry is zero, i.e.,

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}_{n \times n}.$$

Next we define the inverse of a matrix. Given an $(n \times n)$ matrix A such that $\det. A \neq 0$, we wish to find another matrix B such that $Bx = Ax = I_n$, where I_n is the identity matrix of order $(n \times n)$. The matrix B is called the inverse of matrix A and is usually denoted by $B \equiv A^{-1}$.

Finally we state an essential theorem:

Theorem: Given an $(n \times n)$ matrix A such that $\det. A \neq 0$,

$$A^{-1} = \frac{1}{\Delta} [\Delta_{ij}]^T, \quad \Delta \equiv \det. A, \quad \Delta_{ij} \text{ is the cofactor}$$

for the (i, j) entry of A .

Examples:

$$(1) A = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}, \quad \det. A = 4 + A^{-1} = \frac{1}{4} \begin{bmatrix} 4 & 0 \\ -3 & 1 \end{bmatrix}.$$

$$A \equiv \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \det. A = 1 + A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$A \equiv \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & -2 \\ 1 & 0 & -3 \end{bmatrix}, \det. A = 17 \rightarrow A^{-1} = \frac{1}{17} \begin{bmatrix} 3 & 6 & -4 \\ 7 & -3 & 2 \\ 1 & 2 & -7 \end{bmatrix}.$$

Given the definitions of determinant, transpose and inverse, together with the above theorem, we are able to find the equilibrium point for the three nation model given in (4.10). The equilibrium point in an arms race represents the level of arms such that the dynamics of the race cease. It is the value of the armaments which results in a no change (i.e., increase or decrease) in the armaments of the two nations.

Let X_e be the equilibrium armament which represents no change in the level of arms represented by the equation (4.10). To find an expression defining such an arms level let us subtract $X(n-1)$ from both sides of the equation (4.10), i.e.,

$$X(n) - X(n-1) = (A-I)X(n-1) + V.$$

The above equation can be rewritten compactly as:

$$\Delta X(n) = A_o X(n-1) + V \quad (4.13)$$

where $\Delta X(n)$ represents the change in armaments at the n^{th} instant and $A_o = A - I$. At the equilibrium point this change, $\Delta X(n)$, is zero and the equilibrium armament is: X_e , i.e.,

$$0 = A_o X_e + V$$

so that

$$X_e = -A_o^{-1} V. \quad (4.14)$$

We shall now compute the equilibrium point for an arms race represented by the following equation:

$$X(n) = \begin{bmatrix} 2 & 0 \\ 3 & 5 \end{bmatrix} X(n-1) + \begin{bmatrix} .1 \\ .5 \end{bmatrix}. \quad (4.15)$$

In view of the above equation we can write:

$$A_o = A - I = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} \rightarrow A_o^{-1} = \frac{1}{4} \begin{bmatrix} 4 & 0 \\ -3 & 1 \end{bmatrix}.$$

The equilibrium point for the arms race (4.15) is now

computed as:

$$\begin{aligned} X_e &= -A_o^{-1} V \\ &= -\frac{1}{4} \begin{bmatrix} 4 & 0 \\ -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} .1 \\ .5 \end{bmatrix} \\ &= -\begin{bmatrix} .1 \\ .05 \end{bmatrix}. \end{aligned} \quad (4.16)$$

We have shown how Richardson translated the fears and rationales of statesmen into a model of an arms race between two nations. The model, when constructed, was a coupled system of linear difference equations. We then showed how such a coupled system could be decoupled and represented as two independent second order difference equations. Finally, we saw that yet a third representation could be obtained for the arms race model by using matrix algebra and how such a representation allowed for an easy expansion of the model to n nations and for a simpler solution of

such properties as the equilibrium point. We turn then from these issues of how to represent the model to a discussion of what we can learn from the model--what are its stability and sensitivity properties?

5. STABILITY PROPERTIES OF THE RICHARDSON ARMS RACE MODEL

To meaningfully discuss the stability conditions for the arms race model it is necessary to define two relevant concepts: (1) initial condition and (2) solution. For simplicity, consider the following first order difference equation

$$y(n) = \rho y(n-1), n = 1, 2, \dots \quad (5.1)$$

An initial condition is the value of y at the time the process described by (5.1) began, viz.

$$y(0) = y_0$$

where y_0 is the initial condition. A solution describes the time path or trajectory, $y(n)$, in terms of the parameters of the system and the initial condition. A solution for (5.1) is given by:

$$y(n) = y_0 \rho^n. \quad (5.2)$$

This can be shown by substituting (5.2) into (5.1):

$$\text{LHS} = y(n) = y_0 \rho^n$$

$$\text{RHS} = \rho y(n-1) = \rho y_0 \rho^{n-1} = y_0 \rho^n$$

hence, the two sides are equal, and therefore,

$y(n) = y_0 \rho^n$ satisfies (5.1) and can be called a solution.

