

LP Methods.S3

The Dual Linear Program

When a solution is obtained for a linear program with the revised simplex method, the solution to a second model, called the dual problem, is readily available and provides useful information for sensitivity analysis as we have just seen. There are several benefits to be gained from studying the dual problem, not the least of which is that it often has a practical interpretation that enhances the understanding of the original model. Moreover, it is sometimes easier to solve than the original model, and likewise provides the optimal solution to the latter at no extra cost. Duality also has important implications for the theoretical basis of mathematical programming algorithms. In this section, the dual problem is defined, the properties that link it to the original (called the primal) are listed, and the procedure for identifying the dual solution from the tableau is presented.

Definition of the Dual LP Model

In discussing duality, it is common to depart from the standard equality form of the LP given in Section 4.1 in order to highlight the symmetry of the primal-dual relationships. The dual model is derived by construction from the standard inequality form of linear programming model as shown in Tables 1 and 2. All constraints of the primal model are written as *less than or equal to*, and right-hand-side constants may be either positive or negative. In the primal model there are assumed to be n decision variables and m constraints, thus \mathbf{c} and \mathbf{x} are n -dimensional vectors. The matrix of structural coefficients, \mathbf{A} , has m rows and n columns. The *dual model* uses the same arrays of coefficients but arranged in a symmetric fashion. The dual vector π has m components.

Table 1. Matrix definition of primal and dual problems

(P) Maximize $z_p = \mathbf{c}\mathbf{x}$ subject to $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ $\mathbf{x} \geq \mathbf{0}$	(D) Minimize $z_D = \pi\mathbf{b}$ subject to $\pi\mathbf{A} \leq \mathbf{c}$ $\pi \geq \mathbf{0}$
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Table 2. Algebraic definition of primal and dual problems

(P) Maximize $z_p = c_1x_1 + c_2x_2 + \dots + c_nx_n$ subject to $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$ $a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2$ \vdots \vdots	(D) Minimize $z_D = b_1\pi_1 + b_2\pi_2 + \dots + b_m\pi_m$ subject to $a_{11}\pi_1 + a_{21}\pi_2 + \dots + a_{m1}\pi_m \leq c_1$ $a_{12}\pi_1 + a_{22}\pi_2 + \dots + a_{m2}\pi_m \leq c_2$ \vdots \vdots
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$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m$ $x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0$	$a_{1n}x_1 + a_{2n}x_2 + \dots + a_{mn}x_m \leq c_n$ $x_1 \geq 0, x_2 \geq 0, \dots, x_m \geq 0$
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Example 4

<p>(P) Maximize $z_P = 2x_1 + 3x_2$</p> <p>subject to</p> $-x_1 + x_2 \leq 5$ $x_1 + 3x_2 \leq 35$ $x_1 \leq 20$ $x_1 \geq 0, x_2 \geq 0$	<p>(D) Minimize $z_D = 5x_1 + 35x_2 + 20x_3$</p> <p>subject to</p> $-x_1 + x_2 + x_3 \leq 2$ $x_1 + 3x_2 \leq 3$ $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$
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The optimal solution to the primal including slacks is $\mathbf{x}^* = (20, 5, 20, 0, 0)^T$ with $z_P = 55$. The corresponding dual solution including slacks is $\boldsymbol{\pi}^* = (0, 1, 1, 0, 0)$ with $z_D = 55$. Note that $z_P = z_D$. This is always the case as will be shown presently.

From an algorithmic point of view, solving the primal problem with the dual simplex method is equivalent to solving the dual problem with the primal simplex method. When written in inequality form, the primal and dual models are related in the following ways.

- a. When the primal has n variables and m constraints, the dual has m variables and n constraints.
- b. The constraints for the primal are all *less than or equal to*, while the constraints for the dual are all *greater than or equal to*.
- c. The objective for the primal is to maximize, while the objective for the dual is to minimize.
- d. All variables for either problem are restricted to be nonnegative.
- e. For every primal constraint, there is a dual variable. Associated with the i th primal constraint is dual variable π_i . The dual objective function coefficient for π_i is the right-hand side of the i th primal constraint, b_i .
- f. For every primal variable, there is a dual constraint. Associated with primal variable x_j is the j th dual constraint whose right-hand side is the primal objective function coefficient c_j .

- g. The number a_{ij} is, in the primal, the coefficient of x_j in the i th constraint, while in the dual, a_{ij} is the coefficient of y_i in the j th constraint.

