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Linear Programming
Computer Interactive Course

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The book contains some fundamental facts of Linear Programming theory and 70 simplex method case study assignments offered to students for implementing in their own software products. For students in Applied Mathematics, Mathematical Information Technology and others who wish to acquire deeper knowledge of the simplex method through designing their study projects and making wide range numerical experiments with them.

Contents

Preface

The book develops a general methodological framework for teaching Computational Linear Algebra (in the part concerning Linear Programming) so that some minimum learning outcomes are achieved for all students and at the same time student-dependent learning outcomes due to individual creativity and effort are also possible.

Although the theoretical ideas and pivotal concepts behind the book, such as motivation, feedback, reinforcement and others, are well known, specific implementation of these ideas may present a real challenge to practical teachers. The book reflects a comprehensive detailed view of one aspect of differentiated teaching: behavior modification. This aspect is taken as a prime necessity in circumstances when Computational Mathematics is being taught to a large student audience. The methodological conditions and technological practices necessary to implement such an approach are discussed below in the Preface.

Key words: Mathematics Education in Universities, Creativity in Mathematics Education, Competitions in Mathematics Education, Linear Programming.

Introduction

As Russian mathematician Yakov Tsypkin wrote jokingly in the preface to one of his works, reading mathematical books results in the three levels of knowledge. The first level means that a reader has understood the author's argumentation. The second level means that the reader has become capable of reproducing the author's arguments. And the third level means that the reader has acquired a capacity to refute the author's argumentation.

This joke reflects the fact that mathematics as a subject has a specific feature that it can not simply be put in memory. It means that a mathematics student can not stop at the first level of understanding. He needs to reach at least the second level. To do this the student has to pass all the information through his mind by solving a large number of tasks independently, thus as we say "adjust his head and hand". But even this is not enough, because as Hungarian mathematician Alfred Renyi said, "who learns the solution without understanding the matter can not use it properly". Independence, critical approach and

creativeness – these are the third level features and only such knowledge has real value when learning mathematics.

Of late years, mathematics community has been really feeling the need for novel teaching methods to stir the students to greater activity. One would expect a wealth of methods to choose from and apply. But upon examining methods that are practically being used, one finds a lack of appealing and interesting approaches that would create and hold students' interest and make them continue to study.

Here it is pertinent to note that mathematics has another feature. Mathematics may be defined as "chamber" science by the nature and mathematicians are often told to be "piece-goods". However many universities traditionally practice teaching to a large student audience. For example, large audience-oriented teaching has become the trait of some universities to the extent that the large audience have been assigned a special term, "a stream".

It is almost evident that the great size of the stream stands on the way, – it erects obstacles to awaken and hold students' interest in learning mathematics. It is a handicap to independent thinking, as many of students get used to "flow over the stream" and prefer to be as ordinary as their classmates. How can teacher transform this obstacle into advantage? How can he encourage students' independence? How can we help them to understand their outstanding abilities? And finally, how can we prove that mathematics is a live, beautiful science but not a collection of incontestable proofs, unquestionable facts and irrefutable arguments?

Fortunately, mathematics itself often prompts us how to achieve these goals. One of such methods termed the Frontal Competitive Approach (FCA) is discussed below. We consider it mainly in circumstances when the subject is being taught to a large student audience where students' behavior modification is a prime necessity.

Main idea of FCA

It may be explained by the definition itself. "Frontal" means general, involving all students to meet one common goal. "Competitive" means opportunity for a success due to individual's creative and non-standard solutions or actions. To implement FCA we need to do the following: (1) Organize creative environment. (2) Encourage students' creative potential. (3) Give start to students' instinct of competition. (4) Ensure transparency of assessment.

Let us examine these components in detail in the context of teaching Computational Mathematics.

Creative environment

Most of the existing educational materials on Computational Mathematics provide main theoretical data and sometimes theoretical instructions on how to program a numerical method or algorithm. However, it seems to be inadequate to the end. We believe that the true understanding of a numerical method may be achieved if: (a) a student

completes assignments related to a challenging programming project; (b) each project results in practical use of that particular method assigned for the student; (c) the student conducts a set of extensive computational experiments with the program he developed independently; and finally (d) frontal rating of the projects is carried out by the teacher together with the students.

Programming in itself is beneficial for student due to a number of reasons. First, it provides an opportunity to understand and learn a numerical method "from inside". This is quite different from utilizing ready-made software and significant for any creative professional. Second, it improves student's computer proficiency, as it requires keen programming. And finally, it develops general analytic and solution seeking performance and implants practical skill to attack and solve computationally oriented problems.

To organize creative environment while teaching numerical methods to a large audience requires to make programming assignments as varied as possible in terms of the methods' algorithmic significance rather than their initial data. However, the number of variations on every method is usually limited. In these conditions, finding as many as possible versions of every numerical method becomes a matter of great methodological importance for each teacher.

Organizing creative environment means, also, that we should evaluate any laboratory programming project as a single study objective which possesses all the features of a completed software product. Among them are modular structure, convenient interface, efficient utilizing computer resources (memory and time), and possibility to implement a wide plan of computational experiments. This differs definitely from a widely used technique when the students work on one and the same ready-made software when they only enter their initial data and wait passively for a result. The approach we apply makes them perform valuable creative operations, stimulates each student's competitiveness, prevents cheating and helps to improve overall class performance.

A classic example of how to find as many as possible variant forms of a numerical method is the topic "Elimination and Matrix Inversion Methods". First of all, the teacher should systemize a set of Gauss and Gauss-Jordan elimination specific characteristics. They are: (1) direction of elimination of unknowns, (2) mode of access to the matrix entries, (3) mode of updating the active sub-matrix, (4) pivoting strategy etc. Then independence of these characteristics will result in a significant number of different variants of assignments on the same topic being studied.

Over the course on many years, our work is focused on the possible ways of applying FCA to teaching numerical methods in Linear Algebra, Least Squares, Optimal Filtering, Optimal Control, and Linear Programming. As a result, we recommend that teachers use our textbooks that offer a good choice of various project assignments. This book presents a collection of 70 different case study assignments in standard Linear Programming methods.

Student's creative potential

Lectures usually prove one variant of a numerical method in a certain topic. For example, we prove: LU-factorization theorem, LU-factorization theorem with the choice of pivots, LU-factorization algorithm replacing the original matrix by factors L and U, and so on. However, practically always for each proven theorem or algorithm there exists a dual variant. In our case the dual variants are those with $UL-$factorization (LDU and UDL factorizations are also possible). Therefore it will be expedient if each assignment contains formulation and proof of the theorem/algorithm for the assigned variant, which has to be made independently. Thus students are trained to understand subject of mathematics in a wider sense, their creative abilities and potential become more active and may be well evaluated especially for the gifted students. The same approach may be used while teaching Linear Programming methods, too.

Instinct of competition

As a rule, students of mathematical departments are very eager to gain and demonstrate professional knowledge of computers and modern programming technology. They are not interested in "hanging about" the initial level of computer proficiency. More to it, they express definitely their wish to show their skills for creating "outstanding" software. FCA ideally supports this instinct of competition. Indeed, multiplicity of assignment variants fortunately comprises two features of the variants: their resemblance and difference. Due to the resemblance between variants students' projects can be compared, and due to the difference they are of individual nature. Teachers applying FCA note surprising cases when a student who has got already a credit on the project, for example in sparse system solution, continues upgrading the software by changing access mode to matrix entries in order to achieve faster operation. Sometimes students arrange a kind of competition between them, in whose software has better interface or faster operation. Sometimes in holidays we hold a presentation of the best programs developed by the students. At first we required that a software should be written in Pascal but now we accept usage of various tools: Visual Basic, Delphi, Builder C++ etc.

Transparency of assessment

Teacher's role in FCA application is very important. Besides all above mentioned, a teacher should put forward a precise and definite system of requirements and evaluation criteria for students' projects. A student should know definitely what mark and for what work quantity and quality he will get. Starting his work student chooses by himself the level of assessment he initially pretends for. System of assignments should be designed in such a way that allows each student to move independently from one assessment level to another according to his own work. For example, the system of assignments on simplex method given in this book, contains 70 different variants divided into three groups dependent on their complexity: basic level (20 variants), advanced level (30 variants),

and higher level (20 variants). The marks are given accordingly: sufficient, good, and excellent. This transparency of assessment have a notable effect on the students' activity.

Technological summary of FCA

Some general tools indicative of the FCA are the following.

1. *Creative environment.* A broad assortment of assignments and tasks is offered to students together with the clearly differentiated scenarios of their accomplishment.

2. *Goal setting.* A clear formulation of both short-term and long-term goals is offered: (1) marked improvement of programming skills together with the deeper understanding a particular numerical method, and (2) profound understanding the subject of Computational Mathematics and ability to attack computationally oriented problems.

3. *Challenge.* The environment including non-trivial assignments and tasks with increasing levels of difficulty, challenges the student to keep self-independent working.

4. *Student-controlled navigation.* Putting the locus of control in the hands of the student has a great psychological effect: even the weak students put in a claim for higher grades and try to move to the upper level of difficulty while choosing the assignment.

5. *Competition.* Competition by "playing" against others appears to be naturally embedded in the teaching process because the above tools are in excellent agreement with this human instinct.

6. *Rewards.* Rewards, such as higher grades or "automatic" credit for the course, may be offered as students show an obvious increment in skill and success.

Conclusion

Designing efficient education process in a large audience is complicated and time consuming. This book is intended to touch on a few basic teaching tools to exploit when motivating students to learn Computational Mathematics. Called FCA and applied in practice, this approach has proved that students have a generally positive response to it. Individual, self-dependent work within a large audience encourages students' sound competition, desire for creative solutions and better performance indices.

Chapter 1.

General Definitions

> The material of this first chapter is introductory: its goal is to recall those basic concepts which the student may know from Calculus and Linear Algebra and must thoroughly understand before proceeding.

Given any two objects x and y, the **ordered pair** is (x, y) where x is distinguished as the first element and y as the second element of the pair. A set of ordered pairs is called a **relation**. For example, if the set of ordered pairs of real numbers whose values are equal is denoted by A, we write down this relation as $A = \{(x, y) \in R \times R : x = y\}$. If A is a relation then the **domain** of A, abbreviated dmn A, is given by

$$\text{dmn } A = \{x : (x, y) \in A \text{ for some } y\}$$

If A is a relation then the **range** of A, abbreviated rng A, is given by

$$\text{rng } A = \{y : (x, y) \in A \text{ for some } x\}$$

A **function** f is a relation having the property that whenever $(x, y) \in f$ and $(x, y^*) \in f$, then $y = y^*$. A **sequence** is a function whose domain is a set of intergers. A **finite sequence** is one whose domain is a finite set of integers, usually $\{1, 2, \ldots, n\}$ for some positive integer n. It is also called an n-**tuple**. The set of real valued sequences with domain $\{1, 2, \ldots, n\}$ is denoted by R^n. Each element of R^n is a **real** n-**tuple**, or a **real sequence of n elements**, or an n-**dimensional vector**. An n-vector x is written as $x = (x_1, x_2, \ldots x_n)$.

Definition 1.1. *A* **point**, *or a* **vector** *in R^n is a sequence of n real numbers:* $x = (x_1, x_2, \ldots x_n)$.

The j-th position of n-vector $x = (x_1, x_2, \ldots x_n)$ is called the jth coordinate, and x_j is called the jth coordinate value of x, or the jth component of x. Thus, vector $x = (x_1, x_2, \ldots x_n)$ is a real valued n-tuple.

While using vector-matrix operations, it is customarily to identify a vector with a one-column matrix. We shall do this too, although for saving space within the text, it is more convenient to write the vector components in row form, as done in the above. But if it is necessary to emphasize that vectors written as $x = (x_1, x_2, \ldots x_n)$ are thought of as $n \times 1$ matrices, the transpose symbol will be used: $x = (x_1, x_2, \ldots x_n)^T$ or $x^T = (x_1, x_2 \ldots x_n)$.

Definition 1.2. *Given two distinct points P and Q represented by two vectors $p, q \in R^n$, **line segment** $\overleftarrow{PQ}$ is the set of points defined by*

$$\Theta p + (1 - \Theta)q, \qquad 0 \leq \Theta \leq 1$$

The equivalent expression $q + \Theta(p - q), \quad 0 \leq \Theta \leq 1$, has a clear geometric interpretation: vector q is added with the multiple $\Theta \quad (0 \leq \Theta \leq 1)$ of the difference vector $(p - q)$, the latter being directed from the point Q to the point P. Since it is always possible to replace Θ by $\gamma, \quad \gamma = 1 - \Theta$, so the segment direction denoted here by the over left arrow, does not matter. Owing to this, a line segment is usually denoted by the over line: $\overline{PQ}$.

Definition 1.3. *The set of points S is said to be **convex** if $\forall p, q \in S : \Theta p + (1 - \Theta)q \in S$, $0 \leq \Theta \leq 1$, i.e. if the whole of the segment of a straight line joining any two points of S, lies in S.*

Definition 1.4. *An extremal point or a corner point or a **vertex** of a convex set S is any point of it not belonging to a line segment joining two arbitrary points of the set.*

In the other words, point $b \in S$ is a vertex of a convex set S if there do not exist two points $p, q \in S$ such that $b = \Theta p + (1 - \Theta)q$ for some Θ, $0 < \Theta < 1$.

Definition 1.5. The convex span *of points $P_1, P_2, \ldots P_k$ represented by the corresponding vectors $p_1, p_2 \ldots p_k$, is the set of points*

$$y = \Theta_1 p_1 + \Theta_2 p_2 + \ldots \Theta_k p_k, \quad \sum_{i=1}^{k} \Theta_i = 1, \quad 0 \leq \Theta_i, \quad i = \overline{1, k}$$

The convex span can be construstively deduced (built) by the sequential applying the definition of line segment. By doing so, it may be clarified the condition for the weighting coefficients Θ_i. Let us draw such a deduction.

Let P_1, P_2 be arbitrary points in R^n. Then

$$y_2 = k_2 p_2 + (1 - k_2)p_1, \quad 0 \leq k_2 \leq 1$$

is a point in $\overline{P_1 P_2}$. Join it with an arbitrary point P_3 by the line segment. Then

$$y_3 = k_3 p_3 + (1 - k_3)y_2 = k_3 p_3 + (1 - k_3)k_2 p_2 + (1 - k_3)(1 - k_2)p_1, \quad 0 \leq k_3 \leq 1$$

is a point in the triangle $P_1 P_2 P_3$. Join it with an arbitrary point P_4 by the other line segment. Then

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$$y_4 = k_4 p_4 + (1 - k_4)y_3 = k_4 p_4 + (1 - k_4)k_3 p_3$$
$$+ (1 - k_4)(1 - k_3)k_2 p_2 + (1 - k_4)(1 - k_3)(1 - k_2)p_1, \quad 0 \le k_4 \le 1$$

is a point in the tetrahedron $P_1 P_2 P_3 P_4$.

Such a process can be proceeded or stopped after all available points are involved. For instance, if the last of available poins is P_5 then having joined it with y_4 by the line segment, we obtain that

$$y_5 = k_5 p_5 + (1 - k_5)y_4$$

$$= k_5 p_5 + (1 - k_5)(k_4 p_4 + (1 - k_4)(k_3 p_3 + (1 - k_3)(k_2 p_2 + (1 - k_2)p_1))), \quad 0 \le k_5 \le 1$$

is an arbitrary point of the pentahedron $P_1 P_2 P_3 P_4 P_5$.

By the construction, we have

$$y_5 = \Theta_5 p_5 + \Theta_4 p_4 + \Theta_3 p_3 + \Theta_2 p_2 + \Theta_1 p_1$$

where
$\Theta_5 = k_5$
$\Theta_4 = (1 - k_5)k_4$
$\Theta_3 = (1 - k_5)(1 - k_4)k_3$
$\Theta_2 = (1 - k_5)(1 - k_4)(1 - k_3)k_2$
$\Theta_1 = (1 - k_5)(1 - k_4)(1 - k_3)(1 - k_2)$

Let now the weighting coefficients Θ_i be subject to the constraint

$$\Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 + \Theta_5 = 1, \quad \Theta_i \ge 0, \quad i = \overline{1,5}$$

Then, choosing any $\Theta_i, 0 \le \Theta_i \le 1$, under this condition, we find sequentially

$$k_5 = \Theta_5, \quad 0 \le k_5 \le 1$$

$$k_4 = \frac{\Theta_4}{(1 - \Theta_5)} = \frac{\Theta_4}{(\Theta_1 + \Theta_2 + \Theta_3 + \Theta_4)}, \quad 0 \le k_4 \le 1$$

$$k_3 = \frac{\Theta_3}{(1 - \Theta_5 - \Theta_4)} = \frac{\Theta_3}{(\Theta_1 + \Theta_2 + \Theta_3)}, \quad 0 \le k_3 \le 1$$

$$k_2 = \frac{\Theta_2}{(1 - \Theta_5 - \Theta_4 - \Theta_3)} = \frac{\Theta_2}{(\Theta_1 + \Theta_2)}, \quad 0 \le k_2 \le 1$$

These are the same $k_2, k_3, k_4 \ k_5$ used sequentially in the constructing of the arbitrary points of: line segment (k_2), triangle (k_3), tetrahedron (k_4), and pentahedron (k_5). By doing so, it has been proven that the convex span of points $p_1, p_2, \ldots, p_k$ is completely defined by the constraint $\Theta_1 + \Theta_2 + \ldots + \Theta_k = 1$ on the non-negative coefficients Θ_i in the linear combination

$$y = \Theta_1 p_1 + \Theta_2 p_2 + \ldots + \Theta_k p_k$$

Definition 1.6. *If f is a function and A is a subset of dmn f, then f_A,* **the restriction of f to** *A, is the function whose domain is A, given by*

$$f_A(t) = f(t) \quad for\ t \in A$$

Definition 1.7. *A* **level curve of** *f where f is a function of n real variables, is a set $C \subseteq$ dmn f if the restriction of f to C is constant.*

Definition 1.8. *If $f(x)$ is a real function of n real variables $x = (x_1, x_2, \ldots x_n)$, and if the partial derivatives $f_1(x), \ldots, f_n(x)$ exist (where $f_i(x) = \partial f(x)/\partial x_i$), the gradient of f at x is defined as the n-dimensional vector*

$$\operatorname{grad} f(x) \equiv (f_1(x), \ldots, f_n(x))$$

In general, the gradient of a real valued function f of n real variables is a vector having the direction of greatest increase of f at each point in dmn f. Correspondingly, the vector of antigradient, $- \operatorname{grad} f(x)$, shows the direction of steepest descent along the surface f in the domain. For all real valued functions f of n real variables possessing a continuous gradient, a level curve of f is orthogonal to the gradient at each point in dmn f.

Chapter 2.

A Standard Linear Programming Problem

In this section, we use the standard matrix form of a linear programming problem to which any LP problem can be easily reduced. We describe several facts, explaining how to narrow the search for the optimum over the feasible region and providing a better understanding of the simplex method, which is an algebraic method readily programmable for a computer.