The stability of a dynamic system refers to the behavior of the solution, given in the above example by (5.2), as n becomes very large. Generally speaking, a system is said to be stable if the solution $y(n)$ converges to some value and unstable if it does not. If we examine (5.2) we observe that if

$$|\rho| > 1$$

then as n gets larger and larger, $y(n)$ becomes larger and larger: i.e., $n \rightarrow \infty$ implies that $y(n)$ diverges. Thus for $|\rho| > 1$ the system given by (5.1) is unstable. Contrariwise, we note that if

$$|\rho| < 1$$

as n increases, $y(n)$ approaches zero. Thus the condition $|\rho| < 1$ implies that the system given by (5.1) is stable.

Note finally that if $\rho = 1$ we obtain from (5.2):

$$y(n) = y_0$$

i.e., the system "sits" at the initial value and in effect $\rho = 1$ implies that there is no dynamic. However, this latter case is also stable. We distinguish the two stable cases from one another by calling the first, i.e., when $|\rho| < 1$, asymptotically stable; the second is simply denoted as stable.

Consider a slightly more complicated system:

$$y(n) = \rho y(n-1) + c \quad (5.3)$$

where c is a constant. In a completely analogous fashion it can be shown that the solution to this system is given by:

$$y(n) = \rho^n y(0) + \rho^{n-1}c + \rho^{n-2}c + \dots + \rho c + c. \quad (5.4)$$

Exercise: show that (5.4) is a solution to (5.3)

Examining (5.4) we see that if $|\rho| > 1$, as n increases $y(n)$ still diverges, similarly if $|\rho| < 1$ as n increases $y(n)$ converges.

We see then that stability implies that the system converges to some value, zero or otherwise, or that the system never moves from its initial condition. The value to which the system converges is the equilibrium point of the system. As was defined in section 5, the equilibrium point of a dynamic system is the point at which the dynamic stops, or the system comes to rest.

At the equilibrium point the dynamics of the system cease; for the system (5.3) let y_e be the equilibrium point. Then:

$$y_e = -\frac{c}{\rho - 1}$$

i.e, the equilibrium point is given by: $-\frac{c}{\rho - 1}$.

Exercise: Show that zero is the equilibrium point for (5.1)

Consequently, a system is stable if as n increases $y(n)$ converges to the equilibrium point of the system.

Thus far we have only considered a first order system. However, as we showed in section 3, the arms race model is a second order system. We must, therefore, extend our analysis of stability to second order systems. Consider the second order difference equation:

$$y(n) = \alpha_2 y(n-2) + \alpha_1 y(n-1) \quad (5.5)$$

and to simplify our analysis assume

$$\alpha_1 > 0, \alpha_2 > 0.$$

Let

$y(n) = c \rho^n$, where c, ρ are unknown constants and n is an integer, be a solution to (5.5) in an analogous sense to that described above in (5.2).

We can now write

$$y(n-1) = c \rho^{n-1}$$

$$y(n-2) = c \rho^{n-2}$$

and substitute into (5.5) to obtain

$$c \rho^n = \alpha_2 c \rho^{n-2} + \alpha_1 c \rho^{n-1}.$$

Divide by $c \rho^{n-2}$:

$$\begin{aligned}\rho^2 &= \alpha_2 + \alpha_1 \rho \\ 0 &= \rho^2 - \alpha_1 \rho - \alpha_2.\end{aligned}\tag{5.6}$$

Equation (5.6) is a quadratic equation in ρ . The two roots of this equation are

$$\frac{1}{2} [\alpha_1 \pm \sqrt{\alpha_1^2 + 4 \alpha_2}].$$

If $\alpha_1 > 0$ and $\alpha_2 > 0$ then both roots will be real.

Now we can write the solution to (5.5):

$$y(n) = c_1 \rho_1^n + c_2 \rho_2^n\tag{5.7}$$

where c_1 and c_2 are constants determined by the initial conditions. Analyzing the solution given in (5.7) we see that the same type of argument given for the stability of a first order equation applies in a second order system, viz., $|\rho_i| < 1$, $i = 1, 2$ guarantees that the system will be asymptotically stable.