Modifications to Inequality Form

It is rare that a linear program is given in inequality form. This is especially true when the model has definitional constraints that are introduced for convenience, or when it has been prepared for the tableau simplex method where all RHS constants must be positive. Nevertheless, no matter how the primal is stated, its dual can always be found by first converting the primal to the inequality form in Table 1 and then writing the dual accordingly. For example, given an LP in standard equality form

$$\begin{aligned} &\text{Maximize } z_P = \mathbf{c}\mathbf{x} \\ &\text{subject to } \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0} \end{aligned}$$

we can replace the constraints $\mathbf{A}\mathbf{x} = \mathbf{b}$ with two inequalities: $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ and $-\mathbf{A}\mathbf{x} \leq -\mathbf{b}$ so the coefficient matrix becomes $\begin{pmatrix} \mathbf{A} \\ -\mathbf{A} \end{pmatrix}$ and the right-hand-side vector becomes $(\mathbf{b}, -\mathbf{b})^T$. Introducing a partitioned dual row vector (γ_1, γ_2) with $2m$ components, the corresponding dual is

$$\begin{aligned} &\text{Minimize } z_D = \gamma_1\mathbf{b} - \gamma_2\mathbf{b} \\ &\text{subject to } \gamma_1\mathbf{A} - \gamma_2\mathbf{A} \leq \mathbf{c} \\ &\qquad \qquad \gamma_1 \geq \mathbf{0}, \gamma_2 \geq \mathbf{0} \end{aligned}$$

Letting $\pi = \gamma_1 - \gamma_2$ we may simplify the representation of this problem to obtain the pair given in Table 3.

Table 3. Equality form of primal-dual models

(P) Maximize $z_P = \mathbf{c}\mathbf{x}$ subject to $\mathbf{A}\mathbf{x} = \mathbf{b}$ $\mathbf{x} \geq \mathbf{0}$	(D) Minimize $z_D = \pi\mathbf{b}$ subject to $\pi\mathbf{A} \leq \mathbf{c}$
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This is the asymmetric form of the duality relation. Similar transformations can be worked out for any linear program by first putting the primal into inequality form, constructing the dual, and then simplifying the latter to account for special structure. We say two LPs are *equivalent*

if one can be transformed into another so that feasible solutions, optimal solutions and corresponding dual solutions are preserved; e.g., the inequality form in Table 1 and the equality form in Table 3 are equivalent primal-dual representations. This suggests the following result which can be proven by constructing the appropriate models.

Proposition 1: Duals of equivalent problems are equivalent. Let (P) refer to an LP and let (D) be its dual. Let (\hat{P}) be an LP that is equivalent to (P). Let (\hat{D}) be the dual of (\hat{P}). Then (\hat{D}) is equivalent to (D), that is, they have the same optimal objective function values or they are both infeasible.

Table 4 describes more general relations between the primal and dual that can be easily derived from the standard definition. They relate the sense of constraint i in the primal with the sign restriction for \bar{c}_i in the dual, and sign restriction of x_j in the primal with the sense of constraint j in the dual. Note that when these alternative definitions are allowed there are many ways to write the primal and dual problems; however, they are all equivalent.

Table 4. Modifications in the primal-dual formulations

Primal model	Dual model
Constraint i is \leq	$\bar{c}_i \geq 0$
Constraint i is $=$	\bar{c}_i is unrestricted
Constraint i is \geq	$\bar{c}_i \leq 0$
$x_j \geq 0$	Constraint j is \leq
x_j is unrestricted	Constraint j is $=$
$x_j \leq 0$	Constraint j is \geq

Example 5

(P) Maximize $z_P = -3x_1 - 2x_2$	(D) Minimize $z_D = 8\bar{c}_1 + 13\bar{c}_2$
subject to $-x_1 - x_2 = 8$	subject to $-\bar{c}_1 + \bar{c}_2 = -3$
$x_1 + 2x_2 = 13$	$-\bar{c}_1 + 2\bar{c}_2 = -2$
$x_1 \geq 0, x_2$ unrestricted	\bar{c}_1 unrestricted, $\bar{c}_2 \geq 0$

Relations between Primal and Dual Objective Function Values

There are a number of relationships between solutions to the primal and dual problems that are interesting to theoreticians, useful to algorithm developers, and important to analysts for interpreting solutions. We present these relationships as theorems and prove them for the primal–dual pair in Table 1; however, they are true for all primal–dual formulations. In what follows, \mathbf{x} refers to any feasible solution of the primal and π to any feasible solution of the dual; \mathbf{x}^* and π^* are the respective optimal solutions if they exist.