2.1. Problem statement

The LP problem (in the standard form) has:

1) given constraints $Ax = b$, $A = A(m,n)$, $rank A = m$, $(m < n)$

$$x \in R^n, \quad b \in R^m$$

2) natural constraints restricting every component of x to a nonnegative value, $x \geq 0$
(i. e. $\forall i : x_i \geq 0$)

and therewith requires $\min_x c^T x$ where $c \in R^n$. So, the standard LP problem looks as follows:

$$\begin{array}{lrcl} \text{minimize} & z & = & c^T x \\ \text{over those } x \text{ satisfying} & Ax & = & b, \quad x \geq 0 \end{array}$$

where x is $n \times 1$, c is $n \times 1$, A is is $m \times n$, and b is $m \times 1$.

Remark 2.1.1. *Vector $c = (c_1, c_2, \ldots c_n)$ determines the* **objective function** *z, $z = c^T x$, in n variables $(x_1, x_2, \ldots x_n)$ as well as the vector of gradient: $\operatorname{grad} z = c$. This vector have the following property: Given different constant values denoted by "const", vector $\operatorname{grad} z = c$ is orthogonal to the $c^T x = const$ hyperplanes in $dmn\, z = R^n$.*

Definition 2.1. *A standard linear programming problem is said to be* **nondegenerate** *if every $m \times m$ submatrix selected from the $m \times (n+1)$ augmented matrix $A_a = [A \,|\, b]$ is nonsingular. Otherwise the problem is said to be* **degenerate**.

Remark 2.1.2. *The assumption of nondegeneracy is valid in most of the problems encountered in practice and is convenient in theory. How to cope with the task of solving degenerate problems will be considered in Chapter 6.*

General **key characteristics of the LP problem** are as follows: (1) convexity of the set of feasible solutions; (2) existence of feasible basic solutions; (3) identity of feasible basic solutions with vertices of the feasible region; and (4) coincidence of at least one optimal solution (if it exists) with a vertex of feasible region. Let us prove these properties consecutively.

2.2. Convexity of the set of feasible solutions

Definition 2.2. *A set X of points in R^n satisfying both the given and natural constraints of the LP problem, $Ax = b$, $x \geq 0$, is the* **set of feasible solutions** *or, shortly, the* **feasible region**.

Theorem 2.2.1. *The set X of feasible solutions x,*

$$X = \{x \in R^n : Ax = b, \ x \geq 0\} \tag{2.1}$$

in a linear programming problem is convex.

Proof. Let x, y be an arbitrary pair of points in X. Then

$$Ax = b, \ x \geq 0; \quad Ay = b, \ y \geq 0$$

The line segment joining these points is the set of points w,

$$w = \alpha x + \beta y, \ \alpha + \beta = 1, \ \alpha \geq 0, \ \beta \geq 0, \ w \geq 0$$

We have

$$Aw = A(\alpha x + \beta y) = \alpha Ax + \beta Ay = \alpha b + \beta b = (\alpha + \beta)b = b$$

i.e. $Aw = b, \ w \geq 0$, which proves the theorem.

2.3. Existence of feasible basic solutions (FBS)

Definition 2.3. *A* **basic solution** *to the LP problem (in the standard form) is a vector*

$$x = P \begin{pmatrix} B^{-1}b \\ \text{- - -} \\ O \end{pmatrix}$$

where P is a permutation matrix introduced to interchange the columns of A so that $AP = [B \,|\, R]$, B being a $m \times m$-matrix of full rank, O being a block of $(n-m)$ zeros, and R being a block of $(n-m)$ column vectors linearly dependent on those included in B.

All basic solutions are determined by the different linearly independent sets of m columns of A. The permutation matrix P affects matrix A so that a linear independent combination of m column vectors of A appears in submatrix B. Then the constraint $Ax = b$ is eqiuvalent to $APy = b$ where $y = P^{-1}x$.

Splitting column vector y into two patrs, $y = (y^B, y^F)$, where y^B is $m \times 1$ and y^F is $(n-m) \times 1$, we have

$$y^B = B^{-1}b - B^{-1}Ry^F$$

$$y = \begin{pmatrix} B^{-1}b - B^{-1}Ry^F \\ \text{- - -} \\ y^F \end{pmatrix}$$

In such a way, solution y to $APy = b$ is expressed through the free variables y^F. Setting them equal to zero, we obtain vector

$$y = \begin{pmatrix} B^{-1}b \\ O \end{pmatrix}$$

Vector x differs from this y only by the permutation of elements: $x = Py$. It is precisely this vector x that is referred to as a basic solution.

Definition 2.4. *Those $n-m$ zero elements of vector $x = Py$ that happen to be zeros due to setting $y^F = 0$, according to the last paragraph, are called* **nonbasic variables**. *The rest m elements of $x = Py$ are called* **basic variables***, and they form a* **basis***.*

Definition 2.5. *Any basic solution is said to be a* **basic feasible solution** *if all its elements are nonnegative.*

In order to avoid a formal concept of a permutation matrix P in the above definitions, rephrase them in a different way using the following

Definition 2.6. *A* **vector of constraints** $a_i \in R^m$, $i = \overline{1, n}$, *is any column a_j of matrix A:*

$$A = [a_1, a_2, \ldots, a_n]$$

With this notion, the given constraints $Ax = b$ can be written as

$$x_1 a_1 + x_2 a_2 + \ldots + x_n a_n = b \tag{2.2}$$

As the constraint matrix A is of full rank, $rankA = m$, there always exist linearly independent sets of m column vectors of constraints. Number of such sets ranges up to C_n^m, the number of combinations of n objects taken m at a time (without regard to order).

Definition 2.4'. *The variables x_i associated with those columns a_i in (2.2) which form a linearly independent set of m column vectors of constraints are called* **basic variables**. *The other variables are called* **nonbasic**.

Definition 2.7. *Any vector $x = (x_1, x_2, \ldots, x_n)$ satisfying the given constraints (2.2) and the natural constraints $x_i \geq 0$, $i = \overline{1,n}$, is said to be a* **feasible solution** *to the LP problem.*

Definition 2.3'. *A point $x = (x_1, x_2, \ldots, x_n)$ satisfying (2.2) is called a* **basic solution** *if settings $x_i = 0$ are made for those x_i in (2.2) which are associated with the vectors a_i not entering a chosen linearly independent set of m vectors.*

Not every basic solution is feasible. Such situation happens when there are one or more negative elements in a basic solution.

Definition 2.5'. *A basic solution $x = (x_1, x_2, \ldots, x_n)$ is said to be a* **basic feasible solution** *to the LP problem, a BFS for short, if vectors a_i associated with the positive coefficients x_i, $x_i > 0$, in (2.2) form a linearly independent set of vectors.*

It follows from the last definition that the number r of strictly positive elements of a BFS is not greater than m, $r \leq m$, because some x_i in (2.2) associated with the vectors a_i forming a chosen linearly independent set of m vectors, may occur zero.

Definition 2.8. *A basic feasible solution is said to be* **nonsingular** *if it does not contain zero elements. Otherwise the BFS is said to be* **singular**.

Theorem 2.3.1. *If a linear programming problem is nondegenerate, then the basic variables in any basic solution are precisely the nonzero variables.*

Proof. Suppose that there exist q of the basic variables in a basic solution, $q = m-r$, $0 < q < m$, that are zero. Then b would be a linear combination of r columns of B (and thus of A), B being composed of the columns a_i associated with the basic variables x_i in (2.2). It would imply that the problem is degenerate. This contradicts the hypothesis of the theorem and yields the desired result.

Theorem 2.3.2. *If the constraints have a feasible solution, they also have a basic feasible solution.*

Proof. We use the direct approach to build a BFS. Let $(n - r)$ variables in a feasible solution are zero and the rest r variables are positive. This fact, by renumbering the variables if necessary (which can be done without loss of generality), can be written as follows:

$$x_j = 0, \quad j = r + 1, r + 2, \ldots, n$$

$$\sum_{j=1}^{r} x_j a_j = b, \quad x_j > 0, \ j = 1, 2, \ldots, r$$

Case (1): $\{a_j, \ j = 1, 2, \ldots, r\}$ is a linearly independent set. Then $r \leq m$ where $m = rankA$, and thus the given feasible solution is also a basic feasible solution in which $(m - r)$ variables happen to be zero. For this case, the proof is completed.

Case (2): $\{a_j, \ j = 1, 2, \ldots, r\}$ is a linearly dependent set. This means that there exist numbers $\alpha_j, \ j \in \{1, 2, \ldots, r\}$, not all zero, such that

$$\sum_{j=1}^{r} \alpha_j a_j = 0$$

Let $\alpha_k > 0$ for some k (if necessary, this inequality can by multiplied by -1). Then

$$a_k = -\sum_{j=1, j \neq k}^{r} (\frac{\alpha_j}{\alpha_k}) a_j \ .$$

We have the constraint

$$x_1 a_1 + \ldots + x_k a_k + \ldots + x_r a_r = b \tag{2.3}$$

Substituting here the last expression for a_k yields

$$x_1 a_1 + \ldots + x_k(\ldots - \frac{\alpha_{k-1}}{\alpha_k} a_{k-1} - \frac{\alpha_{k+1}}{\alpha_k} a_{k+1} - \ldots) + \ldots + x_r a_r = b$$

After rearrangment, we have

$$\sum_{j \neq k, j=1}^{r} (x_j - \frac{\alpha_j x_k}{\alpha_k}) a_j = b$$

It is always possible to choose k so that

$$\frac{x_k}{\alpha_k} = \min_{j}(\frac{x_j}{\alpha_j}; \alpha_j > 0)$$

By this, the values

$$X_j = x_j - \alpha_j(\frac{x_k}{\alpha_k}), \quad j \neq k, \ j = \overline{1, r} \tag{2.4}$$

turn out positive and

$$X_j = 0, \quad j = k, r + 1, r + 2, \ldots, n \tag{2.5}$$

As a result, the solution $X = (X_1, X_2, \ldots, X_n)$ is feasible, considering that the constraint

$$\sum_{i=1}^{n} X_i a_i = b \tag{2.6}$$

is satisfied and $X_i \geq 0, \ i = \overline{1, n}$, with $(r - 1)$ variables strictly positive (2.4) and $(n - r + 1)$ variables equal to zero (2.5).

It is along this pathway that a_k, which is linearly dependent on the remaining vectors a_j, turns out omitted from the set $\{a_j, \ j = 1, 2, \ldots, r\}$ of vectors associated with positive coefficients in the constraint $Ax = b$, (2.2), (2.3), that is now rewritten as (2.6). Continuing in this way (if necessary), we come to case 1, in which $r \leq m$, that is, to a BFS, and thus we are done.

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2.4. Identity of feasible basic solutions with vertices of the feasible region

Theorem 2.4.1. *The basic feasible solutions to a LP problem coincide with the vertices of the (convex) set of feasible solutions (2.1).*

Proof. Without loss of generality, let us denote a basic feasible solution as

$$x^0 = (x_1, \ldots, x_m, 0, \ldots, 0)^T$$

It means that x^0 is a unique nonnegative solution to $Ax = b$ with the last $(n - m)$ coordinate values being zero.

Remark 2.4.1. *For this theorem, the following equivalent formulation can be used. A feasible solution x^0 to a LP problem is a basic feasible solution if and only if x^0 is a vertex of the feasible region X, (2.1).*

Part I: The condition is necessary. (Given: x^0 is a basic feasible solution. Needed to prove: x^0 is a vertex.) If x^0 were not a vertex, then we could find two different points u and v in X such that $x^0 = \theta u + (1 - \theta)v$ for some θ, $0 < \theta < 1$. In the other words, we would have $Au = b$, $Av = b$, $u, v \geq 0$, and for the last $(n - m)$ coordinate values of u and v we could write:

$0 = \theta u_{m+1} + (1 - \theta)v_{m+1}$
$0 = \theta u_{m+2} + (1 - \theta)v_{m+2}$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$0 = \theta u_n + (1 - \theta)v_n$

In these equations, both multipliers, θ and $(1 - \theta)$, are positive since x^0 is supposed to be an inner point of the line segment joining u and v. Besides, $u_{m+1}, \ldots, u_n \geq 0$ and $v_{m+1}, \ldots, v_n \geq 0$. By these reasons, the last equations hold only if $u_j = 0$ and $v_j = 0$, $j = m + 1, \ldots, n$. It would force us to agree that u and v are both basic feasible solutions with the last $(n - m)$ coordinate values being zero. This contradicts the fact that such a solution to $Ax = b$, $x \geq 0$ is unique, unless we admit that $x^0 = u = v$ is a vertex.

Part II: The condition is sufficient. (Given: x^0 is a vertex of X. Needed to prove: x^0 is a basic feasible solution.) Denote by r the number of positive coordinate values of vector x^0. Prove that $r \leq m$ and that the vectors of constraints associated in (2.2) with these positive values, form a linearly independent set of vectors. To accomplish this, we shall do in the same way as in the proof of Theorem 2.3.1. Again, as above, we have **case 1** and **case 2**. For case 1, the proof is done. Case 2 implies that there exist numbers α_j, $j = \overline{1, r}$, not all zero, such that

$$\sum_{j=1}^{r} \alpha_j a_j = 0 \tag{2.7}$$

Introduce a scalar ρ such that for these $\alpha_j \neq 0$

$$0 < \rho < \min_j \frac{x_j^0}{|\alpha_j|}$$

Then vectors

$$x^1 = x^0 + \rho\alpha$$
$$x^2 = x^0 - \rho\alpha$$

are nonnegative: $x^1 \geq 0$ and $x^2 \geq 0$ (i.e., all their coordinate values are nonnegative) where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r, 0, \ldots, 0)$. Rewrite (2.7) as $A\alpha = 0$. Thus

$$Ax^1 = A(x^0 + \rho\alpha) = Ax^0 + \rho A\alpha = Ax^0 = b$$

$$Ax^2 = A(x^0 - \rho\alpha) = Ax^0 - \rho A\alpha = Ax^0 = b$$

The half-sum of the last equations together with the expression

$$x^0 = \frac{1}{2}(x^1 + x^2) = \theta x^1 + (1-\theta)x^2, \quad \theta = \frac{1}{2}$$

confirms that we have built two different points $x^1, x^2 \in X$, distinct from x^0 and such that x^0 is the inner point of the line segment joining x^1 and x^2 (namely, the middle point of it). This contradicts the hypothesis of Part II of the theorem, unless we admit that all α_j in (2.7) are zero, i.e. a set of vectors $\{a_j, \; j = \overline{1, r}\}$ is linearly independent. But it is known that if $a_j \in R^m$, then the number r of vectors in the set can not exceed m, $r \leq m$. We returned to Case 1 for which the theorem is proven.

Remark 2.4.2. *Theorem 2.4.1 can be rephrased conversely. A point $x^0 = (x_1^0, x_2^0, \ldots x_n^0)$, $x^0 \geq 0$, is a vertex of the feasible region X, (2.1), if and only if vectors of constraints a_j with positive coefficients x_j^0 in the constraints $Ax^0 = b$ form a linearly independent set. It is obvious that in this equivalent formulation the necessity and sufficiency are simply interchanged.*

2.5. Coincidence of the LP problem solution with a vertex of feasible region

The following proposition is the main for the further transition to the practicable computational method having the advantage of very fast arriving at the solution.

Theorem 2.5.1. *If the objective function $z = c^T x$ has a finite minimum, then at least one optimal solution is a basic feasible solution (i.e., a vertex of the feasible hyperpolyhedron X, (2.1)).*

Proof. Let all basic feasible solutions be determined by the vectors $p_1, p_2, \ldots, p_k$ and let the objective function have at these points corresponding values $z_1, z_2, \ldots, z_k$. Since $z = c^T p_i$, then

$$z_i = c^T p_i, \ i = 1, 2, \ldots, k$$

It is known from Theorem 2.4.1 that $p_1, p_2, \ldots, p_k$ are all verices of the set of feasible solutions X and, according to Theorem 2.2.1, this set is convex. By Definition 1.5, any convex span of points represented by vectors $p_1, p_2, \ldots, p_k$ is the set of points

$$x = \theta_1 p_1 + \theta_2 p_2 + \ldots \theta_k p_k, \quad \theta_i \geq 0, \ \sum_{i=1}^{k} \theta_i = 1$$

At any such point x, $x \in X$, we have

$$z = c^T x = \theta_1 c^T p_1 + \theta_2 c^T p_2 + \ldots \theta_k c^T p_k = \theta_1 z_1 + \theta_2 z_2 + \ldots \theta_k z_k$$

Consequently, seeking the solution $x \in X$, optimal in the sense of $\min z = \min c^T x$, can be narrowed to the task of finding such numbers $\theta_i \geq 0$ adding up to 1 and giving minimum to

$$z = \theta_1 z_1 + \theta_2 z_2 + \ldots \theta_k z_k$$

where $z_i = c^T p_i$ are real numbers. In the finite set of numbers z_i, there always exists at least one having a minimum value $z_{\min}$. Let, without loss of generality, this value be taken by the first l numbers of the set: $z_j = z_{\min}, \ j = 1, 2 \ldots, l$. Then $z = \theta_1 z_1 + \ldots \theta_k z_k$ will take its minimum value z_{min} if nonnegative real coefficients θ_i are chosen to satisfy $\theta_1 + \ldots \theta_l = 1$ and $\theta_i = 0$ for $i = l + 1, \ldots, k$. Thus, it has been proven that a finite miminum of $z = c^T x$, if it exists, is attained at some vertex p_j or at a convex span of finite number of vertices $p_1, p_2, \ldots, p_l$, i.e., at a point

$$x = \sum_{i=1}^{l} \theta_i p_i, \quad \sum_{i=1}^{l} \theta_i = 1, \ 0 \leq \theta_i$$

within the feasible hyperpolyhedron X.

Conclusion from section 1: In the search for solution to a linear programming problem (given in the standard form), it is sufficient to narrow oneself only by the search for basic feasible solutions and only them, i.e., by the search for vertices of the feasible hyperpolyhedron as candidates for the optimal solution. This is precisely what the simplex method described below does.

Chapter 3.

The Simplex Method

The word **simplex** in the common sense means simple, non-composite as opposed to the word **complex**. As a mathematical notion, a simplex is a hyperpolyhedron with $n + 1$ corners in n dimensions (i.e., a convex span of $n + 1$ points in R^n). For example, 0-dimensional simplex is a point, 1-dimensional simplex is a limited straight line, 2-dimensional simplex is a triangle, 3-dimensional simplex is a tetrahedron, etc. Since it was established (see Chapter 2) that solution to a linear program ought to be soughth at the vertices of the feasible set of solutions, which is, by nature, a simplex, the very seeking procedure was named "the simplex method". It was developed by the American mathematician George B. Dantzig of the University of California, Berkeley, in 1947. This ingenious invention makes it possible to find the optimal solution by evaluating only a tiny fraction of all vertices while moving along the edges on the boundary of the hyperpolyhedron and looking for one particular target vertex. Thus, the simplex method avoids visiting every vertex and moves from one starting vertex to the optimal one through many intermediate vertices. At each vertex, all neighboring vertices are evaluated to see which ones are able to decrease the objective function and which are not. A new vertex is chosen from among the able ones, and at this new vertex the evaluation of neighbors is repeated until we arrive at the target vertex.

3.1. Reduction of a linear programming problem to the canonical form for a basis

The simplex method has got the several forms (or different algorithms). To describe them all, we begin from representing the standard LP problem in another (equivalent) form.