Let us examine the stability properties of the Richardson model. As we saw in section 3, the two coupled equations could be decoupled and represented as a second order difference equation. If we further assumed $g = h = 0$ the two equations were identical. Hence, we need examine only one of these equations:

$$x(n) = [\beta_1 + \beta_2]x(n-1) + [\delta_1 \delta_2 - \beta_1 \beta_2]x(n-2).\tag{5.8}$$

Let us assume specific values for the coefficients:

$$\beta_1 = .9 \quad , \quad \delta_1 = 1.2$$

$$\beta_2 = .8 \quad , \quad \delta_2 = 1.0.$$

These coefficient values indicate that the amount of fatigue felt by both nations and the amount of threat perceived by each are roughly equivalent. Furthermore, neither nation is experiencing or perceiving excessive amounts of fatigue or threat (over or under), as reflected by the fact that the parameter values are nearly unity.

If we substitute these specific values into (5.8) we obtain:

$$x(n) = 1.7 x(n-1) + .48 x(n-2) \quad (5.9)$$

and comparing (5.9) with (5.5) we see that

$$\alpha_1 = 1.7$$

$$\alpha_2 = .48.$$

Hence, we can now use (5.6) and solve for ρ , and the two roots are:

$$\begin{aligned} & \frac{1}{2} [\alpha_1 \pm \sqrt{\alpha_1^2 + 4 \alpha_2}] \\ &= \frac{1}{2} [1.7 \pm \sqrt{1.7^2 + 4 \cdot .48}] \\ &= 1.95, \quad -.25. \end{aligned} \quad (5.10)$$

We find that one of the roots is larger than unity (i.e., $1.95 > 1$). Thus (5.9) represents an unstable system.

As a second, contrasting example, assume the coefficient values to be:

$$\beta_1 = .50 \quad , \quad \delta_1 = .65$$

$$\beta_2 = .36 \quad , \quad \delta_2 = .30 .$$

In this realization of the model we see that two nations are experiencing fatigue (β) and perceiving threat (δ) differently. But of greater significance is the fact that these parameter values are less than unity. In the case of the threat parameter, δ , this indicates that both nations are perceiving rather little threat. Or to put it somewhat differently, the amount of hostility between the two is low. An interpretation of β must be done more cautiously. Initially this parameter measured fatigue but was assumed to be positive and subtracted from the equation. When we rewrote Richardson's model to express it more generally, we put no constraint on β and added it in the equation. Consequently, $0 < \beta < 1$ implies that a nation is experiencing some fatigue. Perhaps this interpretation can be seen more easily by changing the interpretation of β from "fatigue" to a "depreciation" coefficient. Thus the above values for β imply that the effectiveness of the weapons is depreciating at a yearly rate of .5 and .36 respectively.

Let us now consider whether this realization of the Richardson model is stable. As before, we determine α_1 and

$$\alpha_2: \quad \alpha_1 = \beta_1 + \beta_2 = .86$$

$$\alpha_2 = \delta_1 \delta_2 - \beta_1 \beta_2 = .195 - .18 = .015$$

and then solve for the two values of ρ :

$$\begin{aligned} & \frac{1}{2} [\alpha_1 \pm \sqrt{\alpha_1^2 + 4\alpha_2}] \\ &= \frac{1}{2} [.86 \pm \sqrt{.74 + .06}] \\ &= .88, \quad -.02. \end{aligned}$$

These values allow us to conclude that the system is asymptotically stable.

Exercises:

- (1) Is the armament race, represented by:
 $y(n) = 3 y(n-1) + 4 y(n-2)$ asymptotically stable?
 - (2) Choose the coefficient α in the armament equation:
 $y(n) = \alpha y(n-1) - 10 y(n-2)$
so that the armament race represented by the above equation is asymptotically stable.
 - (3) In an armament race between two nations, the behavior of the arms buildup can be represented by the equation: $y(n) = + 10 y(n-2)$, note that a similar equation is true for the other nation. Comment on whether the race is asymptotically stable.
-

6. A SENSITIVITY ANALYSIS OF THE RICHARDSON ARMS RACE MODEL

In this last section we turn to an analysis of the sensitivity properties of the arms race model. This analysis, however, requires some familiarity with partial derivatives. For those unfamiliar with partial derivatives we provide the following brief background.