Theorem 1 (Weak Duality) In a primal–dual pair of LPs, let \mathbf{x} be a primal feasible solution and $z_P(\mathbf{x})$ the corresponding value of the primal objective function that is to be maximized. Let π be a dual feasible solution and $z_D(\pi)$ the corresponding dual objective function that is to be minimized. Then $z_P(\mathbf{x}) \leq z_D(\pi)$.

This theorem shows that the objective value for a feasible solution to the dual will always be greater than or equal to the objective function for a feasible solution to the primal. The following sequence demonstrates this result.

1. The primal solution is feasible by hypothesis: $\mathbf{Ax} = \mathbf{b}$
2. Premultiply both sides by π : $\pi\mathbf{Ax} = \pi\mathbf{b}$
3. The dual solution is feasible by hypothesis: $\pi\mathbf{A} \leq \mathbf{c}$
4. Postmultiply both sides by \mathbf{x} : $\pi\mathbf{Ax} \leq \mathbf{c}\mathbf{x}$
5. Combine the results of 2 and 4: $\mathbf{c}\mathbf{x} \geq \pi\mathbf{Ax} = \pi\mathbf{b}$ or
 $z_P(\mathbf{x}) \leq z_D(\pi)$

There are a number of useful relationships that can be derived from Theorem 1. In particular,

- The value of $z_P(\mathbf{x})$ for any feasible \mathbf{x} is a lower bound to $z_D(\pi^*)$.
- The value of $z_D(\pi)$ for any feasible π is an upper bound to $z_P(\mathbf{x}^*)$.
- If there exists a feasible \mathbf{x} and the primal problem is unbounded, there is no feasible π .

- If there exists a feasible π and the dual problem is unbounded, there is no feasible \mathbf{x} .
- It is possible that there is no feasible \mathbf{x} and no feasible π .

The last point is demonstrated by the following example.

Maximize $z = x_1 + 3x_2$ subject to $x_1 - x_2 \leq 3$ $-x_1 + x_2 \leq -5$ x_1, x_2 unrestricted	Minimize $z_D = 3\pi_1 - 5\pi_2$ subject to $\pi_1 - \pi_2 = 1$ $-\pi_1 + \pi_2 = 3$ $\pi_1, \pi_2 \geq 0$
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Theorem 2 (Sufficient Optimality Criterion) In a primal-dual pair of LPs, let $z_p(\mathbf{x})$ be the primal objective function and $z_D(\pi)$ be the dual objective function. If $(\hat{\mathbf{x}}, \hat{\pi})$ is a pair of primal and dual feasible solutions satisfying $z_p(\hat{\mathbf{x}}) = z_D(\hat{\pi})$, then $\hat{\mathbf{x}}$ is an optimal solution of the primal and $\hat{\pi}$ is an optimal solution of the dual.

The proof can be seen in the following steps:

1. Definition of optimality for primal: $z_p(\hat{\mathbf{x}}) = z_p(\mathbf{x}^*)$
2. Feasible dual solution bound on z_p : $z_p(\mathbf{x}^*) \leq z_D(\pi^*)$
3. Definition of optimality for dual: $z_D(\pi^*) = z_D(\hat{\pi})$
4. Combine the results of 1, 2 and 3: $z_p(\hat{\mathbf{x}}) = z_p(\mathbf{x}^*) \leq z_D(\pi^*) = z_D(\hat{\pi})$
5. Objectives are equal by hypothesis: $z_p(\hat{\mathbf{x}}) = z_D(\hat{\pi})$
6. Combine 4 and 5: $z_p(\hat{\mathbf{x}}) = z_p(\mathbf{x}^*) = z_D(\pi^*) = z_D(\hat{\pi})$

Therefore, $\hat{\mathbf{x}}$ and $\hat{\pi}$ are optimal.

This theorem states that equality of objective values implies optimality; moreover, we have:

- h. Given feasible solutions \mathbf{x} and π for a primal-dual pair, if the objective values are equal, they are both optimal.

- i. If \mathbf{x}^* is an optimal solution to the primal, a finite optimal solution exists for the dual with objective value $z_p(\mathbf{x}^*)$.
- j. If π^* is an optimal solution to the dual, a finite optimal solution exists for the primal with objective value $z_D(\pi^*)$.