As it follows from definitions 2.3' and 2.5', in order to find a BFS we should select firstly a linearly independent set of m vectors a_j among columns of matrix A. Let they be already found and (with possible renumbering the variables x_j) be factors at the first m variables $x_1, x_2, \ldots, x_m$ in (2.2). These columns form a nonsingular matrix

$$B = [a_1, a_2, \ldots, a_m]$$

Hence, $A = [B \mid R]$, where $R = [a_{m+1}, \ldots, a_n]$ is a matrix composed of columns, each of them being linearly dependent on the column vectors of B. Introduce the corresponding notation for the basic and nonbasic variables: $x^B = (x_1, x_2, \ldots, x_m)^T$ and $x^F = (x_{m+1}, x_{m+2}, \ldots, x_n)^T$. (Using the simbol "$F$" reminds us that nonbasic variables are treated as "free" to enable moving from one vertex to another.) Thus, $x = (x^B, x^F)$. Correspondingly, $c_B = (c_1, \ldots, c_m)^T$, $c_F = (c_{m+1}, \ldots, c_n)^T$ and $c = (c_B, c_F)$.

We consider the following standard LP problem:

$$\begin{cases} \text{minimize} & z = c^T x \\ \text{subject to} & Ax = b, \quad x \geq 0 \end{cases}$$

With the introduced notations, we rewrite the problem in the other form taking into account that B^{-1} exists.

First we rewrite the constraints $Ax = b$:

$$Bx^B + Rx^F = b, \quad x^B + B^{-1}Rx^F = b', \quad b' = B^{-1}b$$

where matrix $B^{-1}R$ is $m \times (n - m)$. Denote it by A'. Written through the columns a'_j of A', it takes the form:

$$A' = B^{-1}R = [a'_{m+1} : a'_{m+2} : \ldots : a'_n]$$
$$a'_j = B^{-1}a_j, \quad j = m + 1, \ldots, n$$

Next, rewrite the objective function $z = c^T x$:

$$z = c_B^T x^B + c_F^T x^F = c_B^T(b' - A'x^F) + c_F^T x^F = c_B^T b' + (c_F^T - c_B^T A')x^F = z_0 + c'^T x^F$$

where new coefficients are denoted by c', $c'^T = c_F^T - c_B^T A'$, or by elements:

$$c'_j = c_j - c_B^T a'_j, \ j = m + 1, \ldots, n$$

A value of z at $x^F = 0$ is denoted by z_0, $z_0 = c_B^T b'$.

Thus, the standard LP problem

$$\min_x(z = c^T x), \ Ax = b, \quad x \geq 0$$

is reduced to the following equivalent form known as the **canonical form for a basis** $x^B = (x_1, \ldots, x_m)$:

$$\begin{cases} \text{minimize} & z - z_0 & = & c'^T x^F \\ \text{subject to} & x^B + A'x^F & = & b', & x^B \geq 0, \quad x^F \geq 0 \end{cases}$$

where

$$z_0 = c_B^T b'; \ b' = B^{-1}b; \ A' = \{a'_j\}, \ j = m+1, \ldots, n$$

$$c'_j = c_j - c_B^T a_j, \ a'_j = B^{-1}a_j, \ j = m+1, \ldots, n$$

Here, it is taken into account that

$$B = \{a_j\}, \ j = \overline{1, m}, \ rankB = m; \ A = \{a_j\}, \ j = \overline{1, n}$$

The characteristic features of this form are: (1) the basic variables x^B appear in the constraints with the identity matrix I, and yet they do not appear in the objective function; and (2) the objective function $(z - z_0)$ is expressed only through the nonbasic (free) variables x^F. This representation of the LP problem is convenient to derive the computational procedure that can succesfully solve the problem and is known as the simplex method, the method most commonly used at a lot of large businesses all round the world.

3.2. The simplex method for the case of obvious basic feasible solution

Reduction of a linear programming problem to the canonical form for a basis means that a basic solution has been found and it has the following form:

the nonbasic variables $x^F = 0$;

the basic variables $x^B = b'$;

the objective function $z = z_0 = c_B^T b'$.

If here it occurrs that $b' \geq 0$, then this basic solution is a basic feasible solution (a vertex of feasible hyperpolyhedron). Otherwise, it is not a feasible solution.

In this section, we consider a nondegenerate linear programming problem and assume that at the beginning or at a recurrent step it occurrs that $b' \geq 0$. The question arises: Is it possible to do one more step and some more decrease the value of the objective function z by moving to another vertex of feasible hyperpolyhedron ?

We get an answer to the question looking at z:

$$z - z_0 = c'^T x^F$$

Theorem 3.2.1.

(i) If all elements of c' are strictly positive, $c' > 0$, then it is impossible to decrease z while moving from $x^F = 0$ and remaining in the feasible region, $x^F \geq 0$. Minimum $z = z_0$ has been found, and the corresponding basic feasible solution is the unique solution of the linear program, that is, it is an optimal basic feasible solution (OBFS).

(ii) If there, among elements of c', exists at least one element, say c'_s, that is less than zero, then there are only two possibilities:

    (a) If $a'_{is} \leq 0$ for all $i = 1, \ldots, m$, then feasible solutions exist such that z can be made as small as desired, and the linear program has no finite solution.

    (b) If $a'_{is} > 0$ for at least one i, then a new basic solution can be found such that the value of z is strictly decreased.

(iii) If all elements of c' are nonnegative, $c' \geq 0$, while at least one of them, say c'_s, is equal to zero, then z attains its minimum value for at least two different points x. If two such optimal solutions are denoted by $x^{(1)}$ and $x^{(2)}$, then

$$x = \theta x^{(1)} + (1 - \theta)x^{(2)}, \qquad 0 \leq \theta \leq 1$$

is also an optimal solution.

(iv) If procedure (ii) (b) is repeated, then either case (ii) (a) is reached after a finite number of steps, in which z can be made as small as desired, or one of cases, (i) or (iii), is reached after a finite number of steps, in which an optimal solution to the linear program has been found.

The proof of this theorem is obvious from the following analysis. If there, among coefficients c', exists at least one strictly negative, then z can be further decreased from value z_0 by making positive those nonbasic (free) variables in the structure of x^F which appear in the objective function $(z - z_0) = c'^T x^F$ with the negative coefficients c', however preserving the residence of the point (x^B, x^F) within the feasible region:

$$x^B + A'x^F = b', \quad x^B \geq 0, \ x^F \geq 0 \tag{3.1}$$

It does not matter how many such variables exist and what of them are selected for making positive, but the simplex method offers to choose only one of nonbasic variables to increase it from 0 to a positive value while keeping the rest of them equal to 0. Obviously, it is advisable to choose from x^F such a variable that appear in $(z - z_0) = c'^T x^F$ with the most negative coefficient. It would guarantee the steepest descend along z among all possible variants of a variable-by-variable descent.

Assume a choice has been made, and x_s is the chosen variable intended to assume a positive value. We increase it from zero keeping the other nonbasic variables constant and equal to zero. Then constraints (3.1) to be looked at take the simple form:

$$x_i^B + a'_{is}x_s = b'_i, \quad x_i^B \geq 0, \ x_s > 0, \ i = \overline{1, m}. \tag{3.2}$$

From this, we see that the change in the value of x_s, which decreases z, forces us to reduce x_i^B in order to preserve the equalities in (3.2). Naturally, x_i^B is reduced if only $a'_{is} > 0$. If $a'_{is} < 0$, this change will increase x_i^B, and x_i^B remains unchanged if $a'_{is} = 0$. In these two cases the constraint $x_i^B \geq 0$ can not be violated. So, the only case that can trouble is $a'_{is} > 0$ because in (3.2) we need to preserve the inequality $x_i^B \geq 0$.

Now one must answer the question: Up to what value it is allowed to increase x_s in case of $a'_{is} > 0$? From the point of view of a single, say ith, constraint, the answer sounds as follows: As increasing x_s in the region $x_s > 0$ causes a reduction in the value of x_i^B and as x_i^B may be reduced only to $x_i^B = 0$, increasing x_s is allowed only to $x_s = b'_i / a'_{is}$. However, from the point of view of all m constraints simultaneously, increasing x_s is permissible only to the lowest value

$$x_s = \min_{i=\overline{1,m}} \left(\frac{b'_i}{a'_{is}} \right)$$

Assume that this lowest value is discovered in the kth row of constraints (3.2), viz.

$$x_s = \min_{i=\overline{1,m}} \left(\frac{b'_i}{a'_{is}} \right) = \frac{b'_k}{a'_{ks}}.$$

Hence, we have the following result: the chosen nonbasic variable x_s has been increased from 0 to b'_k / a'_{ks} and at the same time the basic variable x_k has been decreased to 0. In other words, x_s has been included into a basis and x_k has been excluded from the basis. For this reason, x_s is called the **entering basic variable** and x_k is called the **departing basic variable**. By this manner a transition from the old basic feasible solution to a new basic feasible solution is performed, and in doing so the value of objective function is decreased.

Therein indeed lies the every next in turn step of the simplex method. Because the number of different basic feasible solutions is finite (it does not exceed C_n^m, - see Definition 2.6), minimum of the objective function (if only it exists) will be found in a finite number of steps.

3.3. Algorithm of the simplex method for the case of obvious basic feasible solution

From Section 3.1 it is obvious that the standard LP problem has a variety of representations refered to as different canonical forms for a basis, and each basis relates to its own canonical form. On the other hand, as it was shown in Section 3.2, the every next in turn step of the simplex method, at which we pass to a lower value of the objective function, means that some variable x_s enters a basis instead of another variable x_k departing the basis. Thus, to fulfil a step of the simplex method means formally to rewrite the LP problem in the canonical form for a new basis being adopted next as giving the less value of the objective function. Let us describe the corresponding actions.

Given a linear programming problem in its initial (or current) canonical form for a basis (see Section 3.1), which is looking as the following system of equations with the nonnegative rihgt-hand side $b' \geq 0$:

$$\begin{bmatrix} I & | & A' \\ --- & | & --- \\ O & | & c'^T \end{bmatrix} \begin{bmatrix} x^B \\ --- \\ x^F \end{bmatrix} = \begin{bmatrix} b' \\ --- \\ z - z_0 \end{bmatrix}, \quad x^B \geq 0, x^F \geq 0$$

where the last equation gives the expression of the objective function via nonbasic (free) variables.

In these equations also shown are: I the $m \times m$ identity matrix; $A' = \{a'_j\}$ the $m \times (n - m)$ martix of column vectors a'_j, $j = m + 1, \ldots, n$; O a row of zeros; c' the column vector of coefficients for nonbasic variables x^F; $z_0 = c_B^T b'$ the initial value of the objective function z at the initial (or current) BFS, which is defined as $x = (x^B, x^F)$, $x^B = b'$, $x^F = 0$.

Action 1°. Find what variable in x^F, say x_s, can be increased from zero in order to decrease z from z_0. Indicative of any such variable is a negative coefficient in the last row of the above partitioned matrix, or more precisely among elements of c'^T. Locate the negative element among c'^T that is largest in magnitude. (If two or more entries share this property, any one of these can be selected.) If all such entries are nonnegative, the system is in the final form and solution is obtained because the desired x_s does not exist any more. The selected variable x_s is called the **entering variable** because it is prepared to be introduced into a basis, and the corresponding (s-th) column is called the **pivot column**.

Action 2°. Find what row of m rows of constraints (they are the first m rows of the above system) provides

$$\frac{b'_k}{a'_{ks}} = \min_{i:\, a'_{is} > 0} \left(\frac{b'_i}{a'_{is}} \right)$$

Designate the number of this row by k and call this row the **key row**. (If two or more rows share this property, any one of these can be selected as key.) The coefficient a'_{ks} of x_s in the key row is called a **pivot element**. The variable being basic in the key row is called the **departing variable** because it is prepared to be excluded from the basis. Both introducing the entering variable and excluding the departing variable are being made by the following two actions.

Action 3°. Normalize the kth equation of the system through dividing it by the pivot a'_{ks}. As a result, the entering variable x_s takes coefficient 1.

Action 4°. Do a series of subtractions of the normalized k-th row taken with the suitable multipliers, from all the other rows of the system in order to eliminate the pivot variable x_s from all system equations except the k-th. (The suitable multipliers have been, by that time, in the system matrix at the crossings of all $m + 1$ rows, excepting the k-th, with the pivot column numbered s.)

As a result of these four actions, the s-th column becomes a column of the identity matrix, and the $NB(k)$-th column stops being such where $NB(k)$ is the number of a basis (departing) variable in the k-th constraint (row). This just means that x_s has been included into a (new) basis while $x_{NB(k)}$ has been excludud from the (old) basis. By this means, a transition to a new canonical form for a basis has been made. Next step of the algorithm will repeat these four actions or will stop at action 1° if no entering variable can be found.

Obviously, actions 3° and 4° coincide with the procedure of Gauss-Jordan elimination of x_s from the given system of equations while actions 1° and 2° forego this procedure with a specific selection of: a pivot column (action 1°) and a key row (action 2°).

Example 3.3.1. Minimize
$$-2x_1 - 4x_2 = z$$
subject to $x_1, x_2 \geq 0$ and
$$2x_1 + 3x_2 \leq 120$$
$$3x_1 + 9x_2 \leq 270$$

First, we write down this LP problem in the standard form introducing two **slack** variables $x_3, x_4 \geq 0$:

$$
\begin{array}{rrrrrr}
2x_1 & + & 3x_2 & + & x_3 & & & = & 120 \\
3x_1 & + & 9x_2 & & & + & x_4 & = & 270 \\
-2x_1 & - & 4x_2 & & & & & = & z
\end{array}
$$

In the matrix notation we have

$$
\begin{bmatrix}
2 & 3 & 1 & 0 \\
3 & 9 & 0 & 1 \\
-2 & -4 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
=
\begin{bmatrix}
120 \\
270 \\
z
\end{bmatrix}
$$

Simultaneously, it serves the initial canonical form of the problem for a basis $x^B = (x_3, x_4)$. The rest variables, x_1 and x_2, are nonbasic: $x^F = (x_1, x_2)$. The initial basic feasible solution is $x = (x_1, x_2, x_3, x_4) = (0, 0, 120, 270)$. The value of the objective function at this point is $z = z_0 = 0$.

To make the simplex method function, i.e. to run the above actions $1° - 4°$, it is sufficient to operate only with the extended matrix. In this example it has the following initial form:

$$
\begin{bmatrix}
2 & 3 & 1 & 0 & | & 120 \\
3 & 9 & 0 & 1 & | & 270 \\
-2 & -4 & 0 & 0 & | & z
\end{bmatrix}
$$

Such matrix is called a **simplex tableau**.

Let is run all the necessary actions $1° - 4°$ in this tableau.

Step 1.

1° $x_s = x_2$ (the pivot column is the second one; correpondingly, x_2 is the entering variable).

2° $\min(\frac{120}{3}, \frac{270}{9}) = \min(40, 30) = 30$; $\quad k = 2$ (the key row is the second one; the basic variable in this row is x_4, and it is the departing variable).

3° After normalizing:

$$
\rightarrow
\begin{bmatrix}
2 & 3 & 1 & 0 & | & 120 \\
1/3 & 1 & 0 & 1/9 & | & 30 \\
-2 & -4 & 0 & 0 & | & z
\end{bmatrix}
$$

$$\uparrow$$

(the pivot column and key row are marked for illumination by arrows).

4° After subtractions:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1/3 & 30 \\ 1/3 & 1 & 0 & 1/9 & 30 \\ -2/3 & 0 & 0 & 4/9 & z+120 \end{array}\right]$$

This is the next canonical form for a basis $x^B = (x_2, x_3)$. Nonbasic variable are $x^F = (x_1, x_4)$. Basic feasible solution is $x = (x_1, x_2, x_3, x_4) = (0, 30, 40, 0)$. The value of the objective function at this point is $z + 120 = 0$, i.e. $z = z_0 = -120$.

Step 2.

1° $x_s = x_1$ (the pivot column is the first)

2° $\min(\frac{40}{1}, \frac{30}{1/3}) = \min(40, 90) = 40$;   (the key row is the first)

    $k = 1$

3° After normalizing:

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1/3 & 30 \\ 1/3 & 1 & 0 & 1/9 & 30 \\ -2/3 & 0 & 0 & 4/9 & z+120 \end{array}\right]$$
$$\uparrow$$

4° After subtractions:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1/3 & 30 \\ 0 & 1 & -1/3 & 2/9 & 20 \\ 0 & 0 & 2/3 & 2/9 & z+140 \end{array}\right]$$

This is the next form for a basis $x^B = (x_1, x_2)$. Nonbasic variables are $x^F = (x_3, x_4)$. Basic feasible solution is $x = (x_1, x_2, x_3, x_4) = (30, 20, 0, 0)$. The value of the objective function for the solution is $z + 140 = 0$, i.e. $z = z_0 = -140$.

Step 3.

1° There is no free (nonbasic) variables having a negative coefficient among the first n elements, $n = 4$, in the last row of the matrix. The process of solution has finished. The solution of the problem is $x_1 = 30, x_2 = 20$ and $z = -140$.

Remark 3.3.1. *If one represents the above problem geometrically on the (Ox_1x_2)-plane, it will be seen that solution started from the origin $(x_1 = x_2 = 0)$. After step 1° the solution moved into the point $x_1 = 0, x_2 = 30$. After step 2° the solution moved along the border line $3x_1 + 9x_2 = 270$ into the point $x_1 = 30, x_2 = 20$. If there is chosen another variant of action 1°, $x_s = x_1$, at step 1° , that is also possible, then at such step 1° the solution moves to the point $x_1 = 60, x_2 = 0$, and then at step 2° , it will move along the border line $2x_1 + 3x_2 = 120$ to the same final point $x_1 = 30, x_2 = 20$ and $z = -140$. Here, for these two variants of action the number of steps from the initial to the final solution occured to be the same: 2 steps. In general case, such coincidence does not occur, and it appears impossible, to foresee what variant of actions leads faster to the final solution.*

Remark 3.3.2. *In the simplex tableau, the lowest right corner position is for the value of objective function. In the above symbol z was written there only for simplicity and from now on there is no need to write it. In doing so, we have the value of the objective function alwayse eqnal to the element of this position taken with the opposite sign, $-z_0$, because at every basic feasible solution we have $z + (-z_0) = 0$. At the initial state, this element equals 0.*

3.4. Computational aspects of the simplex method with the obvious basic solution

We have a simplex tableau in the form of two dimensional array, i.e. as $(m+1) \times (n+1)$ matrix. The first m rows of the tableau contain coefficients and the right-hand sides of constraints represented in a canonical form for a basic. These constraints can be written as a system of equations

$$[I \mid A'] \begin{bmatrix} x^B \\ x^F \end{bmatrix} = b'$$

In this section we abandone the requirement $b' \geq 0$. The last, $(m+1)$-th, row of the matrix contains the coefficients of the objective function having the following expression in the canonical form for a basis:

$$[0 \mid c'^T] \begin{bmatrix} x^B \\ x^F \end{bmatrix} = z - z_0$$

The last, $(n+1)$-th, element of this row is for the right-hand side of this equation. It equals $-z_0$, because we omit the symbol "z" when writing the tableau, and z_0 is the value of z at a current basic solution ($x^B = b', x^F = 0$). Before computation, this element equals 0.

