Suggested Solution to Project 1

Model of Water Quality Control in Schweriner Lake

1) Dividing equation (1) by $V\Delta t$ and taking a limit when $\Delta t \to 0$ we will obtain

$$\frac{dC_N}{dt} = \frac{N}{V} - \frac{QC_N}{V} - KC_N \,. \qquad (2)$$

2') Solving equation (2) by hand.

The equation (2) is an equation with separable variables. Separating the variables and integrating both sides we obtain:

$$\frac{dC_N}{dt} = \frac{N}{V} - C_N\left(\frac{Q+VK}{V}\right)$$

$$\int \frac{dC_N}{\frac{N}{V} - C_N\left(\frac{Q+VK}{V}\right)} = \int dt$$

$$-\frac{V}{Q+VK}\ln\left|\frac{N}{V} - C_N\left(\frac{Q+VK}{V}\right)\right| = t + C$$

Making C_N the subject we get:

$$C_N(t) = \frac{V}{Q+VK}\left(\frac{N}{V} - Ce^{-\frac{Q+VK}{V}t}\right)$$

Substituting the initial condition $C_N(0) = 0$ we get:

$$C_N(t) = \frac{N}{Q+KV}\left(1 - e^{-t(Q/V+K)}\right) . \qquad (3)$$

2'') Solving equation (2) with Matlab

```
>> dsolve('Dy=N/V-(Q+V*K)/V*y','y(0)=0','t')
ans =
(N/(Q+V*K)*exp((Q+V*K)*t/V)-1/(Q+V*K)*N)*exp((-Q-V*K)*t/V)
>> pretty(ans)
```

```
/          (Q + V K) t             \
|N exp(-----------)                |
|               V          N       |       (-Q - V K) t
|------------------ - -------      | exp(------------)
\     Q + V K       Q + V K   /               V
```

3) The equilibrium concentration $C_{N\,e}$ can be obtained from formula (3) when $t \to \infty$ or from equation (2) when $dC_N/dt = 0$. It is equal to

$$C_{N\,e} = \frac{N}{Q + KV}.$$

4) Time needed for reaching p portion ($p = C_N(t)/C_{Ne}$) of equilibrium concentration is obtained by solving the equation:

$$pC_{Ne} = \frac{N}{Q + KV}(1 - e^{-t(Q/V+K)}) \text{ or } p = (1 - e^{-t(Q/V+K)})$$

a) Solving for t by hand:

$$t_p = \frac{-V}{Q + KV}\ln(1 - p). \qquad (4)$$

b) Solving with Matlab:

```
>> syms p t Q V K
>> solve('p=1-exp(-t*(Q/V+K))','t')
ans =
-log(-p+1)*V/(Q+V*K)
>> pretty(ans)
                         log(-p + 1) V
                       - -------------
                            Q + V K
```

In case when pollution is not decomposed (K=0) it takes more time to reach p portion of the equilibrium concentration.

5) If at the initial moment $t = 0$ the amount of pollution entered the lake was C_0 and after that pollution is not coming in ($N = 0$) them the equation for the concentration of pollution has the following form:

$$\frac{dC}{dt} = -C(\frac{Q}{V} + K)$$

6) The solution to the above equation is

$$C(t) = C_0\, e^{-t(Q/V+K)}.$$

7) Time needed to reach $1 - p$ portion ($C(t)/C_0 = 1 - p$) of the initial concentration C_0 is determined by formula (4).