Taking these results one step farther lead to the Fundamental Duality Theorem.

Theorem 3 (Strong Duality) In a primal-dual pair of LPs, if either the primal or the dual problem has an optimal feasible solution, then the other does also and the two optimal objective values are equal.

We will prove the result for the primal and dual problems given in Table 3. Solving the primal problem by the simplex algorithm yields an optimal solution $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$, $\mathbf{x}_N = \mathbf{0}$ with $\bar{\mathbf{c}} = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_N = \mathbf{0}$, which can be written $[\mathbf{c}_B, \mathbf{c}_N - \mathbf{c}_B\mathbf{B}^{-1}(\mathbf{B}, \mathbf{N})] = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} = \mathbf{0}$. Now if we define $\pi = \mathbf{c}_B\mathbf{B}^{-1}$ we have $\pi\mathbf{A} = \mathbf{c}$ and $z_p(\mathbf{x}) = \mathbf{c}_B\mathbf{x}_B = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} = \pi\mathbf{b} = z_D(\pi)$. By the sufficient optimality criterion, Theorem 2, π is a dual optimal solution. This completes the proof when the primal and dual are as stated.

In general, every LP can be transformed into an equivalent problem in standard equality form. This equivalent problem is of the same type as the primal in Table 3, hence the proof applies. Also, by Proposition 1, the dual of the equivalent problem in standard form is equivalent to the dual of the original problem. Thus the theorem must hold for it too.

Complementary Solutions

For purposes of this section it is helpful to repeat the definition of the primal and dual problems given in Table 1 in a slightly different but equivalent form. Table 5 contains the modified representation, where \mathbf{I}_m is an $m \times m$ identity matrix and $\mathbf{x}_s = (x_{s1}, \dots, x_{sm})^T$ an m -dimensional vector of slack variables.

Table 5. Equivalent primal-dual pair

(P) Maximize $z_p = \mathbf{c}\mathbf{x}$ subject to $(\mathbf{A}_1, \dots, \mathbf{A}_n)\mathbf{x} + \mathbf{I}_m\mathbf{x}_s = \mathbf{b}$ $\mathbf{x} \geq \mathbf{0}, \mathbf{x}_s \geq \mathbf{0}$	(D) Minimize $z_D = \pi\mathbf{b}$ subject to $\pi\mathbf{A}_j \leq c_j, j = 1, \dots, n$ $\pi \geq \mathbf{0}$
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Each structural variable x_j is associated with the dual constraint j , and each slack variable x_{s_i} is associated with dual variable i . Recall that a basic solution is found selecting a set of basic variables, constructing the basis matrix \mathbf{B} , and setting the nonbasic variables to zero. This gives the primal solution

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} \text{ with } z_p = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}.$$

The complementary dual solution associated with this basis is defined to be

$$\boldsymbol{\pi} = \mathbf{c}_B\mathbf{B}^{-1} \text{ with } z_D = \boldsymbol{\pi}\mathbf{b} = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}.$$

Every basis defines complementary primal and dual solutions with identical objective function values.

Theorem 4 (Optimality of Feasible Complementary Solutions) Given the solution \mathbf{x}_B determined from the basis \mathbf{B} , when \mathbf{x}_B is optimal to the primal, the complementary solution $\boldsymbol{\pi} = \mathbf{c}_B\mathbf{B}^{-1}$ is optimal to the dual.

The proof of Theorem 4 can be seen in the following sequence.

1. Primal and dual objective values are equal by construction:

$$z_p(\mathbf{x}_B) = \mathbf{c}_B\mathbf{x}_B = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$$

$$z_D(\boldsymbol{\pi}) = \boldsymbol{\pi}\mathbf{b} = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$$

2. Primal objective value for a basic solution when nonbasic variable x_k is allowed to increase: $z_p = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} - (\boldsymbol{\pi}\mathbf{A}_k - c_k)x_k$

3. Since the primal solution is optimal:

$$\bar{c}_k = \boldsymbol{\pi}\mathbf{A}_k - c_k \leq 0 \text{ or}$$

$$\boldsymbol{\pi}\mathbf{A}_k \leq c_k$$

4. From 3, when x_k is a structural variable, dual constraint k is satisfied:

$$\boldsymbol{\pi}\mathbf{A}_k \leq c_k$$

5. From 3, when the nonbasic variable is a slack variable x_{s_i} , π_i is nonnegative: $c_{s_i} = 0$ and $\mathbf{A}_{s_i} = \mathbf{e}_i$ so $\pi_i = 0$