For computation, it is convenient to introduce two pointers: (1) the pointer $NB(i)$, $i = \overline{1, m}$; $NB(i)$ is number of a basic variable in the i-th constraint, and (2) the pointer $NF(j)$, $j = 1, 2, \ldots, n-m$; $NF(j)$ is number of a nonbasic variable. Thus, these pointers store the basic and nonbasic variable numbers.

At each step of the algorithm the following actions $1° - 4°$ are done:

1° Look through $NF(j)$ whether there exist or not negative values among elements

$$a(m+1, NF(j)), \ j = 1, 2, \ldots, n - m$$

If "not", the desired solution has been found and given by the equations: $x^B = b'$, $x^F = 0$, $z_{\min} = z_0$. We have the following values:

$$x(NB(i)) = a(i, n+1), \ i = 1, 2, \ldots, m$$

for basic variables,

$$x(NF(j)) = 0, \quad j = 1, 2, \ldots, n - m$$

for nonbasis variables, and

$$z_{\min} = -a(m + 1, n + 1)$$

for the objective function.

If "yes", then select one element from the mentioned negative elements, for example, the most negative. Let it have the number s, $s = NF(l)$ for some l. By this, one column, its number s, is announced the pivot column, and the corresponding variable $x(s)$ is determined to be the entering basic variable.

2° Look through the row numbers, $i = 1, 2, \ldots, m$, in order to select the key row, its number k, according to a predetermined rule. This rule is formulated so that the basic variable having the number $NB(i)$, does not go out of the feasible region due to increasing the entering variable $x_s = x(s)$. To do this, write down the i-th constraint:

$$x(NB(i)) + a'_{is}x(s) = b'_i$$

where a'_{is} is the element of the above submatrix A'; b'_i is the element of the right-hand side b' of constraints. In our notation for the simplex tableau (matrix A, its elements $a(i, j)$), the same constraint has the form:

$$x(NB(i)) + a(i, s)x(s) = a(i, n + 1)$$

Now, in dependence of a combination of signs and critical (zero) values for a'_{is} and $b'_i = a(i, n + 1)$, determine the limiting value $\max(x_s)$ up to which the entering variable $x(s)$ may be increased with keeping $x(NB(i))$ nonnegative. The complete set of variants is shown in Table 3.4.1.

It is necessary to use this table for each row of constraints, $i = 1, 2, \ldots, m$. If in doing so one encounters one or more situations of "prohibited", then one should refuse $x(s)$ as the entering variable and return to action 1° in order to find another entering variable $x(s)$ and to do with it action 2°. If no situation of "prohibited" is encountered for all constraints number k should be determined by rule:

$$k = \arg\min_{i:a'_{i,s}>0, b'_i>0} \left[\frac{b'_i}{a'_{is}} \right] = \arg\min_{i:a'_{i,s}>0, b'_i>0} \left[\frac{a(i, n + 1)}{a(i, s)} \right]$$

As a result, coefficient $a'_{ks} = a(k, s)$ is announced the pivot and the k-th row is announced the key row, in order to complete a circle of two actions, 3° and 4°, constituting one circle of Gauss-Jordan elimination for systems of linear algebraic equations. The fact of such elimination has a towfold effect: variable $x(s), s = NF(l)$,

Table 3.4.1. Limiting value $\max(x_s)$ for the entering variable x_s for the cases when the basic variable $x(NB(i))$ appears in the i-th constraint with coefficient $+1$

$a'_{is} = a(i, s)$	$b'_i = a(i, n+1)$	$\max(x_s)$	
> 0	> 0	b'_i/a'_{is}	
< 0	> 0	∞	
$= 0$	> 0	∞	
> 0	< 0	$-\infty$	(prohibited)
< 0	< 0	∞	
$= 0$	< 0	∞	(*)
> 0	$= 0$	$-\infty$	(prohibited)
< 0	$= 0$	∞	
$= 0$	$= 0$	∞	

(*) - the solution remains in the infeasible region

enters a basis, and variable $x(NB(k))$ leaves the basis. This can be registered by the mutual exchage of numbers in the two pointers:

$$NF(l) \longleftrightarrow NB(k)$$

3° Normalize the k-th row through dividing it by the pivot $a(k, s)$.

4° Substract the k-th row times $a(i, s)$ from the other m rows, $i = 1, 2, \ldots, m+1$ except $i = k$.

Remark 3.4.1. *It is obvious that only $(n-m+1)$ columns of the simplex tableau (matrix A) are re-calculated at each step. They are:*

(1) the last column, having number $n + 1$, with the right-hand sides b' of constraints;

(2) all columns for nonbasic variables except the pivot column that is replaced by a trivial column (all zeros and one unit instead of the pivot element $a(k, s)$), totaling $(n - m - 1)$ columns;

(3) the column for the departing variable $x(NB(k))$.

The other m columns remain unchanged and so may not be re-calculated. But on the whole, this algorithm is the Gauss-Jordan elimination with the special choice of pivot $a(k, s)$.

Example 3.4.1. Find a solution $x_1 \geq 0, x_2 \geq 0$ of the following LP-problem:

$$
\begin{aligned}
x_1 &+ x_2 &\leq 50 \\
-x_1 &+ x_2 &\leq 10 \\
x_1 & &\geq 20 \\
&x_2 &\geq 10 \\
-2x_1 &- 3x_2 &= z \quad \rightarrow \quad \min
\end{aligned}
$$

Geometrical plotting shows that the feasible region is a quadrangle on the ox_1x_2–plane with the vertices at points: $P = (20, 10)$; $Q = (20, 15)$; $R = (30, 20)$; and $S = (40, 10)$. Moving the line $z = const$ along the antigradient $-\text{grad}(z) = (2, 3)$, we find that the solution is attained in vertex R where $z_{min} = -120$. Having this result for the checking, let us fulfil the above algorithm formally.

First, write down the given problem in the standard form:

$$
\begin{array}{rrrrrrrcl}
x_1 & + & x_2 & + & x_3 & & & = & 50 \\
-x_1 & + & 2x_2 & & & + & x_4 & = & 10 \\
x_1 & & & & & - & x_5 & = & 20 \\
& & x_2 & & & & - x_6 & = & 10 \\
-2x_1 & - & 3x_2 & & & & & = & z \quad \rightarrow \quad min
\end{array}
$$

where x_3, x_4, x_5 and x_6 are the slack variables, each nonnegative, introduced in order to replace inequalities by equalities. Fill in the arrays $A(5, 7)$; $NB(4)$ and $NF(2)$ with the initial data:

$$
\begin{array}{c}
A \\
\begin{array}{|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 0 & 0 & 0 & 50 \\
\hline
-1 & 2 & 0 & 1 & 0 & 0 & 10 \\
\hline
-1 & 0 & 0 & 0 & 1 & 0 & -20 \\
\hline
0 & -1 & 0 & 0 & 0 & 1 & -10 \\
\hline
-2 & -3 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
\end{array}
\qquad
\begin{array}{c}
NB \\
\begin{array}{cc}
(1) & \boxed{3} \\
(2) & \boxed{4} \\
(3) & \boxed{5} \\
(4) & \boxed{6} \\
\end{array}
\end{array}
\qquad
\begin{array}{c}
NF \\
\begin{array}{cc}
(1) & \boxed{1} \\
(2) & \boxed{2} \\
\end{array}
\end{array}
$$

(the arrow $\rightarrow$ is to the left of row 2, the arrow $\uparrow$ is below column 1)

$$BS = (0, 0, 50, 10, -20, -10) \neq BFS$$

The last line shows that the initial Basic Solution (BS) is not Basic Feasible Solution (BFS).

Step 1

$1°$ s $\boxed{2}$ $\qquad$ l $\boxed{2}$

$2°$ $\min(50/1, 10/2, \infty, \infty) = 5$; $\qquad$ k $\boxed{2}$

As a result of actions $1°$ and $2°$, we have found the pivot column, $s = 2$, and the key row, $k = 2$ (above and below, they are marked by arrows near the tableau), and the pivot element $a_{ks} = a_{22} = 2$ is determined at their crossing.

$3°$ After normalizing the key row:

$$
\begin{array}{c}
A \\
\begin{array}{|rrrrrr|r|}
\hline
1 & 1 & 1 & 0 & 0 & 0 & 50 \\
-1/2 & 1 & 0 & 1/2 & 0 & 0 & 5 \\
-1 & 0 & 0 & 0 & 1 & 0 & -20 \\
0 & -1 & 0 & 0 & 0 & 1 & -10 \\
\hline
-2 & -3 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
\end{array}
\qquad
\begin{array}{c}
NB \\
\begin{array}{cc}
(1) & \boxed{3} \\
(2) & \boxed{4} \\
(3) & \boxed{5} \\
(4) & \boxed{6} \\
\end{array}
\end{array}
\qquad
\begin{array}{c}
NF \\
\begin{array}{cc}
(1) & \boxed{1} \\
(2) & \boxed{2} \\
\end{array}
\end{array}
$$

(the arrow $\rightarrow$ is to the left of row 2, the arrow $\uparrow$ is below column 1)

4° After subtraction and switching $NF(l) \longleftrightarrow NB(k)$:

	A								NB		NF
$\rightarrow$	3/2	0	1	−1/2	0	0	45	(1)	3	(1)	1
	−1/2	1	0	1/2	0	0	5	(2)	2	(2)	4
	−1	0	0	0	1	0	−20	(3)	5		
	−1/2	0	0	1/2	0	1	−5	(4)	6		
	−7/2	0	0	3/2	0	0	15				

$\uparrow$

$$BS = (0, 5, 45, 0, -20, -5) \neq BFS$$

Step 2

1° $s \boxed{1}$ $l \boxed{1}$

2° $\min(\frac{45}{3/2}, \infty, \infty, \infty) = 30;$ $k \boxed{1}$

Found are: a pivot column, $s = 1$, and a key row, $k = 1$. In the above tableau they are marked by arrows. The pivot element $3/2$ is at their crossing.

3° After normalizing the key row:

	A								NB		NF
$\rightarrow$	1	0	2/3	−1/3	0	0	30	(1)	3	(1)	1
	−1/2	1	0	1/2	0	0	5	(2)	2	(2)	4
	−1	0	0	0	1	0	−20	(3)	5		
	−1/2	0	0	1/2	0	1	−5	(4)	6		
	−7/2	0	0	3/2	0	0	15				

$\uparrow$

4° After subtraction and switching $NF(l) \longleftrightarrow NB(k)$:

A								NB		NF	
1	0	2/3	−1/3	0	0	30	(1)	1	(1)	3
0	1	1/3	1/3	0	0	20	(2)	2	(2)	4
0	0	2/3	−1/3	1	0	10	(3)	5		
0	0	1/3	1/3	0	1	10	(4)	6		
0	0	7/3	1/3	0	0	120				

Step 3

1° There are no negative coefficients in the last row. The solution has been found: $x_1 = 30$, $x_2 = 20$ and $z_{\min} = -120$.

In geometrical interpretation, the solution was found by moving from the origin $(0,0)$ to the point $(0,5)$ and then to the point $R = (30, 20)$. The first and second

points were not in the feasible region. Thus, increasing the entering basic variable yields not only decreasing the value of the objective function but introducing the solution into the feasible region, too.

Remark 3.4.2. *As for the situation $-\infty$ ("prohibited") that appears in Table 3.4.1, see example 4.1.4 in the below Section 4.1.*

3.5. The simplex-method free from generation the initial BFS

The simplex-algorithm considered in Sections 3.3 and 3.4 is suitable for the case when a basic feasible solution is readily available from the very beginning. Such a solution holds if $b \geq 0$ and all m constraints are stated with $\geq$ signs. In such an event, each constraint given originally as an inequality of "less then or equal to" type is to be converted into an equality in order to create the *standard* linear programming problem. We do this by introducing one new variable in each inequality, called a **slack variable** since it takes up all of the slack in the original inequality given as "the left side" $\leq$ "the right side".

Since all the variables that occur in the problem must be greater than or equal to zero, the slack variable for such inequality have to be determined by the difference:

"the slack variable" = "the right side" - "the left side".

This formula shows that the desired equality is now obtained:

"the left side" + "the slack variable" = "the right side".

The slack variables introduced in this case and in such a manner into each inequality are nonnegative and completely independent of one another. Also, the formula shows that, in such situation, slack variables appear in the converted constraints *with the multiplier* "$+1$" (see Table 3.4.1). Variables added by this way are just the basic variables x^B in the initial $BFS \geq 0$ because they appear in the standard LP problem with the identity matrix, I, and do not appear in the formula for the given objective function, z.

If among constraints there are those of "greater than or equal to" type, then, as it can be seen from Example 3.4.1, the given algorithm does also work althougt it has to start not from the feasible region but from some non-obligatory nonnegative basic solution (BS). Such situation appears when, by preliminary multiplying such constraints by "-1", these constraints are formally being transformed into the "less than or equal to" cathegory because in doing so the inequality $b \geq 0$ changes to $(-b) \leq 0$ for the new right hand side $(-b)$.

Remark 3.5.1. *Certainly, one can notice that the above mentioned multiplying, as such, is not a cause for the situation to appear. If the situation exists, it exists by its nature, i.e. by the nature of the problem under consideration (confer Example 3.4.1 and the below Example 3.5.2).*

As for the slack variables in the case of "greater than or equal to" constraints, they obviously, are to be determined by the formula:

"the slack variable" = "the left side" - "the right side" ≥ 0

in order to guarantee the given constraint and to meet the requirement that all the variables must be nonnegative. The formula shows that the constraint is reduced to the equality:

"the left side" - "the slack variable" = "the right side"

so the slack variable appears in the converted constraint *with the multiplier* " -1 ".

If there exists a constraint of "equal to" type, then one can introduce a so-called *artificial* variable in it and then solve the problem, as it is considered below in Examles 3.5.1 and 3.5.2. However, it must be noticed that if a solution for such cases exists, then the introduced artificial variables are zero in this solution. Accordingly, if they can not be zero, no solution exists.

Example 3.5.1. For $x_1, x_2 \geq 0$, solve the problem:

$$
\begin{array}{rrrcr}
x_1 & + & x_2 & \leq & 6 \\
-2x_1 & + & 3x_2 & \leq & 3 \\
2x_1 & + & 3x_2 & \leq & 21 \\
-2x_1 & - & 5x_2 & = & z \quad \rightarrow \quad \min
\end{array}
$$

Step 0. Input the numerical values of the coefficients from the given problem and automatically set the rest values to generate the following initial form of the arrays:

$$A \qquad NB \qquad NF$$

1	1	0	0	1	6
−2	3	1	0	0	3
2	3	0	1	0	21
−2	−5	0	0	0	0

NB: 5, 3, 4 $\qquad NF$: 1, 2

$$BS = (0, 0, 3, 21, 6) = BFS, \quad 0 = z.$$

Step 1.

1° $s = 2, \quad l = 2$

2° $\min(6, 1, 7) = 1, \quad k = 2$

3° After normalization:

$$A$$

1	1	0	0	1	6
−2/3	1	1/3	0	0	1
2	3	0	1	0	21
−2	−5	0	0	0	0

4° After subtraction:

$$A \qquad\qquad NB \quad NF$$

5/3	0	−1/3	0	1	5
−2/3	1	1/3	0	0	1
4	0	−1	1	0	18
−16/3	0	5/3	0	0	5

→ (first row), ↑ (first column)

NB: 5, 2, 4 NF: 1, 3

$$BS = (0, 1, 0, 18, 5) = BFS; \qquad -5 = z$$

Step 2.

1° $s = 1, \quad l = 1$

2° $\min(3, \infty, 9/2) = 3, \quad k = 1$

3° After normalization:

$$A$$

1	0	−1/5	0	3/5	3
−2/3	1	1/3	0	0	1
4	0	−1	1	0	18
−16/3	0	5/3	0	0	5

→ (first row), ↑ (first column)

4° After subtraction:

$$A \qquad\qquad NB \quad NF$$

1	0	−1/5	0	3/5	3
0	1	1/5	0	2/5	3
0	0	−1/5	1	−12/5	6
0	0	3/5	0	16/5	21

NB: 1, 2, 4 NF: 5, 3

$$BS = (3, 3, 0, 6, 0) = BFS; \quad -21 = \min z$$

Step 3.

1° No negative elements among $a(m + 1, NF(l))$, $l = 1, 2$, and NF contains 5 (the number of artificial variable x_5 introduced at Step 0 for the first constraint; thus $x_5 = 0$).

Therefore, Basic Optimal Solution (BOS) can be output (omitting the artificial variable):

$$BOS = (3, 3, 0, 6); \quad -21 = \min z$$

Table 3.5.1. Limiting value $\max(x_s)$ for the entering variable x_s for the cases when the basic variable $x(NB(i))$ appears in the i-th constraint with coefficient -1

$a'_{is} = a(i, s)$	$b'_i = a(i, n+1)$	$\max(x_s)$
> 0	> 0	∞
< 0	> 0	$-\infty$ (*prohibited*)
$= 0$	> 0	∞ (*)
> 0	< 0	∞
< 0	< 0	b'_i / a'_{is}
$= 0$	< 0	∞
> 0	$= 0$	∞
< 0	$= 0$	$-\infty$ (*prohibited*)
$= 0$	$= 0$	∞

(*) - the solution remains in the infeasible region

Geometrically, here was a movement through the points $(0,0) \rightarrow (0,1) \rightarrow (3,3)$ on the plane ox_1x_2.

Remark 3.5.2. *The distinction between the slack and artificial variables is that the latter must be zero in the final solution while the first have not (cf. Examples 3.4.1 and 3.5.1).*

If among the given constraints are those of "greater than or equal to" type, then there is no need to convert them into the "less than or equal to" inequality through preliminary multiplication by "-1". As above discussed, it is sufficient to introduce a slack (nonnegative) variable *with the coefficient* "-1" into each such a constraint. For such constraints, the following Table 3.5.1 can be shown to replace Table 3.4.1 while doing Action 2° of the simplex algorithm. That this is the case can be easily seen from the corresponding equation for such a constraint:

$$-x(NB(i)) + a'_{is}x(s) = b'_i$$

Indeed, slack variables introduced in the above manner are the basic variables x^B in the initial basic solution (IBS) because each of them appears only in one constraint and does not appear in the formula for the given objective function, with the only difference that it appears there with the multiplier "-1" instead of "$+1$" and that this IBS may occur to lie outside the feasible region (if $b'_i > 0$).

Example 3.5.2. For $x_1, x_2 \geq 0$ solve the problem:

$$
\begin{array}{rcrclcl}
2x_1 & & & \geq & 3 \\
& & x_2 & \geq & 1 \\
x_1 & + & x_2 & = & 6 \\
-2x_1 & + & 3x_2 & \leq & 3 \\
2x_1 & + & 3x_2 & \leq & 21 \\
-2x_1 & - & 5x_2 & = & z & \rightarrow & \min
\end{array}
$$

Step 0. Input the numerical values of the coefficients from the given problem and automatically set the rest values to generate the following initial form of the arrays:

$$A \qquad\qquad NB \quad NF$$

2	0	−1	0	0	0	0	3		3		1
0	1	0	−1	0	0	0	1		4		2
1	1	0	0	0	0	1	6		7		
−2	3	0	0	1	0	0	3		5		
2	3	0	0	0	1	0	21		6		
−2	−5	0	0	0	0	0	0				

$→$ (left of table), $↑$ (under column 2)

$$BS = (0, 0, -3, -1, 3, 21, 6) \neq BFS; \qquad 0 = z$$

Step 1.