Let $z = f(x,y)$ be a function of x and y and let (x_1, y_1) be any point. The function $f(x, y_1)$ then depends on x alone and is defined in an interval about x_1 . Hence its derivative with respect to x at $x = x_1$ may exist. If it does, its value is called the partial derivative of $f(x,y)$ with respect to x at (x_1, y_1) , and is denoted by:

$$\frac{\partial f}{\partial x}(x_1, y_1) \text{ or } \left. \frac{\partial z}{\partial x} \right|_{(x_1, y_1)} .$$

Another notation used for the partial derivative is:

f_x or more explicitly-- $f_x(x_1, y_1)$. The partial derivative: $\frac{\partial f}{\partial y}(x_1, y_1)$ is defined similarly: one holds x constant equal to x_1 and differentiates $f(x_1, y)$ with respect to y .

Examples:

(i) If $z = xuv + u - 2v$, then

$$\frac{\partial z}{\partial x} = uv, \quad \frac{\partial z}{\partial u} = xv + 1, \quad \frac{\partial z}{\partial v} = xu - 2.$$

(ii) If $x^2 + y^2 - z^2 = 1$, then

$$2x - 2z \frac{\partial z}{\partial x} = 0 \rightarrow \frac{\partial z}{\partial x} = \frac{x}{z}$$

$$2y - 2z \frac{\partial z}{\partial y} = 0 \rightarrow \frac{\partial z}{\partial y} = \frac{y}{z} .$$

(iii) $z^2 = x^2 + y^2 - 1$, this is the same equation as in (ii) above.

$$2z \frac{\partial z}{\partial x} = 2x \rightarrow \frac{\partial z}{\partial x} = \frac{x}{z}$$

$$2z \frac{\partial z}{\partial y} = 2y \rightarrow \frac{\partial z}{\partial y} = \frac{y}{z}$$

For simplicity consider again the first order system:

$$y(n) = \rho y(n-1), n = 1, 2, \dots \quad (5.1)$$

Its solution is given by the expression (5.2). We would like to assess the impact of a slight change in ρ , on the solution trajectory $y(n)$. This is accomplished by evaluating the first partial derivative of $y(n)$ with respect to ρ :

$$\frac{\partial y(n)}{\partial \rho} = y_0 n \rho^{n-1}. \quad (6.1)$$

From elementary calculus we know that the change in $y(n)$ is related to the first partial by the following relation:

$$\Delta y(n) \approx \frac{\partial y(n)}{\partial \rho} \cdot \Delta \rho, \quad (6.2)$$

where $\Delta \rho$ is the differential change in the parameter ρ . The quantity $\frac{\partial y(n)}{\partial \rho}$, the first partial of $y(n)$ with respect to ρ , is called the sensitivity coefficient of $y(n)$ with respect to ρ , it measures the effect on $y(n)$ of a differential (or small) change in ρ . Knowing the sensitivity coefficient and the differential change in ρ , the change in $y(n)$ can be computed.

Now suppose that the quantity of interest, i.e., $y(n)$, is a function of two parameters, ρ_1 and ρ_2 . We would like to compute the change in $y(n)$ as these parameters change by amounts $\Delta\rho_1$ and $\Delta\rho_2$. Following the same line of argument, we can write:

$$\Delta y(n) \approx \frac{\partial y(n)}{\partial \rho_1} \cdot \Delta\rho_1 + \frac{\partial y(n)}{\partial \rho_2} \cdot \Delta\rho_2. \quad (6.3)$$

$\frac{\partial y(n)}{\partial \rho_1}$ and $\frac{\partial y(n)}{\partial \rho_2}$ are the two sensitivity coefficients of the quantity $y(n)$ with respect to the parameters ρ_1 and ρ_2 .

From (6.2) it can be seen that if the quantity $\frac{\partial y(n)}{\partial \rho}$ is large, a change in the quantity ρ will have a large effect on $y(n)$, i.e., we will obtain a large $\Delta y(n)$. Thus the size of the sensitivity coefficient determines the effect a change in the parameter will have on the quantity of interest. A knowledge of various sensitivity coefficients will indicate which parameters have the largest effect on the quantity being investigated.