6. For basic variables: $\mathbf{c}_B - \boldsymbol{\pi} \mathbf{B} = \mathbf{c}_B - \mathbf{c}_B \mathbf{B}^{-1} \mathbf{B} = \mathbf{0}$

7. From 6, when x_k is a structural variable and basic, the k th dual constraint is satisfied as an equality: $c_k - \boldsymbol{\pi} \mathbf{A}_k = 0$

8. From 6, when x_{s_i} is a slack variable and basic, π_i is zero:

$$c_{s_i} = 0 \text{ and } \mathbf{A}_{s_i} = \mathbf{e}_i \text{ so } \pi_i = 0$$

All constraints are satisfied so $\boldsymbol{\pi}$ is feasible. By Theorem 2 it must be optimal.

From Step 3 of the proof, it can be inferred that the reduced cost, \bar{c}_k , for the primal variable x_k is equivalent to the dual slack, s_k , for dual constraint k . Moreover, Steps 7 and 8 illustrate an important property known as *complementary slackness*. Given the primal-dual pair in Table 1, we have the following.

Complementary solutions property: For a given basis, when a primal structural variable is basic, the corresponding dual constraint is satisfied as an equality (the dual slack variable is zero), and when a primal slack variable is basic (the primal constraint is loose), the corresponding dual variable is zero.

This property holds whether or not the primal and dual solutions are feasible. We have already seen this in the simplex tableau. That is, when a primal structural variable is basic, its reduce cost (dual slack) is zero; when a primal slack variable is basic, the corresponding structural dual variables is zero (Step 8 of proof).

Illustration of Complementary Solutions

Tables 6 and 7 respectively show the 10 basic solutions for the primal and dual problems given in Example 4. In equality form, the primal problem has 5 variables and 3 constraints, while the dual has 5 variables and 2 constraints. Thus there are $\frac{n}{m} = \frac{5}{3} = \frac{5}{2} = 10$ potential bases in each case. The numbers in the leftmost column of the tables identify the complementary solutions; e.g., No. 1 in Table 6 is complementary to No. 1 in Table 7. Note that for No. 7, no solution exists because the columns associated with the variables (x_{s1}, x_{s2}, x_2) as well as the columns associated with (s_1, s_3) do not form a basis.

Table 6. Basic solutions for primal problem

No.	Basic variables	Nonbasic variables	x_1	x_2	x_{s1}	x_{s2}	x_{s3}	z_p	Primal status
1	x_{s1}, x_{s2}, x_{s3}	x_1, x_2	0	0	5	35	20	0	Feasible
2	x_1, x_{s2}, x_{s3}	x_2, x_{s1}	-5	0	0	40	25	-10	Infeasible
3	x_2, x_{s2}, x_{s3}	x_1, x_{s1}	0	5	0	20	20	15	Feasible
4	x_{s1}, x_1, x_{s3}	x_2, x_{s2}	35	0	40	0	-15	70	Infeasible
5	x_{s1}, x_2, x_{s3}	x_1, x_{s2}	0	11.67	-6.67	0	20	35	Infeasible
6	x_{s1}, x_{s2}, x_1	x_2, x_{s3}	20	0	25	15	0	40	Feasible
7	x_{s1}, x_{s2}, x_2	x_1, x_{s3}	No solution						
8	x_1, x_2, x_{s3}	x_{s1}, x_{s2}	5	10	0	0	15	40	Feasible
9	x_1, x_{s2}, x_2	x_{s1}, x_{s3}	20	25	0	-60	0	115	Infeasible
10	x_{s1}, x_1, x_2	x_{s2}, x_{s3}	20	5	20	0	0	55	Feasible

Table 7. Basic solutions for dual problem

No.	Basic variables	Nonbasic variables						z_D	Dual status
			s_1	s_2	1	2	3		
1	s_1, s_2	1, 2, 3	-2	-3	0	0	0	0	Infeasible
2	$s_2, 1$	$s_1, 2, 3$	0	-5	-2	0	0	-10	Infeasible
3	$s_1, 1$	$s_2, 2, 3$	-5	0	3	0	0	15	Infeasible
4	$s_2, 2$	1, $s_1, 3$	0	3	0	2	0	70	Feasible
5	$s_1, 2$	1, $s_2, 3$	-1	0	0	1	0	35	Infeasible
6	$s_2, 3$	1, 2, s_1	0	-3	0	0	2	40	Infeasible
7	$s_1, 3$	1, 2, s_2	No solution						
8	1, 2	$s_1, s_2, 3$	0	0	-0.75	1.25	0	40	Infeasible
9	1, 3	$s_1, 2, s_2$	0	0	3	0	5	115	Feasible
10	2, 3	1, s_1, s_2	0	0	0	1	1	55	Feasible