$1°$ $s = 2, \quad l = 2$

$2°$ $\min(\infty, \infty, 6, 1, 7) = 1, \quad k = 4$

$3°$ After normalization:

$$A$$

2	0	−1	0	0	0	0	3
0	1	0	−1	0	0	0	1
1	1	0	0	0	0	1	6
−2/3	1	1	0	1/3	0	0	1
2	3	0	0	0	1	0	21
−2	−5	0	0	0	0	0	0

$→$ (left), $↑$ (under column 2)

$4°$ After subtraction:

$$A \qquad\qquad NB \quad NF$$

2	0	−1	0	0	0	0	3		3		1
2/3	0	0	−1	−1/3	0	0	0		4		5
5/3	0	0	0	−1/3	0	1	5		7		
−2/3	1	0	0	1/3	0	0	1		2		
4	0	0	0	−1	1	0	18		6		
−16/3	0	0	0	5/3	0	0	5				

$→$ (left), $↑$ (under column 1)

$$BS = (0, 1, -3, 0, 0, 18, 5) \neq BFS; \qquad -5 = z$$

Step 2.

$1°$ $s = 1, \quad l = 1$

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2° $\min(\infty, \infty, 3, \infty, 9/2) = 3, \quad k = 3$

3° After normalization:

$$A$$

$$\to \begin{array}{|ccccccc|c|}
\hline
2 & 0 & -1 & 0 & 0 & 0 & 0 & 3 \\
2/3 & 0 & 0 & -1 & -1/3 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1/5 & 0 & 3/5 & 3 \\
-2/3 & 1 & 0 & 0 & 1/3 & 0 & 0 & 1 \\
4 & 0 & 0 & 0 & -1 & 1 & 0 & 18 \\
\hline
-16/3 & 0 & 0 & 0 & 5/3 & 0 & 0 & 5 \\
\hline
\end{array}$$

$\uparrow$

4° After subtraction:

$$A \qquad\qquad NB \quad NF$$

$$\begin{array}{|ccccccc|c|}
\hline
0 & 0 & -1 & 0 & 2/5 & 0 & 0 & -3 \\
0 & 0 & 0 & -1 & -1/5 & 0 & 0 & -2 \\
1 & 0 & 0 & 0 & -1/5 & 0 & 3/5 & 3 \\
0 & 1 & 0 & 0 & 1/5 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 & -1/5 & 1 & 0 & 6 \\
\hline
0 & 0 & 0 & 0 & 3/5 & 0 & 0 & 21 \\
\hline
\end{array} \qquad \begin{array}{|c|} \hline 3 \\ 4 \\ 1 \\ 2 \\ 6 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 7 \\ 5 \\ \hline \end{array}$$

$$BS = (3, 3, 3, 2, 0, 6, 0) = BFS; \qquad -21 = \min z$$

Step 3.

1° No negative elements among $a(m + 1, NF(l)), l = 1, 2$; and NF contains 7 (the number of the artificial variable introduced at Step 0 for the third constraint; thus $x_7 = 0$). Therefore, BOS can be output (omitting x_7) as follows: $BOS = (3, 3, 3, 2, 0, 6), \ -21 = \min z$.

Geometrically, there was a movement through the points $(0, 0) \to (0, 1) \to (3, 3)$ on the plane $ox_1 x_2$, the first two of them lying outside the feasible region.

Definition 3.1. *A basis is said to be correct, if each basic variable appears only in one constraint with the coefficient "+1" and does not appear in the objective function equation. If there exists at least one basic variable appearing there with the coefficient "−1" instead, such a basis is called* **"noncorrect"**.

So, for the case of correct basis we operate only with Table 3.4.1 while for the case of noncorrect basis we address sometimes to Table 3.5.1.

3.6.   The simplex-method with generation an initial BFS

The preceding algorithm (see Sections 3.4 and 3.5) does not rule out the appearance of impermissible solutions at the initial state or at some intermediate steps. In order to exclude such a situation from the very beginning, i.e. to generatate an initial basic feasible solution, the following algorithm has been developed.

The idea is to define a new linear programming problem, for which an initial basic feasible vector is easily found, and whose solution gives the desired initial BFS for the original problem.

Let us illustrate the idea by the previous Example 3.5.2:

$$
\begin{array}{rcrcrl}
\text{minimize} & z & = & -2x_1 & - & 5x_2 \\
\text{subject to} & 2x_1 & & & \geq & 3 \\
& & & x_2 & \geq & 1 \\
& x_1 & + & x_2 & = & 6 \\
& -2x_1 & + & 3x_2 & \leq & 3 \\
& 2x_1 & + & 3x_2 & \leq & 21 \\
& & & x_i \geq 0 & & (i = 1, 2)
\end{array}
$$

It is obvious from Step 0 (see the last example in Section 3.5) that the point $x_1 = 0$, $x_2 = 0$ does not satisfy the constraints owing to the requirements $2x_1 \geq 3$, $x_2 \geq 1$ and $x_1 + x_2 = 6$, so we must find a different initial vector. If, as usual, we introduce slack variables x_3 and x_4 for the first and second offending inequalities, we obtain instead two equalities

$$
\begin{array}{rcrcl}
2x_1 & - & x_3 & = & 3 \\
x_2 & - & x_4 & = & 1
\end{array}
$$

with $x_3 \geq 0$, $x_4 \geq 0$. Our inability to let $x_1 = x_2 = 0$ for the inequalities shows here in the fact that $x_1 = x_2 = 0$ yields $x_3 = -3$ and $x_4 = -1$ in violation of $x_3 \geq 0$ and $x_4 \geq 0$.

It is worth noting that slack variables x_5 and x_6 introduced in the fourth and fifth (non-offending) inequalities do not cause such a trouble because we have

$$
\begin{array}{rcrcrcrcl}
-2x_1 & + & 3x_2 & + & x_5 & & & = & 3 \\
2x_1 & + & 3x_2 & & & + & x_6 & = & 21
\end{array}
$$

and $x_1 = x_2 = 0$ does not violate $x_5 \geq 0$, $x_6 \geq 0$.

The trick with the inequalities of "greater than or equal to" type (and, as it will be seen, with the constraints of "equal to" type) is to introduce one or more another variables, in this example x_7, x_8 and x_9, correspondingly, for the first, second and third offending constraints, the so-called *artificial variables*, and then to: (1) replace x_3 in the first equation $2x_1 - x_3 = 3$ by $x_3 - x_7$ so that we can write the first constraint $2x_1 \geq 3$ as $2x_1 - x_3 + x_7 = 3$ with $x_3 \geq 0$ and $x_7 \geq 0$; (2) replace x_4 in the second equation $x_2 - x_4 = 1$ by $x_4 - x_8$ so that we can write the second constraint $x_2 \geq 1$ as $x_2 - x_4 + x_8 = 1$ with $x_4 \geq 0$ and $x_8 \geq 0$; and (3) write the third constraint $x_1 + x_2 = 6$ as $x_1 + x_2 + x_9 = 6$ with

$x_9 \geq 0$. Now we can let $x_1 = x_2 = 0$ since we can easily have $x_7 - x_3 = 3$, $x_8 - x_4 = 1$ and $x_9 = 6$ while the slack variables (x_3, x_4, x_5 and x_6) and artificial variables (x_7, x_8 and x_9) are all nonnegative. We therefore consider the extended constraints:

$$
\begin{array}{rrrrrrrcl}
2x_1 & & -3x_3 & & & +x_7 & & = & 3 \\
& x_2 & & -x_4 & & & +x_8 & = & 1 \\
x_1 & + & x_2 & & & & & x_9 = & 6 \\
-2x_1 & + & 3x_2 & & +x_5 & & & = & 3 \\
2x_1 & + & 3x_2 & & & +x_6 & & = & 21
\end{array}
\tag{3.3}
$$

for which a basic feasible vector is obvious as $x_1 = x_2 = x_3 = x_4 = 0$, $x_5 = 3$, $x_6 = 21$, $x_7 = 3$, $x_8 = 1$, $x_9 = 6$. If we can somehow start from this vector and eventually find another BFS for the extended constraints (3.3) but for which all the artificial variables (x_7, x_8 and x_9) are zero, then the first six components of this vector will form a basic feasible vector for the original LP problem (with slack variables only), namely for the original constraints written as equalities

$$
\begin{array}{rrrrrrcl}
2x_1 & & -x_3 & & & & = & 3 \\
& x_2 & & -x_4 & & & = & 1 \\
x_1 & + & x_2 & & & & = & 6 \\
-2x_1 & + & 3x_2 & & +x_5 & & = & 3 \\
2x_1 & + & 3x_2 & & & +x_6 & = & 21 \\
\multicolumn{8}{c}{x_i \geq 0 \quad (\text{for} \quad i = 1, \ldots, 6)}
\end{array}
\tag{3.4}
$$

Conversely, any BFS for the last set of constraints (3.4), including only original and slack variables, generates a basic feasible vector for the above extended set of constraints (3.3) (extended with the artificial variables) with all artificial variables being zero: $x_7 = x_8 = x_9 = 0$. Therefore the constraints we really care about, namely the original ones, have a BFS if and only if the extended set of constraints has a basic feasible vector for which all artificial variables are zero, in this example $x_7 = x_8 = x_9 = 0$. Since of course all they must be nonnegative, we are really trying to *minimize* the sum of artificial variables. In this example:

$$
\begin{array}{ll}
\text{minimize} & w = x_7 + x_8 + x_9 \\
\text{subject to} & (3.3) \text{ and } x_i \geq 0 \quad (\text{for } i = 1, \ldots, 9)
\end{array}
$$

Thus, we have to solve this auxiliary LP problem with an artificial objective function w before solving the original one. Since artificial variables must be nonnegative, $x_7 \geq 0$, $x_8 \geq 0$, $x_9 \geq 0$, there are two possibilities:

(1) The minimum value of w is in fact 0. This means that we are able to find a BFS of (3.3) in which $x_7 = x_8 = x_9 = 0$, that is, we can find a BFS of (3.4).

(2) The minimum value of w is greater than 0. This means that no BFS of (3.4) exists because, assuming such a solution did exist, we could find a BFS of (3.3) with $x_7 = x_8 = x_9 = 0$, that is, $w = 0$, and it would contradict what we have just assumed.

Now it is possible to continue our exposition of the method by the same example and see how it is performed from the very beginning to the end.

Example 3.6.1. We solve the problem of Example 3.5.2 and at the same time illustrate the general case implementation when constraints of all types are involved.

Step 0. Data input/calculation:

np, the number of original (prime) variables, here $np = 2$.

ng, the number of constraints of "$\geq$" type, here $ng = 2$.

ne, the number of constraints "$=$" type, here $ne = 1$.

nl, the number of constraints "$\leq$" type, here $nl = 2$.

$ns = ng + nl$, the number of slack variables, here $ns = 4$.

$na = ng + ne$, the number of artificial variables, here $na = 3$.

$n = np + na + ns$, the total number of variables, here $n = 9$.

$m = na + nl$, the total number of constraints, here $m = 5$.

Array memory assignment:

(1) $A(1..m + 2, 1..n + 1)$, here $A(1..7, 1..10)$ is an (7×10) – matrix.

(2) $NB(1..m)$, here $NB(1..5)$ is a 5 dimensional vector.

(3) $NF(1..n - m)$, here $NF(1..4)$ is a 4 dimensional vector.

Coefficients entry from the problem statement in part relating to matrix A:

$A(1..m + 1, 1..np)$, here $A(1..6, 1..2)$

$A(1..m + 1, n + 1)$, here $A(1..6, 10)$

and the rest array elements automatic fill; as a result the arrays become

				A							NB	NF
2	0	−1	0	0	0	1	0	0	3	7	1
0	1	0	−1	0	0	0	1	0	1	8	2
1	1	0	0	0	0	0	0	1	6	9	3
−2	3	0	0	1	0	0	0	0	3	5	4
2	3	0	0	0	1	0	0	0	21	6	
−2	−5	0	0	0	0	0	0	0	0		
0	0	0	0	0	0	1	1	1	0		

$$BS = (0, 0, 0, 0, 3, 21, 3, 1, 6) = BFS, \quad 0 = z_{\min}, \quad 0 = w$$

Remark 3.6.1. *From this result, it is evident that variables are ordered as follows:*

x_1 *and* x_2 *are original (prime) variables,*

x_3, x_4, x_5 *and* x_6 *are slack variables and*

x_7, x_8 *and* x_9 *are artificial variables.*

Such ordering may be fixed by the program design. Artificial variables are introduced only for "$\geq$" and "$=$" constraints. There is an extra row added to the bottom of matrix A; this last row represents the artificial problem. It contains the coefficients of the artificial objective function w and this function value located in the lower right corner (the southeast

corner) of A matrix, with the sign changed, $(-w)$. The second row from the bottom of A matrix represents, as previously, the original problem: it contains the coefficients of the original objective function z and this function value located at the extreme right position, also with the sign changed, $(-z)$.

Remark 3.6.2. *The simplex method under consideration is composed of two stages: Stage I intended to solve the artificial LP problem and Stage II intended to solve the original one. Step 0, considered above in the example, is only a setup step. It is also a preparatory step. To be prepared to the main (working) steps of Stage I, we must express the artificial objective function w only through nonbasic variables. For this example, $w = x_7 + x_8 + x_9$ by definition, but all artificial variables (x_7, x_8, x_9) are to appear in the initial BFS in the capacity of basic variables. By virtue of this requirement, Step 0 must be comleted by elimination of the artificial variables from the bottom row. It is made by subtraction of all those rows where artificial variables were introduced from the bottom row. Given the accepted ordering of variables (see Remark 3.6.1), the first na rows should be subtracted from the last row. Here $na = 3$, and after such a preparatory subtraction we have*

$$A$$

$\rightarrow$	2	0	-1	0	0	0	1	0	0	3
	0	1	0	-1	0	0	0	1	0	1
	1	1	0	0	0	0	0	0	1	6
	-2	3	0	0	1	0	0	0	0	3
	2	3	0	0	0	1	0	0	0	21
	-2	-5	0	0	0	0	0	0	0	0
	-3	-2	1	1	0	0	0	0	0	-10
	$\uparrow$									

As a result we obtain also a correct initial value of the artificial objective function: $10 = w$.

Remark 3.6.3. *Given the constraints ordering: "$\geq$" first, "$=$" second and "$\leq$" third, then basic variables in the inicial BFS have numberes $(np + ng + 1)$ to $n = np + na + ns$, in this example 5 to 9.*

From this time on, Stage I begins where the conventional simplex method (see Sections 3.3 and 3.4) has to be used to solve the auxiliary (artificial) problem $w \to \min$.

Step 1.

    1° $s = 1, \quad l = 1$

    2° $\min(3/2, \infty, 6, \infty, 21/2) = 3/2, \quad k = 1$

3° After normalization:

$$A$$

1	0	−1/2	0	0	0	1/2	0	0	3/2
0	1	0	−1	0	0	0	1	0	1
1	1	0	0	0	0	0	0	1	6
−2	3	0	0	1	0	0	0	0	3
2	3	0	0	0	1	0	0	0	21
−2	−5	0	0	0	0	0	0	0	0
−3	−2	1	1	0	0	0	0	0	−10

$\uparrow$

4° After subtractions:

$$A \qquad NB \quad NF$$

											NB	NF
1	0	−1/2	0	0	0	1/2	0	0	3/2		1	7
0	1	0	−1	0	0	0	1	0	1		8	2
0	1	1/2	0	0	0	−1/2	0	1	9/2		9	3
0	3	−1	0	1	0	1	0	0	6		5	4
0	3	1	0	0	1	−1	0	0	18		6	
0	−5	−1	0	0	0	1	0	0	3			
0	−2	−1/2	1	0	0	3/2	0	0	−11/2			

$\uparrow$

$$BS = (3/2, 0, 0, 0, 6, 18, 1, 9/2) = BFS; \quad -3 = z; \quad 11/2 = w$$

Step 2.

1° $s = 2, \quad l = 2$

2° $\min(\infty, 1, 9/2, 2, 6) = 1, \quad k = 2$

3° After normalization (since the pivot located in above matrix by the intersection of arrows, happens to be 1, division by it makes no change in the key row): the matrix holds unchanged.

4° After subtractions:

$$A \qquad NB \quad NF$$

											NB	NF
1	0	−1/2	0	0	0	1/2	0	0	3/2		1	7
0	1	0	−1	0	0	0	1	0	1		2	8
0	0	1/2	1	0	0	−1/2	−1	1	7/2		9	3
0	0	−1	3	1	0	1	−3	0	3		5	4
0	0	1	3	0	1	−1	−3	0	15		6	
0	0	−1	−5	0	0	1	5	0	8			
0	0	−1/2	−1	0	0	3/2	2	0	−7/2			

$\uparrow$

$$BS = (3/2, 1, 0, 0, 3, 15, 0, 0, 7/2) = BFS; \quad -8 = z; \quad 7/2 = w$$

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Step 3.

1° $s = 4, \quad l = 4$

2° $\min(\infty, \infty, 7/2, 1, 5) = 1, \quad k = 4$

3° After normalization:

$$A$$

1	0	−1/2	0	0	0	1/2	0	0	3/2
0	1	0	−1	0	0	0	1	0	1
0	0	1/2	1	0	0	−1/2	−1	1	7/2
→ 0	0	−1/3	1	1/3	0	1/3	−1	0	1
0	0	1	3	0	1	−1	−3	0	15
0	0	−1	−5	0	0	1	5	0	8
0	0	−1/2	−1	0	0	3/2	2	0	−7/2

↑

4° After subtractions:

$$A \qquad NB \qquad NF$$

										NB	NF
1	0	−1/2	0	0	0	1/2	0	0	3/2	1	7
0	1	−1/3	0	1/3	0	1/3	0	0	2	2	8
→ 0	0	5/6	0	−1/3	0	−5/6	0	1	5/2	9	3
0	0	−1/3	1	1/3	0	1/3	−1	0	1	4	5
0	0	2	0	−1	1	−2	0	0	12	6	
0	0	−8/3	0	5/3	0	8/3	0	0	13		
0	0	−5/6	0	1/3	0	11/6	1	0	−5/2		

↑

$$BS = (3/2, 2, 0, 1, 0, 12, 0, 0, 5/2) = BFS; \quad -13 = z; \quad 5/2 = w$$

Step 4.

1° $s = 3, \quad l = 3$

2° $\min(\infty, \infty, 3, \infty, 6) = 3, \quad k = 3$

3° After normalization:

$$A$$

1	0	−1/2	0	0	0	1/2	0	0	3/2
0	1	−1/3	0	1/3	0	1/3	0	0	2
→ 0	0	1	0	−2/5	0	−1	0	6/5	3
0	0	−1/3	1	1/3	0	1/3	−1	0	1
0	0	2	0	−1	1	−2	0	0	12
0	0	−8/3	0	5/3	0	8/3	0	0	13
0	0	−5/6	0	1/3	0	11/6	1	0	−5/2

↑

4° After subtractions:

$$
\begin{array}{c}
A
\end{array}
$$

1	0	0	0	−1/5	0	0	0	3/5	3
0	1	0	0	1/5	0	0	0	2/5	3
0	0	1	0	−2/5	0	−1	0	6/5	3
0	0	0	1	1/5	0	0	−1	2/5	2
0	0	0	0	−1/5	1	0	0	−12/5	6
0	0	0	0	3/5	0	0	0	16/5	21
0	0	0	0	0	0	1	1	1	0

NB: 1, 2, 3, 4, 6 NF: 7, 8, 9, 5

$$BS = (3,3,3,2,0,6,0,0,0) = BFS; \quad -21 = \min(z); \quad 0 = \min(w)$$

By this point, Stage I has been successfully accomplished as indicated by: (1) the zero value of w (and hence by zero values of artificial variables) and (2) the absence of negative entries in the bottom row. Stage II is not needed here because of the absence of negative entries in the second row from the bottom. The minimums for w and z have been reached simultaneously. In geometrical interpretation, there was a movement through the following points: $(0,0) \to (3/2,0) \to (3/2,1) \to (3/2,2) \to (3,3)$ on the ox_1x_2 plane.