Let us consider the armament behavior of a nation described by the equation (5.5). The solution of this equation is given by equation (5.7). We want to investigate which of the two parameters (α_1 or α_2) has the larger effect on the trajectory when it is changed from its nominal value. To see this we compute the sensitivity coefficients: $\frac{\partial y(n)}{\partial \alpha_1}$ and $\frac{\partial y(n)}{\partial \alpha_2}$.

$$\frac{\partial y(n)}{\partial \alpha_1} = c_1 n \rho_1^{n-1} \frac{\partial \rho_1}{\partial \alpha_1} + c_2 n \rho_2^{n-1} \frac{\partial \rho_2}{\partial \alpha_1}$$

$$\frac{\partial y(n)}{\partial \alpha_2} = c_1 n \rho_1^{n-1} \frac{\partial \rho_1}{\partial \alpha_2} + c_2 n \rho_2^{n-1} \frac{\partial \rho_2}{\partial \alpha_2} .$$

In the above two expressions ρ_1 and ρ_2 are defined by equation (5.6). For given values of the parameters α_1 , α_2 , c_1 and c_2 , the above two coefficients can be evaluated and the relative size of the two determined. The larger coefficient results in the larger effect on the trajectory and hence can be thought of as the more "critical" parameter. Thus, in the context of an arms race model, this would be the parameter one would avoid changing if at all possible.

7. CONCLUSION

This module has sought to demonstrate the use of second order linear difference equations for modeling arms races among nations. Based on the work of Lewis Fry Richardson, this module has shown how a reaction system between two nations can be represented in three alternate but equivalent mathematical forms and how various properties of the system -- the equilibrium, stability, and sensitivity properties -- can be examined. While the model does not ultimately allow Richardson to predict wars, the mathematical formulation and subsequent analysis permits an understanding of dynamics that are related to the outbreak of wars. Thus the model represents a first step towards the goal of understanding international conflict.

8. REFERENCES

- Richardson, L. F. (1960) Arms and Insecurity.
Pittsburgh: Boxwood Press.
- Richardson, L. F. (1960) Statistics of Deadly Quarrels.
Pittsburgh: Boxwood Press.
- Zinnes, Dina A., John V. Gillespie and R. Michael Rubison
(1976) "A Reinterpretation of the Richardson Arms
Race Model," in Dina A. Zinnes and John V. Gillespie,
eds., Mathematical Models in International Relations.
New York: Praeger Publishers, 189-217.

9. ANSWERS TO EXERCISES

PAGE 10:

Exercise 1: $x(n) = 2 x(n-1) + 7 y(n-1)$ (1)

$$y(n) = 4 x(n-1) + 5 y(n-1)$$

In the above equation let $n \rightarrow n-1$:

$$y(n-1) = 4 x(n-2) + 5 y(n-2)$$

From equation (1): $x(n) = 2 x(n-1) + 7 y(n-1)$

$$\begin{aligned} &= 2 x(n-1) + 7 [4 x(n-2) + 5 y(n-2)] \\ &= 2 x(n-1) + 28 x(n-2) + 35 y(n-2) \end{aligned}$$

(2)

Also from equation (1): $y(n-1) = \frac{1}{7} [x(n) - 2 x(n-1)]$;

changing $(n-1)$ to $(n-2)$ in this equation we obtain:

$$y(n-2) = \frac{1}{7} [x(n-1) - 2 x(n-2)];$$

substitute this in equation (2) and simplify to obtain:

$$x(n) = 7 x(n-1) + 18 x(n-2) \quad (3)$$

Also from the equation (1) one obtains:

$$y(n) = 4 x(n-1) + 5 y(n-1) ; \text{ (substitute equation (1) for } x(n-1)\text{)}$$

$$= 4 [2 x(n-2) + 7 y(n-2)] + 5 y(n-1)$$

$$= 8 x(n-2) + 28 y(n-2) + 5 y(n-1) \quad (4)$$

from the second equation in (1):

$$y(n) - 5 y(n-1) = 4 x(n-1)$$

let $(n-1) \rightarrow (n-2)$:

$$y(n-1) - 5 y(n-2) = 4 x(n-2)$$

substitute the above in (4) and simplify:

$$y(n) = 7 y(n-1) + 18 y(n-2).$$

PAGE 10:

Exercise 2:

$$\begin{aligned}x(n) &= 7 y(n-1) \rightarrow x(n+1) = 7 y(n) \text{ (changing } n \rightarrow (n+1)\text{)} \\ &= 7 \cdot 4 x(n-1) \text{ (substitute for } y(n) \text{ from the} \\ &\quad \text{second equation)} \\ &= 28 x(n-1)\end{aligned}$$

$$\begin{aligned}y(n) &= 4 x(n-1) \rightarrow y(n+1) = 4 x(n) \\ &= 4 \cdot 7 y(n-1) \\ &= 28 y(n-1).\end{aligned}$$

Comment: $x(n+1) = 28 x(n-1)$

let $(n+1) \rightarrow n$ then: $x(n) = 28 x(n-2)$.