The conditions derived in this section are illustrated by the data in the tables. Fig. 1 shows the objective function values for the solutions that are feasible for the primal and dual problems. As can be seen, the objective value for every primal feasible solution provides a lower bound for the optimal dual objective ($z_D = 55$). The objective value for every feasible dual solution provides an upper bound for the optimal primal objective ($z_P = 55$). These bounds converge to the optimum as might be expected; however, $z_P = z_D$ for all points.

The complementary solutions property is similarly exhibited by all points. For example, consider No. 9. Here, x_1 is basic in the primal and s_1 is nonbasic in the dual so the first dual constraint is satisfied as an equality. Also, x_{s_2} is basic and z_2 , the corresponding dual variable, is nonbasic. These two observations can be written mathematically as $x_1 - s_1 = 0$ and $x_{s_2} - z_2 = 0$. In the next section, we provide a general statement of this result for all complementary pairs.

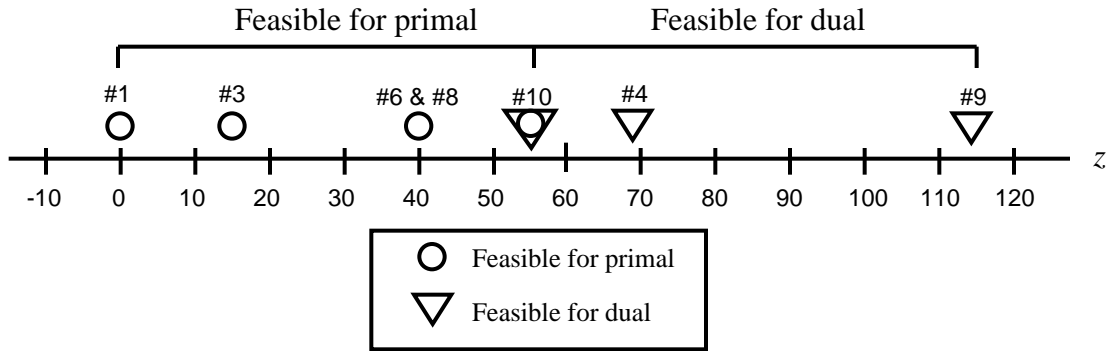


Figure 1. Objective values of primal and dual solutions

Finding Complementary Solutions for Standard Inequality Forms

The simplex algorithm solves the primal and dual problems simultaneously. This is obvious for the revised simplex method which uses the complementary dual solution directly in the computations. When the primal and dual problems are in the standard inequality form given in Table 1, the tableau method provides all dual values in row 0. Fig. 2 shows row 0 of the general tableau. Note that the x 's are labels but the entries in row 0 are numbers corresponding the values of the dual variables. The dual slacks appear under the primal structural variable labels, and the dual structural variables appear under the primal slack variable labels.

Row no.	Basic variables	Coefficients									RHS
		z	x_1	x_2	...	x_n	x_{s1}	x_{s2}	...	x_{sm}	
0	z	1	s_1	s_2	...	s_n	1	2	...	m	$z_D = z_P$

Figure 2. Dual variables shown in the simplex tableau

To illustrate, solution No. 3 from Tables 6 and 7 is displayed in the tableau below. The primal solution is shown as the RHS vector. The complementary dual solution is given in row 0. In particular, $s_1 = -5$, $s_2 = 3$ and $z_D = 15$, while all other dual variables are zero.

Row	Basic	Coefficients						RHS
		z	x_1	x_2	x_{s1}	x_{s2}	x_{s3}	
0	z	1	-5	0	3	0	0	15
1	x_2	0	-1	1	1	0	0	5
2	x_{s2}	0	4	0	-3	1	0	20
3	x_{s3}	0	1	0	0	0	1	20

The optimal tableau for the example is shown next. From this tableau we can read both the primal and dual solutions. For the dual problem the optimum is $z^* = 15$, $x_1^* = 1$, and $x_2^* = 0$.