In order to show how Stage II is to be executed, let us choose another path after Step 2.

Step 3.

1° $s = 3, \quad l = 3$

2° $\min(\infty, \infty, 7, \infty, 15) = 7, \quad k = 3$

3° After normalization:

$$
\begin{array}{c}
A
\end{array}
$$

1	0	−1/2	0	0	0	1/2	0	0	3/2
0	1	0	−1	0	0	0	1	0	1
0	0	1	2	0	0	−1	−2	2	7
0	0	−1	3	1	0	1	−3	0	3
0	0	1	3	0	1	−1	−3	0	15
0	0	−1	−5	0	0	1	5	0	8
0	0	−1/2	−1	0	0	3/2	2	0	−7/2

4° After subtractions:

$$A \qquad\qquad NB \quad NF$$

1	0	0	1	0	0	0	−1	1	5
0	1	0	−1	0	0	0	1	0	1
0	0	1	2	0	0	−1	−2	2	7
0	0	0	5	1	0	0	−5	2	10
0	0	0	1	0	1	0	−1	−2	8
0	0	0	−3	0	0	0	3	2	15
0	0	0	0	0	0	1	1	1	0

NB: 1, 2, 3, 5, 6 NF: 7, 8, 9, 4

$$BS = (5, 1, 7, 0, 10, 8, 0, 0, 0) = BFS; \quad -15 = z; \quad 0 = \min(w)$$

Indicative of successful completing of Stage I is 0 for w and nonnegativeness of all coefficients in the bottom row. If w did not become 0 with these coefficients nonnegative, it would be evidence that Stage II could not started because the originaly given constraints did not have any BFS. That would be the case of *contradictory constraints*. (see also Chapter 6)

Remark 3.6.4. *The formal reason for the case of contradictory constraints to take place is that the BFS obtained after Stage I contains at least one of the artificial variables among basic variables (hence $w \neq 0$).*

Turning back to the example, we pass on to Stage II. In this version of steps, Stage II is necessary because z did not reach its minimum as withnessed by one negative coefficient, -3, in the last but one row of matrix A. We need neither artificial variables nor artificial objective function any more at Stage II. Thus, they can be deleted from all the upcoming actions. To be deleted are those columns of matrix A received a value of unity in the last row, and the last row as well. For the assumed ordering of the variables (see Remark 3.6.1), they are: columns with the numbers $(np + ns + 1)$ to n, here 7 to 9. Among deleted elements are also elements of NF containing the same numbers; for this ordering these are always the first na elements, here $na = 3$.

Stage II. Step 4.

1° $s = 4, \quad l = 4$

2° $\min(5, \infty, 7/2, 2, 8) = 2, \quad k = 4$

3° After normalization: **4°** After subtractions:

$$A \qquad\qquad\qquad\qquad A \qquad\qquad NB \quad NF$$

1	0	1	1	0	0	5
0	1	0	−1	0	0	1
0	0	1	2	0	0	7
0	0	0	1	1/5	0	2
0	0	0	1	0	1	8
0	0	0	−3	0	0	15

$\uparrow$

1	0	1	0	−1/5	0	3
0	1	0	0	1/5	0	3
0	0	1	0	−2/5	0	3
0	0	0	1	1/5	0	2
0	0	0	0	−1/5	1	6
0	0	0	0	3/5	0	21

NB: 1, 2, 3, 4, 6 NF: 5

$$BS = (3, 3, 3, 2, 0, 6) = BFS; \quad -21 = \min(z)$$

Geometrically, there was a movement through the points $(0, 0) \rightarrow (3/2, 0) \rightarrow (3/2, 1) \rightarrow (5, 1) \rightarrow (3, 3)$ on the plane ox_1x_2.

Chapter 4.

The Dual Simplex Method

The set of points $x \in R^n$ which yield the minimum value z_{min} of $z = c^T x$ is called the "hyperplane of $z = z_{min}$". This hyperplane divides R^n into two parts: in the first part $z > z_{min}$ and there exists the feasible region; in the second part $z < z_{min}$ and there is no feasible region. The above expounded conventional simplex method works if it starts in the region where $z > z_{min}$. However, in case the initial basic solution lies in the second part, where $z < z_{min}$, it is necessery not to decrease but increase z while approaching the value z_{min} through the non-feasible basic solutions. For such cases, the dual simplex method is designed. Another situation which the method has been devised for may appear as a result of imposing an additional constraint onto the problem just solved. The situation will be certain to appear if the additionally imposed constraint does not contain entirely the original feasible region but parts off a portion from it. Then, naturally, the new z_{min} is greater than the former one. The latter is used as the initial point for the dual simplex method to start and solve the problem as the new one.

By virtue of the fact that it is necessary to increase z while moving from the region of $z < z_{min}$, being in this region is indicated by the *condition* that all coefficients of the objective function $z = c^T x$ represented in terms of canonical form for a basis, are nonnegative. Moving towards z_{min} is assured by sequential elimination of negative variables from the basic solutions. The absence of such variables makes a basic solution the feasible basic solution and, on keeping up the above condition, the optimal one.

4.1. Algorithm with the correct form of a basis

It is unimportant for what reason a situation of readiness to start the dual simplex method has occurred. It is determined by two indications: (1) there is no negative coefficient of the objective function in the canonical form for a basis and (2) the corresponding

basic solution is not a feasible one (it contains one or more negative values). In order to facilitate the detection of the second indication, the appearance of basic variables (x^B) in the constraints should be always made with the coefficient $+1$. This is so called a *correct form of a basis* (see also Section 3.5). Such form is obtained from the very beginning after prior multiplication by -1 those constraints containing a basic variable with the coefficient -1. Hence, a linear programming problem is taken in the following canonical form for a basis:

$$
\begin{bmatrix} I & | & A' \\ -- & | & -- \\ O^T & | & c'^T \end{bmatrix} \begin{bmatrix} x^B \\ --- \\ x^F \end{bmatrix} = \begin{bmatrix} b' \\ --- \\ z - z_0 \end{bmatrix}, \qquad c' \geq 0
$$

where $BS = \{x^B = b', x^F = 0\}$. By doing so, checking of the second indication is reduced to answering the question: Are there negative values among elements of vector b'? From this point, any step of the dual simplex method is started. But in the whole, every step of it falls into four actions as follows:

Action 1° Check elements of b'. If there are no negatives among them, then $z_{\min} = z_0$ has been found with the $BFS = \{x^B = b', x^F = 0\}$, which is the sought optimal solution. If there are one or more negative elements in b', select that row of constraints, its number k, containing the most negative element b'_k of vector b'. By doing so, a basic variable x_r is found, its munber $r = NB(k)$, which has to be removed from x^B owing to $x_r = b'_k$ being less than zero.

After that, the next question arises: What nonbasic variable, say x_s, is to be selected from among the elements of x^F for entering a basis by increasing x_s from zero, instead of the departing variable x_r? Let us assume this question has been answered: the nonbasic variable x_s has been selected and intended for elimination from all the equations of canonical form except the $k-$th equation, which must be normalized prior to this elimination through division by the pivot element $a'_{ks} < 0$. Due to such normalization, one negative element in column b' is removed and correct form of a basis is preserved. As for the elimination in the row of objective function, an important point is that it must preserve the first of above mentioned indications for further applicability of the dual simplex method. To clarify this, let us write the formula to update the $j-$th coefficient of objective function while eliminating x_s:

$$
c_j^+ = c_j' - c_s' a''_{kj}
$$

Here a''_{kj} is the $j-$th coefficient in the $k-$th row after the row normalization:

$$
a''_{kj} = a'_{kj}/a'_{ks}
$$

The first indication requires that the updated value c_j^+ is nonnegative, $c_j^+ \geq 0$, while $c_j' \geq 0$ and $c_s' \geq 0$. It is obvious that only that $j-$th column threatens to disobey the required inequality $c_j^+ \geq 0$ for which $a''_{kj} > 0$. Since $c_j^+ \geq 0$ is equivalent to

$$
\frac{c_j'}{|a'_{kj}|} \geq \frac{c_s'}{|a'_{ks}|}
$$

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the selection of s, the number of pivot variable x_s, turns out to be subject to the condition:

$$\frac{c'_s}{\mid a'_{ks} \mid} = \min_{j:\, a'_{kj}<0} \frac{c'_j}{\mid a'_{kj} \mid} \tag{4.1}$$

Thus, we continue to do the following actions.

Action 2° Scan through the k−th row to find negative coefficients a'_{kj} only for non-basic variables. If there are no such a'_{kj}, there does not exist a basic feasible solution (quit the algorithm with this message). If such elements a'_{kj} exist, select among them one, say a'_{ks}, whose number is s, for which the condition (4.1) holds. Call a'_{ks} the pivot.

Action 3° Normalize the k−th row through dividing it by the pivot a'_{ks}.

Action 4° Subtract the k−th row times a'_{is} from each i−th row $(i \neq k)$.

From this, it is obvious that at each step the dual simplex method is equivalent to the Gauss-Jordan elimination in actions 3° and 4° preceded by special choice of pivot a'_{ks} in actions 1° and 2°.

Example 4.1.1. Find nonnegative solution $x_1, x_2 \geq 0$ to the problem

$$
\begin{array}{rrrcrcl}
2x_1 & + & 3x_2 & \leq & 120 & & \\
3x_1 & + & 9x_2 & \leq & 270 & & \\
-2x_1 & - & 4x_2 & = & z & \rightarrow & \min
\end{array}
$$

(considered in Example 3.3.1) by the dual simplex method.

First of all, write down the problem in standard canonical form for a basis:

2	3	1	0	120
3	9	0	1	270
−2	−4	0	0	0

$$A = \begin{pmatrix} 2 & 3 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{pmatrix} \quad , \quad b = \begin{pmatrix} 120 \\ 270 \end{pmatrix}$$

$$c^T = \begin{pmatrix} -2 & -4 & 0 & 0 \end{pmatrix} \quad , \quad Ax = b$$

$$c^T x = z$$

This form corresponds to $BS = (0, 0, 120, 270) = BFS$, $z = 0$, which is imaged by the point $(0, 0)$ on Ox_1x_2-plane. In totality, there are $C_4^2 = 6$ variants to choose a linearly independent set of two vectors of four: (1,2), (1,3), (1,4), (2,3), (2,4), (3,4).

Variant (3,4) has been shown above. Let us write down the rest variants in order to have the complete set of basic solutions with their point images on the plane.

Variant (1,2)

$$B = \begin{pmatrix} 2 & 3 \\ 3 & 9 \end{pmatrix} \quad , \quad B^{-1} = \begin{pmatrix} 1 & -1/3 \\ -1/3 & 2/9 \end{pmatrix}$$

$$B^{-1}A = \begin{pmatrix} 1 & 0 & 1 & -1/3 \\ 0 & 1 & -1/3 & 2/9 \end{pmatrix} \quad , \quad b' = B^{-1}b = \begin{pmatrix} 30 \\ 20 \end{pmatrix}$$

1	0	1	−1/3	30
0	1	−1/3	2/9	20
0	0	2/3	2/9	140

$BS = (30, 20, 0, 0) = BFS$
$-140 = z_{\min}$
the point $(30, 20)$

Variant (1,3)

$$B = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} \quad, \quad B^{-1} = \begin{pmatrix} 0 & 1/3 \\ 1 & -2/3 \end{pmatrix}$$

$$B^{-1}A = \begin{pmatrix} 1 & 3 & 0 & 1/3 \\ 0 & -3 & 1 & -2/3 \end{pmatrix} \quad, \quad b' = B^{-1}b = \begin{pmatrix} 90 \\ -60 \end{pmatrix}$$

1	3	0	1/3	90
0	-3	1	-2/3	-60
0	2	0	2/3	180

BS=$(90, 0, -60, 0) \neq$ BFS
$-180 = z < z_{\min}$
the point $(90, 0)$

Variant (1,4)

$$B = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} \quad, \quad B^{-1} = \begin{pmatrix} 1/2 & 0 \\ -3/2 & 1 \end{pmatrix}$$

$$B^{-1}A = \begin{pmatrix} 1 & 3/2 & 1/2 & 0 \\ 0 & 3/2 & -3/2 & 1 \end{pmatrix} \quad, \quad b' = B^{-1}b = \begin{pmatrix} 60 \\ 90 \end{pmatrix}$$

1	3/2	1/2	0	60
0	3/2	-3/2	1	90
0	-1	1	0	120

BS=$(60, 0, 0, 90)$=BFS
$-120 = z > z_{\min}$
the point $(60, 0)$

Variant (2,3)

$$B = \begin{pmatrix} 3 & 0 \\ 9 & 1 \end{pmatrix} \quad, \quad B^{-1} = \begin{pmatrix} 0 & 1/9 \\ 1 & -1/3 \end{pmatrix}$$

$$B^{-1}A = \begin{pmatrix} 1/3 & 1 & 0 & 1/9 \\ 1 & 0 & 1 & -1/3 \end{pmatrix} \quad, \quad b' = B^{-1}b = \begin{pmatrix} 30 \\ 30 \end{pmatrix}$$

1/3	1	0	1/9	30
1	0	1	-1/3	30
-2/3	0	0	4/9	120

BS=$(0, 30, 30, 0)$=BFS
$-120 = z > z_{\min}$
the point $(0, 30)$

Variant (2,4)

$$B = \begin{pmatrix} 3 & 0 \\ 9 & 1 \end{pmatrix} \quad, \quad B^{-1} = \begin{pmatrix} 1/3 & 0 \\ -3 & 1 \end{pmatrix}$$

$$B^{-1}A = \begin{pmatrix} 2/3 & 1 & 1/3 & 0 \\ -3 & 0 & -3 & 1 \end{pmatrix} \quad, \quad b' = B^{-1}b = \begin{pmatrix} 40 \\ -90 \end{pmatrix}$$

2/3	1	1/3	0	40
-3	0	-3	1	-90
2/3	0	4/3	0	160

BS=$(0, 40, 0, -90)$
$-160 = z < z_{\min}$
the point $(0, 40)$

In order to apply the dual simplex method it is necessary to start from the region $z < z_{\min}$, that is, from any $BS \neq BFS$. In this example, we have two such points: $(90, 0)$ or $(0, 40)$. Let us take, for instance, variant $(1, 3)$ as initial one:

$$
\begin{array}{c}
A \\
\begin{array}{|cccc|c|}
\hline
1 & 3 & 0 & 1/3 & 90 \\
0 & -3 & 1 & -2/3 & -60 \\
\hline
0 & 2 & 0 & 2/3 & 180 \\
\hline
\end{array}
\end{array}
\qquad
\begin{array}{c}
NB \\
\begin{array}{|c|}
\hline
1 \\
3 \\
\hline
\end{array}
\end{array}
\qquad
\begin{array}{c}
NF \\
\begin{array}{|c|}
\hline
2 \\
4 \\
\hline
\end{array}
\end{array}
$$

Step 1.

1° $k = 2$. Choosing the second row, $k = 2$, as the key row is shown above by the arrow on the left of the tableau.

2° $\min\limits_{l=1,2;\ j=NF(l)} \left(\frac{2}{|-3|}, \frac{2/3}{|-2/3|} \right) = \frac{2}{3}; \quad l = 1; \quad s = 2$

Choosing the second column, $s = 2$, as the pivot column is shown above by the arrow under the tableau.

3° After normalization:

$$
\begin{array}{c}
A \\
\begin{array}{|cccc|c|}
\hline
1 & 3 & 0 & 1/3 & 90 \\
0 & 1 & -1/3 & 2/9 & 20 \\
\hline
0 & 2 & 0 & 2/3 & 180 \\
\hline
\end{array}
\end{array}
$$

4° After subtractions:

$$
\begin{array}{c}
A \\
\begin{array}{|cccc|c|}
\hline
1 & 0 & 1 & -1/3 & 30 \\
0 & 1 & -1/3 & 2/9 & 20 \\
\hline
0 & 0 & 2/3 & 2/9 & 140 \\
\hline
\end{array}
\end{array}
\qquad
\begin{array}{c}
NB \\
\begin{array}{|c|}
\hline
1 \\
2 \\
\hline
\end{array}
\end{array}
\qquad
\begin{array}{c}
NF \\
\begin{array}{|c|}
\hline
3 \\
4 \\
\hline
\end{array}
\end{array}
$$

Step 2.