This is a second-order difference equation, it resembles the equation (3) except that the term denoting the value at the instant $(n-1)$ is zero.

PAGE 14:

Exercise 1: In view of the definitions (4.4) we can rewrite the two equations as:

$$X(n) = \begin{bmatrix} 7 & 0 \\ 2 & 4 \end{bmatrix} X(n-1).$$

Exercise 2: In view of the definitions (4.4) we can rewrite the two equations as:

$$X(n) = \begin{bmatrix} 0 & 5 \\ 2 & 0 \end{bmatrix} X(n-1) + v.$$

PAGE 14: Exercises (Continued):

Differences:

- (1) The matrices A are different, the main Diagonal entries in Exercise 1 are nonzero while these are zero in Exercise 2.
- (2) The term V is zero in the Exercise 1.

PAGE 17:

Exercise: $\det. A = a_{21} \Delta_{21} + a_{22} \Delta_{22} + a_{23} \Delta_{23}$ - second row expansion
 $= a_{13} \Delta_{13} + a_{23} \Delta_{23} + a_{33} \Delta_{33}$ - third column expansion

PAGE 23:

Exercise: The difference equation is: $y(n) = \rho y(n-1) + c$

The solution (5.4) is: $\rho^n y(0) + \sum_{j=0}^{n-1} \rho^j c$

LHS of the equation (5.3) = $\rho^n y(0) + \sum_{j=0}^{n-1} \rho^j c$

RHS = $\rho y(n-1) + c$

$$= \rho [\rho^{n-1} y(0) + \sum_{j=0}^{n-2} \rho^j c]$$

$$= \rho^n y(0) + \rho \sum_{j=0}^{n-2} \rho^j c$$

$$= \rho^n y(0) + \sum_{j=0}^{n-1} \rho^j c$$

= LHS.

PAGE 24:

Exercise: $y(n) = \rho y(n-1)$, $n = 1, 2, \dots$

At the equilibrium point the dynamics of the system cease, i.e.,

$$\rho y_e = 0 \rightarrow y_e = 0$$

0 is the equilibrium point of the system.

PAGE 28:

Exercise 1: $y(n) = 3 y(n-1) + 4 y(n-2)$.

Following the procedure indicated below equation (5.5)

we obtain the algebraic equation in ρ i.e.,

$$\rho^2 - 3\rho - 4 = 0$$

The two roots of the above equation are:

$$\rho_1, \rho_2 = \frac{1}{2} [3 \pm \sqrt{9 + 16}]$$

$$= \frac{1}{2} [3 \pm 5]$$

$$= 4, -1$$

since the root 4 is larger than one, the system is unstable.

Exercise 2: $y(n) = \alpha y(n-1) - 10 y(n-2)$.

The corresponding algebraic equation needed for

stability analysis is:

$$\rho^2 - \alpha\rho + 10 = 0$$

PAGE 28: Exercise 2 (Continued)

its two roots should be less than one for asymptotic stability;

$$\rho_1, \rho_2 = \frac{1}{2} [\alpha \pm \sqrt{\alpha^2 - 40}]$$

$$\frac{1}{2}[\alpha + \sqrt{\alpha^2 - 40}] < 1 \quad \text{and} \quad \frac{1}{2}[\alpha - \sqrt{\alpha^2 - 40}] < 1.$$

From the above two inequalities we obtain the restrictions on α for asymptotic stability.

$$\frac{1}{2}[\alpha + \sqrt{\alpha^2 - 40}] < 1 \rightarrow \alpha + \sqrt{\alpha^2 - 40} < 2$$

$$\rightarrow \alpha - 2 < -\sqrt{\alpha^2 - 40} \rightarrow (\alpha - 2)^2 < \alpha^2 - 40$$

$$\rightarrow 4 + \alpha^2 - 4\alpha < \alpha^2 - 40 \rightarrow -4\alpha < -44 \rightarrow \alpha > 11.$$

Following the above procedure we obtain the same result from the other inequality, i.e., $\alpha > 11$.

Exercise 3: $y(n) = 10 y(n-2)$.

The corresponding algebraic equation needed for stability analysis is:

$$\rho^2 - 10 = 0 \rightarrow \rho = \pm \sqrt{10}$$

since $|\rho| > 1$, the system is unstable.