Row	Basic	Coefficients						RHS
		z	x_1	x_2	x_{s1}	x_{s2}	x_{s3}	
0	z	1	0	0	0	1	1	55
1	x_2	0	0	1	0	1/3	-1/3	5
2	x_1	0	1	0	0	0	1	20
3	x_{s1}	0	0	0	1	-1/3	4/3	20

Finding Complementary Solutions for Nonstandard Forms

When the primal linear programming problem is in a nonstandard form with equality or *greater than or equal to* constraints, the dual variables do not appear directly in the tableau. For an equality in the primal, there is no slack variable in the tableau; however, if a unit vector is inserted in phase 1 to represent the artificial variable for that constraint in the initial tableau, the dual variable will appear in row 0 under that artificial variable (assuming the artificial is given a zero objective coefficient in phase 2). For a *greater than or equal to* constraint the dual variable associated with that constraint is the negative of the value appearing in row 0 of the column of the slack variable for the constraint. Finally, because the primal simplex method requires that all variables be restricted to be nonnegative, nonstandard forms that contain unrestricted variables or variables constrained to be nonpositive are not allowed.

The foregoing developments are neatly summarized in the following theorem.

Theorem 5 (Necessary and Sufficient Optimality Conditions) Consider a primal-dual pair of LPs. Let \mathbf{x} and $\boldsymbol{\pi}$ be the primal and dual variables and let $z_P(\mathbf{x})$ and $z_D(\boldsymbol{\pi})$ be the corresponding objective functions. If \mathbf{x} is a primal feasible solution, it is optimal iff there exists a dual feasible solution $\boldsymbol{\pi}$ satisfying $z_P(\mathbf{x}) = z_D(\boldsymbol{\pi})$.

The “if” (sufficient) part of the proof follows directly from the sufficient optimality conditions of Theorem 2. The “only” (necessary) part follows from the optimality of complementary solutions stated in Theorem 4. It also follows from the Fundamental Duality Theorem.

Complementary Slackness

We have observed the complementary slackness property of complementary basic solutions which holds for all bases whether optimal or not. To present this property in mathematical terms, we first recast the primal and dual problems given in Table 1 by introducing slack variables. Table 8 defines the revised models.

Table 8. Primal and dual problems with slack variables added

(P) Maximize $z_P = \mathbf{c}\mathbf{x}$ subject to $\mathbf{A}\mathbf{x} + \mathbf{I}_m\mathbf{x}_s = \mathbf{b}$ $\mathbf{x} \geq \mathbf{0}, \mathbf{x}_s \geq \mathbf{0}$	(D) Minimize $z_D = \boldsymbol{\pi}\mathbf{b}$ subject to $\boldsymbol{\pi}\mathbf{A} - \boldsymbol{\pi}_s\mathbf{I}_n = \mathbf{c}$ $\boldsymbol{\pi} \geq \mathbf{0}, \boldsymbol{\pi}_s \geq \mathbf{0}$
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The vector of slacks for the primal is $\mathbf{x}_s = (x_{s1}, x_{s2}, \dots, x_{sm})$, with x_{si} the slack variable for the i th constraint. The vector of slacks for the dual is $\boldsymbol{\pi}_s = (\pi_{s1}, \pi_{s2}, \dots, \pi_{sn})$, with π_{sj} the slack variable for the j th constraint. \mathbf{I}_m and \mathbf{I}_n are identity matrices of size m and n , respectively. Both problems in equality form have $n + m$ variables.

The primal and dual variables are linked by identifying $n + m$ pairs with one variable in the pair from each problem.

Pair (x_j, π_{sj}) for $j = 1$ to n : The primal variable x_j is paired with the slack variable π_{sj} associated with the j th dual constraint.

Pair (x_{si}, π_i) for $i = 1$ to m : The dual variable π_i is paired with the slack variable x_{si} associated with the i th primal constraint.

Complementary slackness is the condition that at least one member of each pair is zero. For a particular pair (x_j, π_{sj}) , the property implies that

either x_j is zero or the corresponding dual constraint is satisfied as an equality. For the pair (x_{si}, π_i) , it implies that either primal constraint i is satisfied as an equality, or the corresponding dual variable is zero.

Theorem 6 (Complementary Slackness) The pairs $(\mathbf{x}, \mathbf{x}_s)$ and $(\boldsymbol{\pi}, \boldsymbol{\pi}_s)$ of primal and dual feasible solutions are optimal to their respective problems iff whenever a slack variable in one problem is strictly positive, the value of the associated nonnegative variable of the other problem is zero.