1° The solution has been found: $BS = (30, 20, 0, 0) = BFS, \ -140 = z_{\min}$

Example 4.1.2. Assume an additional constraint to be imposed on the problem of Example 4.1.1:

$$5x_1 + 3x_2 \leq 150$$

The problem can be solved as a whole by the conventional simplex method. However, one can save time proceeding from that very place where the solution of Example 4.1.1 had been found by simply adding one more constraint. The scheme of adding may be as follows:

A b

NB *NF*

c $-z$
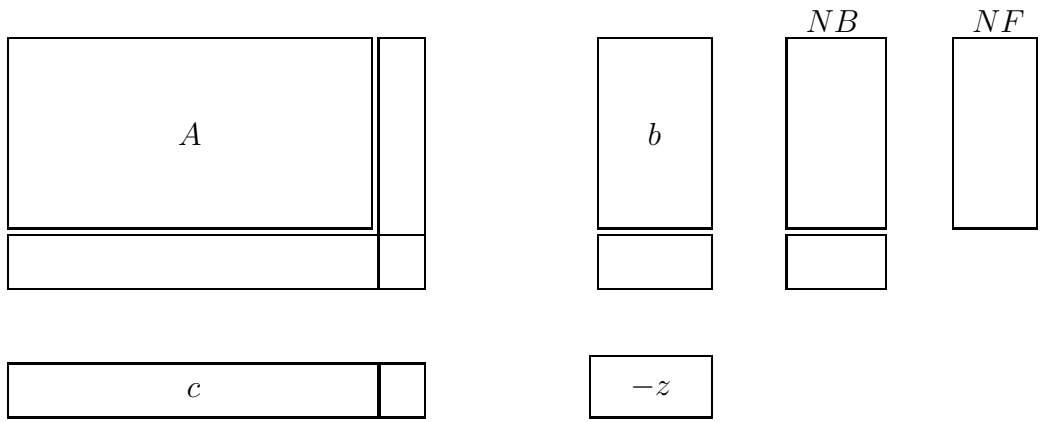
However, techically it may not be convenient if $(A, b, c, -z)$ are stored in a single array. In this case it would be more labour-saving either to store arrays under separate names or to place them in a single array of the following form:

NB $-z$ *c* *NF*

b *A*
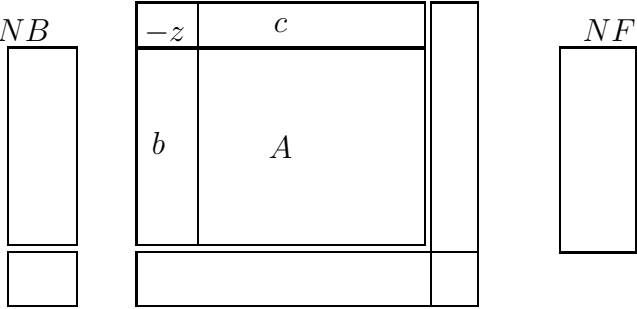
If the programming language permits for array rows and columns to be enumerated from zero, then row number 0 can be used for $(-z, c)$ and column number 0 for b. Doing so and adding one more row for the additional constraint and one more column for one more slack variable x_5, we have the following continuation of Example 4.1.1:

Step 0. Input the additional row of constraints:

NB							*NF*
	140	0	0	2/3	2/9	0	
1	30	1	0	1	−1/3	0	3
2	20	0	1	−1/3	2/9	0	4
(5)	150	5	3	0	0	1	

Here the bottom row does not correspond yet to canonical form for a basis, and so the number 5 is placed, for the time being, in parentheses. To legalize this form, the elimination of the two former basic variables, x_1 and x_2, must be done from the just added row. We do it here by the two preparatory subtractions: (1) subtract the row 1 times 5 and (2) subtract the row 2 times 3 from the bottom row. As a result, we have

$$
NB \quad
\begin{array}{c|cccccc}
140 & 0 & 0 & 2/3 & 2/9 & 0 \\
\hline
30 & 1 & 0 & 1 & -1/3 & 0 \\
20 & 0 & 1 & -1/3 & 2/9 & 0 \\
-60 & 0 & 0 & -4 & 1 & 1 \\
\end{array}
\quad NF
$$

$$
NB = \begin{array}{c} 1 \\ 2 \\ \rightarrow 5 \end{array}
\qquad
NF = \begin{array}{c} 3 \\ 4 \end{array}
$$

$$BS = (30, 20, 0, 0, -60) \neq BFS, \quad -140 = z$$

Step 1.

1° $k = 3$

2° $\min\left(\frac{2/3}{|-4|}\right) = \frac{1}{6}; l = 1, s = NF(l) = 3$

3° After normalization:

$$
\begin{array}{c|ccccc}
140 & 0 & 0 & 2/3 & 2/9 & 0 \\
\hline
30 & 1 & 0 & 1 & -1/3 & 0 \\
20 & 0 & 1 & -1/3 & 2/9 & 0 \\
15 & 0 & 0 & 1 & -1/4 & -1/4 \\
\end{array}
$$

4° After subtractions and switching $NB(k) \longleftrightarrow NF(l)$:

$$
NB \quad
\begin{array}{c|ccccc}
130 & 0 & 0 & 0 & 7/18 & 1/6 \\
\hline
15 & 1 & 0 & 0 & -1/12 & 1/4 \\
25 & 0 & 1 & 0 & 5/36 & -1/12 \\
15 & 0 & 0 & 1 & -1/4 & -1/4 \\
\end{array}
\quad NF
$$

$$
NB = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}
\qquad
NF = \begin{array}{c} 5 \\ 4 \end{array}
$$

$$BS = (15, 25, 15, 0, 0) = BFS, \quad -130 = z_{\min}$$

In geometrical interpretation on the Ox_1x_2-plane, there was the movement from the point $(30, 20)$ in the region of $z < z_{\min}$ to the point $(15, 25)$. That the last point is the problem solution is diagnosed at the beginning of Step 2 by the absence of negative values among $(15, 25, 15)$ in the zero column of the matrix. The solution has been found for one step because the additional constraint was imposed on the final solution of Example 4.1.1. This constraint may be placed upon the other form of the objective function, and in such case the dual simplex method may take more steps to find the solution.

In order to show this, let us place the additional constraint on variant $(2, 4)$ of Example 4.1.1:

Step 0. Input the additional constraint:

$$
NB \quad
\begin{array}{c|ccccc}
160 & 2/3 & 0 & 4/3 & 0 & 0 \\
\hline
40 & 2/3 & 1 & 1/3 & 0 & 0 \\
-90 & -3 & 0 & -3 & 1 & 0 \\
150 & 5 & 3 & 0 & 0 & 1 \\
\end{array}
\quad NF
$$

$$
NB = \begin{array}{c} 2 \\ 4 \\ (5) \end{array}
\qquad
NF = \begin{array}{c} 1 \\ 3 \end{array}
$$

Setting this tableau at ready for the dual simplex method, eliminate, from the added row, the variables x_2 and x_4, which are basic variables befor Step 0. To do this, subtract an appropriate multiple of the $k-$th row so chosen as to eliminate the basic variable $x(NB(k))$ from the last row. First, $k = 1$; thus we eliminate x_2 because $NB(k) = NB(1) = 2$, and the appropriate multiplier for the $k-$th row is taken from position $NB(k) = NB(1) = 2$ of the last row (here it is 3). After subtraction we have

$$
\begin{array}{c}
NB \\
\begin{array}{c} 2 \\ \rightarrow 4 \\ 5 \end{array}
\end{array}
\begin{array}{|c|ccccc|}
\hline
160 & 2/3 & 0 & 4/3 & 0 & 0 \\
\hline
40 & 2/3 & 1 & 1/3 & 0 & 0 \\
-90 & -3 & 0 & -3 & 1 & 0 \\
30 & 3 & 0 & -1 & 0 & 1 \\
\hline
\end{array}
\begin{array}{c}
NF \\
\begin{array}{|c|} \hline 1 \\ 3 \\ \hline \end{array}
\end{array}
$$
$$\uparrow$$

Second, $k = 2$; and we eliminate the basic variable x_4 because $NB(k) = NB(2) = 4$. However, for the coefficient in the $NB(k)-$th position of the last row being zero, subtraction is not needed here. (Such checkings for zero are advisable in the computer programme to avoid unnecessary time consumption.) Thus, readiness for the dual simplex method has been assured:

$$BS = (0, 40, 0, -90, 30) \neq BFS, \qquad -160 = z < z_{\min}$$

Step 1.

1° $k = 2$

2° Look for negative nonbasic variables in the $k-$th row. NF shows that the 1st and the 3rd elements should be checked. They both happen to be negative. Among negatives, select one providing

$$\min_{j=1,3} \left(\frac{c'_j}{\mid a'_{kj} \mid} \right) = \min \left(\frac{2/3}{\mid -3 \mid}, \frac{4/3}{\mid -3 \mid} \right) = \min \left(\frac{2}{9}, \frac{4}{9} \right) = \frac{2}{9}$$

Minimum is provided by the first element, that is, $l = 1$, $j = NB(l) = s = 1$.

It is along action 1° that the key row is found and along action 2° the pivot column is found. Thereby the pivot element a'_{ks}, here -3 marked by arrows near the tableau, is determined to be used in the Gauss-Jordan elimination as follows in actions 3° and 4°.

3° After normalization:

$$
\rightarrow
\begin{array}{|ccccccc|}
\hline
160 & 2/3 & 0 & 4/3 & & 0 & 0 \\
\hline
40 & 2/3 & 1 & 1/3 & & 0 & 0 \\
30 & 1 & 0 & 1 & -1/3 & & 0 \\
30 & 3 & 0 & -1 & & 0 & 1 \\
\hline
\end{array}
$$
$$\uparrow$$

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4° After subtraction and switching $NB(k) \longleftrightarrow NF(l)$:

NB	140	0	0	2/3	2/9	0	NF
2	20	0	1	−1/3	2/9	0	4
1	30	1	0	1	−1/3	0	3
→ 5	−60	0	0	−4	1	1	

$$\uparrow$$

$$BS = (30, 20, 0, 0, -60) \neq BFS, \quad -140 = z < z_{\min}$$

Step 2.

1° $k = 3$

2° $l = 2, j = NF(l) = 3, s = 3$

3° After normalization:

140	0	0	2/3	2/9	0
20	0	1	−1/3	2/9	0
30	1	0	1	−1/3	0
→ 15	0	0	1	−1/4	−1/4

$$\uparrow$$

4° After subtraction and switching $NB(k) \longleftrightarrow NF(l)$:

NB	130	0	0	0	7/18	1/6	NF
2	25	0	1	0	5/36	−1/12	4
1	15	1	0	0	−1/12	1/4	5
3	15	0	0	1	−1/4	−1/4	

$$BS = (15, 25, 15, 0, 0) = BFS, \quad -130 = z_{\min}$$

Step 3.

1° The solution has been found:

$$BOS = (15, 25, 15, 0, 0), \quad z_{\min} = -130$$

Geometrically, there was a movement by two steps through the points: $(0, 40) \rightarrow (30, 20) \rightarrow (15, 25)$.

Example 4.1.3. For $x_1, x_2 \geq 0$ solve the problem (cf. Example 3.5.2)

$$
\begin{array}{rcrcl}
2x_1 & & & \geq & 3 \\
& & x_2 & \geq & 1 \\
-2x_1 & + & 3x_2 & \leq & 3 \\
2x_1 & + & 3x_2 & \leq & 21 \\
-2x_1 & - & 5x_2 & = & z \quad \rightarrow \quad \min
\end{array}
$$

and after that introduce an additional constraint

$$x_1 + x_2 \leq 6$$

Step 0.

$$
\begin{array}{c|ccccccc}
NB & 0 & -2 & -5 & 0 & 0 & 0 & 0 \\
\hline
3 & 3 & 2 & 0 & -1 & 0 & 0 & 0 \\
4 & 1 & 0 & 1 & 0 & -1 & 0 & 0 \\
\rightarrow 5 & 3 & -2 & 3 & 0 & 0 & 1 & 0 \\
6 & 21 & 2 & 3 & 0 & 0 & 0 & 1 \\
\end{array}
\qquad
\begin{array}{c}
NF \\
\hline
1 \\
2 \\
\end{array}
$$

$$BS = (0, 0, -3, -1, 3, 21) \neq BFS, \quad 0 = z$$

We start finding solution by the conventional simplex method as in Section 3.5.

Step 1.

1° $s = 2, l = 2$

2° $\min(\infty, \infty, 3/3, 21/3) = 1, \ k = 3$

3° After normalization:

$$
\begin{array}{c|cccccc}
0 & -2 & -5 & 0 & 0 & 0 & 0 \\
\hline
3 & 2 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & -1 & 0 & 0 \\
1 & -2/3 & 1 & 0 & 0 & 2/3 & 0 \\
21 & 2 & 3 & 0 & 0 & 0 & 1 \\
\end{array}
$$

4° After subtraction and switching $NB(k) \longleftrightarrow NF(l)$:

$$
\begin{array}{c|cccccc}
NB & 5 & -16/3 & 0 & 0 & 0 & 5/3 & 0 \\
\hline
3 & 3 & 2 & 0 & -1 & 0 & 0 & 0 \\
4 & 0 & 2/3 & 0 & 0 & -1 & -1/3 & 0 \\
2 & 1 & -2/3 & 1 & 0 & 0 & 1/3 & 0 \\
\rightarrow 6 & 18 & 4 & 0 & 0 & 0 & -1 & 1 \\
\end{array}
\qquad
\begin{array}{c}
NF \\
\hline
1 \\
5 \\
\end{array}
$$

$$BS = (0, 1, -3, 0, 0, 18) \neq BFS, \quad -5 = z$$

Step 2.

1° $s = 1, l = 1$

2° $\min(\infty, \infty, \infty, 18/4) = 9/2, \ k = 4$

3° After normalization:

5	$-16/3$	0	0	0	$5/3$	0
3	2	0	-1	0	0	0
0	$2/3$	0	0	-1	$-1/3$	0
1	$-2/3$	1	0	0	$1/3$	0
→ 9/2	1	0	0	0	$-1/4$	$1/4$

$\uparrow$

4° After subtraction and switching $NB(k) \longleftrightarrow NF(l)$:

NB								NF
	29	0	0	0	0	$1/3$	$4/3$	
3	-6	0	0	-1	0	$1/2$	$-1/2$	6
4	-3	0	0	0	-1	$-1/6$	$-1/6$	5
2	4	0	1	0	0	$1/6$	$1/6$	
1	$9/2$	1	0	0	0	$-1/4$	$1/4$	

$$BS = (9/2, 4, 6, 3, 0, 0) = BFS, \quad -29 = z_{\min}$$

Now, we introduce the additional constraint and at the same time produce the correct form of a basis (through multiplying the first and second rows by -1). Changing from the conventional simplex method to the dual, we begin the step numbering from the start:

Step 0.

NB										NF
	29	0	0	0	0	$1/3$	$4/3$	0		
3	6	0	0	1	0	$-1/2$	$1/2$	0		6
4	3	0	0	0	1	$1/6$	$1/6$	0		5
2	4	0	1	0	0	$1/6$	$1/6$	0		
1	$9/2$	1	0	0	0	$-1/4$	$1/4$	0		
(7)	6	1	1	0	0	0	0	1		

In the added (the bottom) row of constraints there are variables that were before in the list of basic variables, here $(3, 4, 2, 1)$, and variables that entered the row with non-zero coefficients, here 1 and 2. Making the tableau ready for the dual simplex method, eliminate variables 1 and 2 from the bottom row through Gauss elimination by row. As a result, we obtain

NB									NF
	29	0	0	0	0	$1/3$	$4/3$	0	
3	6	0	0	1	0	$-1/2$	$1/2$	0	6
4	3	0	0	0	1	$1/6$	$1/6$	0	5
2	4	0	1	0	0	$1/6$	$1/6$	0	
1	$9/2$	1	0	0	0	$-1/4$	$1/4$	0	
→ 7	$-5/2$	0	0	0	0	$1/12$	$-5/12$	1	

$\uparrow$

$$BS = (9/2, 4, 6, 3, 0, 0, -5/2) \neq BFS, \quad -29 = z$$

From this point, the dual simplex method starts.

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Step 1.

1° $k = 5$. Variable x_7, $NB(5) = 7$, will be departing the basis because $x_7 = -5/2 < 0$

2° $s = 6$, $l = 1$, because

$$\min_{l,j=NF(l):a'_{kj}<0} \left(\frac{c'_j}{|a'_{kj}|} \right) = \min_{j=6} \left(\frac{4/3}{|-5/12|} \right) = \frac{16}{5}$$

3° After normalization:

29	0	0	0	0	1/3	4/3	0
6	0	0	1	0	$-1/2$	1/2	0
3	0	0	0	1	1/6	1/6	0
4	0	1	0	0	1/6	1/6	0
9/2	1	0	0	0	$-1/4$	1/4	0
6	0	0	0	0	$-1/5$	1	$-12/5$

$\rightarrow$ $\uparrow$

4° After subtraction and switching $NB(k) \longleftrightarrow NF(l)$:

NB

21	0	0	0	0	3/5	0	16/5	NF
3	0	0	1	0	$-2/5$	0	6/5	7
2	0	0	0	1	1/5	0	2/5	5
3	0	1	0	0	1/5	0	2/5	
3	1	0	0	0	$-1/5$	0	3/5	
6	0	0	0	0	$-1/5$	1	$-12/5$	

NB column: 3, 4, 2, 1, 6

$$BS = (3, 3, 3, 2, 0, 6, 0) = BFS, \quad -21 = z_{\min}$$

Example 4.1.4. Find solution to the preceding Example 4.1.3 and then place another constraint

$$x_1 + x_2 = 6$$

on the problem.

Let us continue the solution after Step 2 (see in the above). For the last constraint, we introduce an artificial variable x_7 and artificial objective function $w = x_7$. From this point, the step numbering will be started over again. Writing down the simplex-tableau will be done in the form taken everywhere in the book except Examples 4.1.2 and 4.1.3. Besides, we provide the correct form of basis (an algorithm without such a form is given below, see Section 4.2).

Step 0.

NB NF

0	0	1	0	$-1/2$	$1/2$	0	6
0	0	0	1	$1/6$	$1/6$	0	3
0	1	0	0	$1/6$	$1/6$	0	4
1	0	0	0	$-1/4$	$1/4$	0	$9/2$
1	1	0	0	0	0	1	6
0	0	0	0	$1/3$	$4/3$	0	29
0	0	0	0	0	0	1	0

NB: 3, 4, (2), (1), (7) NF: 5, 6

By virtue of adding the fifth row corresponding to the fifth constraint, variables x_2, x_1 ceased to be basic. Also, the just introduced artificial variable x_7 is, for the time being, not basic with regard to the artificial objective function (see the last row in the above tableau). It is easy to bring x_2, and x_1 back into the set of basic variables by evident subtractions of those rows where x_2 and x_1 were basic variables from the fifth row. At the same time, it is necessary to make x_7 be basic by subtraction of the so modified fifth row from the bottom row. We obtain:

NB NF

0	0	1	0	$-1/2$	$1/2$	0	6
0	0	0	1	$1/6$	$1/6$	0	3
0	1	0	0	$1/6$	$1/6$	0	4
1	0	0	0	$-1/4$	$1/4$	0	$9/2$
0	0	0	0	$1/12$	$-5/12$	1	$-5/2$
0	0	0	0	$1/3$	$4/3$	0	29
0	0	0	0	$-1/12$	$5/12$	0	$5/2$

$\rightarrow$ at the fifth row; $\uparrow$ at the $-1/12$ column

NB: 3, 4, 2, 1, 7 NF: 1 5, 2 6

$$BS = (9/2, 4, 6, 3, 0, 0, -5/2) \neq BFS, \quad z = -29, \ w = -5/2$$

Stage I. The convenient simplex method for the artificial objective function w is used because there exists a negative element $-1/12$ in the bottom row.

Step 1.

1° $l = 1, s = 5$

2° $\min\left(\infty, \frac{3}{1/6}, \frac{4}{1/6}, \infty, -\infty\right)$

According to Table 3.4.1, the token "$-\infty$" means "prohibited". If nevertheless we select the key row by this token, in this example $k = 5$, then it will mean that nonbasic variable x_5 now becomes basic, although it is negative (this is why this case was named "prohibited"). However, x_5, having become negative, increases the artificial objective function w from its value $-5/2$ thanks to negative coefficient

$-1/12$. But namely this is necessary, considering that stage I must be ended in $w = 0$. By virtue of these arguments, the selection $k = 5$ is rightful and must not be considered as prohibited. ("Prohibited" means only an exit out of feasible region where always exists a possibility to return with the usage of the dual simplex method.)