For the primal-dual pair in Table 8, the theorem has the following interpretation. Whenever

$$x_{si} = b_i - \sum_{j=1}^n a_{ij}x_j > 0 \text{ we have } \pi_i = 0 \quad (3)$$

$$s_j = \sum_{i=1}^m a_{ij} \pi_i - c_j > 0 \text{ we have } x_j = 0 \quad (4)$$

Alternatively, we have

$$x_{si} = b_i - \sum_{j=1}^n a_{ij}x_j = 0, i = 1, \dots, m \quad (5)$$

$$x_j s_j = x_j \sum_{i=1}^m a_{ij} \pi_i - c_j x_j = 0, j = 1, \dots, n \quad (6)$$

The proof of the theorem is left as an exercise. In vector notation (5) and (6) can be written collectively as $\boldsymbol{\pi} \mathbf{x}_s = 0$ and $\mathbf{x} \boldsymbol{\pi}_s = 0$, respectively.

Conditions (3) or (5) only require that if $x_{si} > 0$, then $\pi_i = 0$. They do not require that if $x_{si} = 0$, then π_i must be > 0 ; that is, both x_{si} and π_i could be zero and the conditions of the theorem would be satisfied. Moreover, conditions (3) or (5) automatically imply that if $\pi_i > 0$, then $x_{si} = 0$. The same is true for (4) or (6). For instance, if $x_j > 0$, then $s_j = 0$.

The complementary slackness theorem does not say anything about the values of unrestricted variables (corresponding to equality constraints

in the other problem) in a pair of optimal feasible solutions. This is the situation, for example, when the primal is written in equality form as in Table 3. It is concerned only with nonnegative variables of one problem and the slack variables corresponding to the associated inequality constraints in the other problem.

Economic Interpretation

Consider the primal-dual pair given in Table 1. Assume that the primal problem represents a chemical manufacturer that is under direction to limit its output of toxic wastes. Suppose that over a given period of time it produces n different types of chemicals that return a unit profit of c_j each for $j = 1, \dots, n$, and that no more than b_i units of toxic waste i can result from the manufacturing process, $i = 1, \dots, m$. Let a_{ij} be the amount of byproduct i generated by the manufacture of one unit of chemical j . The problem is to decide how many units of j to produce, denoted by x_j , so that no toxic waste levels are exceeded. These constraints can be written as

$$\sum_{j=1}^n a_{ij} x_j \leq b_i \text{ for all } i.$$

To derive an equivalent dual problem, let $\pi_i \geq 0$ be the unit contribution to profit associated with byproduct i . Thus π_i can be interpreted as the amount the company should be willing to pay to be allowed to generate one unit of toxic waste i . Consequently, the term $\sum_{i=1}^m \pi_i a_{ij}$ represents the implied contribution to profit associated with the current mix of toxic wastes when one unit of chemical j is produced. Because the same mix of wastes could probably be generated in other ways as well, no alternative use should be considered if it is less profitable than chemical j . This leads to the constraint $\sum_{i=1}^m \pi_i a_{ij} \leq c_j$ for all j ; however, if a strict inequality holds for some j giving $\sum_{i=1}^m \pi_i a_{ij} < c_j$, better use of the permissible toxic waste levels can be found so it is optimal to set $x_j = 0$. If $x_j > 0$, the implied value of the toxic waste mix should be just equal to the unit profit for chemical j , giving $\sum_{i=1}^m \pi_i a_{ij} = c_j$.

Similarly, if $\sum_{j=1}^n a_{ij} x_j < b_i$ for some i , the marginal contribution to profit associated with the i th toxic waste limit is zero so we should set $\pi_i = 0$. If $\pi_i > 0$ the manufacturer should be generating as much byproduct i as permissible, implying $\sum_{j=1}^n a_{ij} x_j = b_i$. These conditions are nothing more than complementary slackness in Eqs. (3) and (4). When they are satisfied, there is no incentive for the manufacturer to alter its production

plan or to change the implied price structure.

The objective of the dual problem is to minimize pollution costs which can be interpreted as minimizing the total implicit value of toxic wastes generated in the manufacturing process. At optimality, the minimum cost incurred is exactly the maximum revenue realized in the primal. This results in an economic equilibrium where cost = revenue, or

$$\sum_{i=1}^m b_i = \sum_{j=1}^n c_j x_j.$$