3° After normalization:

$$
\rightarrow
\begin{array}{|cccccccc|}
\hline
0 & 0 & 1 & 0 & -1/2 & 1/2 & 0 & 6 \\
0 & 0 & 0 & 1 & 1/6 & 1/6 & 0 & 3 \\
0 & 1 & 0 & 0 & 1/6 & 1/6 & 0 & 4 \\
1 & 0 & 0 & 0 & -1/4 & 1/4 & 0 & 9/2 \\
0 & 0 & 0 & 0 & 1 & -5 & 12 & -30 \\
\hline
0 & 0 & 0 & 0 & 1/3 & 4/3 & 0 & 29 \\
\hline
0 & 0 & 0 & 0 & -1/12 & 5/12 & 0 & 5/2 \\
\hline
\end{array}
$$
$$\uparrow$$

4° After subtraction:

$$
\begin{array}{cc}
NB & NF
\end{array}
$$

$$
\rightarrow
\begin{array}{|cccccccc|c|}
\hline
0 & 0 & 1 & 0 & 0 & -2 & 6 & -9 & 3 \\
0 & 0 & 0 & 1 & 0 & 1 & -2 & 8 & 4 \\
0 & 1 & 0 & 0 & 0 & 1 & -2 & 9 & 2 \\
1 & 0 & 0 & 0 & 0 & -1 & 3 & -3 & 1 \\
0 & 0 & 0 & 0 & 1 & -5 & 12 & -30 & 5 \\
\hline
0 & 0 & 0 & 0 & 0 & 3 & -4 & 39 & \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \\
\hline
\end{array}
\qquad
\begin{array}{|c|}
\hline
3 \\
4 \\
2 \\
1 \\
5 \\
\hline
\end{array}
\quad
\begin{array}{|cc|}
\hline
1 & 7 \\
2 & 6 \\
\hline
\end{array}
$$
$$\uparrow$$

$$BS = (-3, 9, -9, 8, -30, 0, 0) \neq BFS, \quad z = -39, \ w = 0$$

Stage I has been succesfully ended. The bottom row, and the seventh column, and $NF(1) = 7$ may be cut out. After that, the view of the last row coefficients, $(0, 0, 0, 0, 0, 3)$, and the view of right hand side vector, $(-9, 8, 9, -3, -30)$, witness that further must be applied the dual simplex method with respect to the original objective function z.

Stage II. The dual simplex method with a correct form for a basis (cf. Section 4.1).

Step 2.

1° $k = 5$

2° $s = 6$

3° After normalization:

$$
\begin{array}{cccccc|c}
0 & 0 & 1 & 0 & 0 & -2 & -9 \\
0 & 0 & 0 & 1 & 0 & 1 & 8 \\
0 & 1 & 0 & 0 & 0 & 1 & 9 \\
1 & 0 & 0 & 0 & 0 & -1 & -3 \\
\rightarrow \quad 0 & 0 & 0 & 0 & -1/5 & 1 & 6 \\
\hline
0 & 0 & 0 & 0 & 0 & 3 & 39 \\
& & & & & \uparrow &
\end{array}
$$

4° After subtraction:

$$
\begin{array}{c}
& NB & NF
\end{array}
$$

$$
\begin{array}{cccccc|c}
\hline
0 & 0 & 1 & 0 & -2/5 & 0 & 3 \\
0 & 0 & 0 & 1 & 1/5 & 0 & 2 \\
0 & 1 & 0 & 0 & 1/5 & 0 & 3 \\
1 & 0 & 0 & 0 & -1/5 & 0 & 3 \\
0 & 0 & 0 & 0 & -1/5 & 1 & 6 \\
\hline
0 & 0 & 0 & 0 & 3/5 & 0 & 21 \\
\hline
\end{array}
\qquad
\begin{array}{|c|}
\hline
3 \\
4 \\
2 \\
1 \\
6 \\
\hline
\end{array}
\qquad
\begin{array}{|c|}
\hline
(7) \\
5 \\
\hline
\end{array}
$$

$$
BS = (3, 3, 3, 2, 0, 6) = BFS, \quad z_{\min} = -21
$$

Step 3.

1° The above BFS is BOS (basic optimal solution) because there is no negative element in the tableau right hand column, and $z_{\min} = -21$.

In geometrical interpretation on the Ox_1x_2-plane, after placing the constraint $x_1 + x_2 = 6$ upon the problem, there was a movement from point $(9/2, 4)$, that was optimal before the additional constraint placing, into point $(-3, 9)$ where lines $2x_1 + 3x_2 = 21$ and $x_1 + x_2 = 6$ intersect, and it is beyond the feasible region (Step 1). The next movement (Step 2) was into point $(3, 3)$, yielding minimum z.

Remark 4.1.1. *Comparison of Examples 4.1.3 and 4.1.4 shows that every constraint of "=" type requires the introduction of both an artificial variable and artificial objective function into the problem.*

4.2. Algorithm without correct form of a basis

In the above, the dual simplex method is expounded for cases when all inequalities in the constraints have beforehand been brought into the form of "less than or equal to". Then, the added slack variables appear in the system of constraints with the coefficient $+1$ and, thereby, these (as well as all sequent) basic variables have the identity matrix columns in the capacity of constraint vectors. As it was above defined, such a basis

(formed by the identity matrix columns) is called *correct* in order to differentiate it from another situations in which some basic variables appear in constraints with coefficient $+1$ and others with coefficient -1. Obviously, any such situation corresponds to a set of constraints containing both "less than or equal to" and "greater than or equal to" inequalities in the originally given problem.

The dual simplex method for such situations (of non-correct form of a basis) is illustrated by the following example, which also shows the type of constraints for which it is convenient to begin finding the solution by the dual simplex method, namely the type for which the coordinate origin usually used as the starting point for the solution, lies in the semi-space where $z < z_{\min}$.

Remark 4.2.1. *Lying of a current basic solution in the semi-space $z < z_{\min}$ is detected by the condition that the nonbasic variables have in the objective function row only non-negative coefficients and some of the coefficients are positive. Only this condition enables one to apply the dual simplex method.*

Example 4.2.1. For $x_1, x_2 \geq 0$ solve the problem:

$$
\begin{array}{rcrcl}
2x_1 & + & 3x_2 & \geq & 6 \\
2x_1 & + & 6x_2 & \geq & 9 \\
2x_1 & + & 2x_2 & \leq & 7 \\
x_1 & + & 4x_2 & = & z \quad \to \quad \min
\end{array}
$$

Step 0.

	x_1	x_2	x_3	x_4	x_5	b'
	2	3	-1	0	0	6
$\to$	2	6	0	-1	0	9
	2	2	0	0	1	7
	1	4	0	0	0	0

NB: 1 | 3 ; 2 | 4 ; 3 | 5

NF: 1 | 1 ; 2 | 2

$$BS = (0, 0, -6, -9, 7) \neq BFS, \quad 0 = z$$

It is permitted to apply the dual simplex method (see Remark 4.2.1).

Step 1.

1° Look at column b' through $NB(i)$ and find the rows where $a'[i, NB(i)] * b'(i) < 0$. Select from them a row number k where this quantity, $a'[i, NB(i)] * b'(i)$, is most negative. If there is no such a row, the problem has been solved (the end). Here $k = 2$.

2° Look, through $NF(i)$, at those coefficients for nonbasic variables in the $k-$th row which are positive if $a'[k, NB(k)] = -1$, or negative if $a'[k, NB(k)] = +1$. Select

from them a coefficient providing

$$\min_j \left(\frac{c'_j}{\mid a'_{kj} \mid} \right)$$

In this example

$$\min \left(\frac{1}{\mid 2 \mid}, \frac{4}{\mid 6 \mid} \right) = \frac{1}{2}$$

Fix numbers l and $s = NF(l)$ corresponding to the minimum, in this example $l = 1$, $s = 1$.

3° After normalization of the k-th row through division it by the pivot a'_{ks}, here $a'_{ks} = 2$, we obtain:

$$
\rightarrow
\begin{array}{|ccccc|c|}
\hline
2 & 3 & -1 & 0 & 0 & 6 \\
1 & 3 & 0 & -1/2 & 0 & 9/2 \\
2 & 2 & 0 & 0 & 1 & 7 \\
\hline
1 & 4 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
$$
$\uparrow$

4° After subtraction the appropriate multiples of the k-th row (using coefficients a'_{is}) from all other rows $(i \neq k)$ and after switching $NB(k) \leftrightarrow NF(l)$, we obtain:

$$
\begin{array}{cc}
& \begin{array}{cc} NB & NF \end{array}
\end{array}
$$

$$
\rightarrow
\begin{array}{|cccccc|}
\hline
0 & -3 & -1 & 1 & 0 & -3 \\
1 & 3 & 0 & -1/2 & 0 & 9/2 \\
0 & -4 & 0 & 1 & 1 & -2 \\
\hline
0 & 1 & 0 & 1/2 & 0 & -9/2 \\
\hline
\end{array}
\qquad
\begin{array}{c}
_1\,\boxed{3} \\
_2\,\boxed{1} \\
_3\,\boxed{5}
\end{array}
\qquad
\begin{array}{c}
_1\,\boxed{4} \\
_2\,\boxed{2} \\
\end{array}
$$
$\uparrow$

$$BS = (9/2, 0, 3, 0, -2) \neq BFS, \quad 9/2 = z$$

Step 2.

1° $a'[3, NB(3)] * b'(3) < 0, \quad \Rightarrow k = 3$

2° $a'[3, NB(3)] = +1$; one negative coefficient $-4 \Rightarrow l = 2, s = 2$

3° After normalization:

$$
\rightarrow
\begin{array}{|cccccc|}
\hline
0 & -3 & -1 & 1 & 0 & -3 \\
1 & 3 & 0 & -1/2 & 0 & 9/2 \\
0 & 1 & 0 & -1/4 & -1/4 & 1/2 \\
\hline
0 & 1 & 0 & 1/2 & 0 & -9/2 \\
\hline
\end{array}
$$
$\uparrow$

4° After subtractions:

$$
\begin{array}{|ccccc|c|}
\hline
0 & 0 & -1 & 1/4 & -3/4 & -3/2 \\
1 & 0 & 0 & 1/4 & 3/4 & 3 \\
0 & 1 & 0 & -1/4 & -1/4 & 1/2 \\
\hline
0 & 0 & 0 & 3/4 & 1/4 & -5 \\
\hline
\end{array}
\qquad
\begin{array}{cc}
NB & NF \\
\begin{array}{c|c} 1 & 3 \\ 2 & 1 \\ 3 & 2 \end{array}
&
\begin{array}{c|c} 1 & 4 \\ 2 & 5 \end{array}
\end{array}
$$

$$BS = (3, 1/2, 3/2, 0, 0) = BOS, \quad 5 = z = z_{\min}$$

Step 3.

1° All $a'[i, NB(i)] * b'(i) \geq 0, \quad \Rightarrow$ the end

Remark 4.2.2. *Artificial variables with the artificial objective function, as it is in Section 3.6, and the dual simplex method are at first glance to be incompatible. Indeed, all basic solutions will be feasible if artificial variables are introduced for all constraints of "$\geq$" and "$=$" types. As a result, applicability conditions of the dual simplex method vanish because this method works in the region where $BS \neq BFS$ (for more details, see Section 4.3).*

Let us illustrate the last remark by the following problem where the constraint of "$=$" type is introduced not with the artificial variable x_6 (together with the artificial objective function $w = x_6$) but with the replacement of "$=$" by "$\leq$" and "$\geq$" simultaneously.

Example 4.2.2. For $x_1, x_2 \geq 0$ solve the problem

$$
\begin{array}{rcrclcl}
2x_1 & + & 3x_2 & \geq & 6 \\
2x_1 & + & 6x_2 & \geq & 9 \\
x_1 & + & x_2 & \leq & 4 \\
2x_2 & + & 2x_3 & = & 7 \\
x_1 & + & 2x_2 & = & z & \rightarrow & \min
\end{array}
$$

Step 0.

$$
\rightarrow
\begin{array}{|ccccccc|c|}
\hline
2 & 3 & -1 & 0 & 0 & 0 & 0 & 6 \\
2 & 6 & 0 & -1 & 0 & 0 & 0 & 9 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 4 \\
2 & 2 & 0 & 0 & 0 & 1 & 0 & 7 \\
2 & 2 & 0 & 0 & 0 & 0 & -1 & 7 \\
\hline
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
\qquad
\begin{array}{cc}
NB & NF \\
\begin{array}{c|c} 1 & 3 \\ 2 & 4 \\ 3 & 5 \\ 4 & 6 \\ 5 & 7 \end{array}
&
\begin{array}{c|c} 1 & 1 \\ 2 & 2 \end{array}
\end{array}
$$

$$BS = (0, 0, -6, -9, 4, 7, -7) \neq BFS, \quad 0 = z$$

It is permitted to apply the dual simplex method.

Step 1.

1° $a'[1, NB(1)] * b'(1) = -6 < 0$
$a'[2, NB(2)] * b'(2) = -9 < 0$
$a'[5, NB(5)] * b'(5) = -7 < 0, \qquad \Rightarrow \quad k = 2$

2° $a'[2, NB(2)] = -1.$ There are two positive coefficients:
$a'[2, NF(1)] = 2, \quad a'[2, NF(2)] = 6$

$$\min\left(\frac{1}{\mid 2 \mid}, \frac{2}{\mid 6 \mid}\right) = \frac{1}{3} \quad \Rightarrow l = 2, \; s = 2$$

3° After normalization:

$$\rightarrow \begin{array}{|ccccccc|c|}
\hline
2 & 3 & -1 & 0 & 0 & 0 & 0 & 6 \\
1/3 & 1 & 0 & -1/6 & 0 & 0 & 0 & 3/2 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 4 \\
2 & 2 & 0 & 0 & 0 & 1 & 0 & 7 \\
2 & 2 & 0 & 0 & 0 & 0 & -1 & 7 \\
\hline
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}$$
$\uparrow$

4° After subtractions:

								NB		NF	
1	0	−1	1/2	0	0	0	3/2	1	3	1	1
1/3	1	0	−1/6	0	0	0	3/2	2	2	2	4
2/3	0	0	1/6	1	0	0	5/2	3	5		
4/3	0	0	1/3	0	1	0	4	4	6		
4/3	0	0	1/3	0	0	−1	4	5	7		
1/3	0	0	1/3	0	0	0	−3				

$\rightarrow$ at row 5, $\uparrow$ at column 1

$$BS = (0, 3/2, -3/2, 0, 5/2, 4, -4) \neq BFS, \quad 3 = z$$

Step 2.

1° $a'[1, NB(1)] * b'(1) = -3/2 < 0$
$a'[5, NB(5)] * b'(5) = -4 < 0, \quad \Rightarrow k = 5$

2° $a'[5, NB(5)] = -1, \quad \& \quad a'[5, NF(1)] = 4/3 > 0, \quad \&$
$a'[5, NF(2)] = 1/3 > 0$

$$\min\left(\frac{1/3}{\mid 4/3 \mid}, \frac{1/3}{\mid 1/3 \mid}\right) = \frac{1}{4} \quad \Rightarrow l = 1, \; s = 1$$

3° After normalization:

$$
\begin{array}{ccccccc|c}
1 & 0 & -1 & 1/2 & 0 & 0 & 0 & 3/2 \\
1/3 & 1 & 0 & -1/6 & 0 & 0 & 0 & 3/2 \\
2/3 & 0 & 0 & 1/6 & 1 & 0 & 0 & 5/2 \\
4/3 & 0 & 0 & 1/3 & 0 & 1 & 0 & 4 \\
\hline
1 & 0 & 0 & 1/4 & 0 & 0 & -3/4 & 3 \\
\hline
1/3 & 0 & 0 & 1/3 & 0 & 0 & 0 & -3 \\
\end{array}
$$
$\uparrow$

4° After subtractions:

$$
\begin{array}{ccccccc|c}
 & & & & & NB & & NF \\
\end{array}
$$

0	0	−1	1/4	0	0	3/4	−3/2		1	3		1	7
0	1	0	−1/4	0	0	1/4	1/2		2	2		2	4
0	0	0	0	1	0	1/2	1/2		3	5			
0	0	0	0	0	1	1	0		4	6			
1	0	0	1/4	0	0	−3/4	3		5	1			
0	0	0	1/4	0	0	1/4	−4						

$$BS = (3, 1/2, 3/2, 0, 1/2, 0, 0) = BOS, \quad -4 = z_{\min}$$

Step 3.

1° All $a'[i, NB(i)] * b(i) \geq 0, \quad \Rightarrow \quad$ the end

4.3.   Algorithm without correct form of a basis using artificial variables

In this section, we solve the same problem (cf. Example 4.2.2) but by another way. By this we show the compatibility of artificial variables and the dual simplex method.

Example 4.3.1. For $x_1, x_2 \geq 0$ solve the problem

$$
\begin{array}{rcrclcl}
2x_1 & + & 3x_2 & \geq & 6 \\
2x_1 & + & 6x_2 & \geq & 9 \\
x_1 & + & x_2 & \leq & 4 \\
2x_2 & + & 2x_3 & = & 7 \\
x_1 & + & 2x_2 & = & z & \to & \min
\end{array}
$$

Among constraints there is one of "=" type. Solely because of it we introduce an artificial variable, here x_6, and, correspondingly, the artificial objective function $w = x_6$.

Step 0.

2	3	−1	0	0	0	6
2	6	0	−1	0	0	9
1	1	0	0	1	0	4
2	2	0	0	0	1	7
1	2	0	0	0	0	0
0	0	0	0	0	1	0

Here, the last row is intended for coefficients and the value of function w. So far as at the first stage we need to minimize function w, it is necessary to make a preparatory subtraction of the fourth row from the last row. By doing so we enable variables (x_3, x_4, x_5, x_6) to become basic variables in the canonical form with respect to w. As a result, we obtain the initial state:

$$NB \qquad NF$$

2	3	−1	0	0	0	6		3		1
2	6	0	−1	0	0	9		4		2
1	1	0	0	1	0	4		5		
2	2	0	0	0	1	7		6		
1	2	0	0	0	0	0				
−2	−2	0	0	0	0	−7				

$$BS = (0, 0, -6, -9, 4, 7) \neq BFS, \quad 0 = z, \ 7 = w$$

Now, by the last row of the tableau we recognize that minimization of w (Stage I) must be done by means of the conventional (non-dual) simplex method without generation of BFS (see Section 3.5).

Stage I. Step 1.

1° $l = 1, \ s = 1$

2° $\min \left(\infty, \infty, \frac{4}{1}, \frac{7}{2} \right) = 7/2 \quad \Rightarrow \quad k = 4$

3° After normalization:

2	3	−1	0	0	0	6
2	6	0	−1	0	0	9
1	1	0	0	1	0	4
1	1	0	0	0	1/2	7/2
1	2	0	0	0	0	0
−2	−2	0	0	0	0	−7

4° After subtractions:

$$
\begin{array}{|cccccc|c|}
\hline
0 & 1 & -1 & 0 & 0 & -1 & -1 \\
0 & 4 & 0 & -1 & 0 & -1 & 2 \\
0 & 0 & 0 & 0 & 1 & -1/2 & 1/2 \\
1 & 1 & 0 & 0 & 0 & 1/2 & 7/2 \\
\hline
0 & 1 & 0 & 0 & 0 & -1/2 & -7/2 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
\end{array}
$$

(with $\rightarrow$ at the second row, and $\uparrow$ under the first column)

NB: 3, 4, 5, 1 NF: $_1\,6$, $_2\,2$

$$BS = (7/2, 0, 1, -2, 1/2, 0) \neq BFS, \quad 7/2 = z, \ 0 = w$$

Stage I has been successfully completed, thus the artificial objective function (the last row) and artificial variable x_6 (the sixth column) are of no further need. Yet, BS has not been feasible. Further, by the next to the last row we recognize that at Stage II the dual simplex method must be applied to minimize z (cf. Remark 4.2.1 in Section 4.2).

Stage II. Step 2.

$\mathbf{1°}$ $a'[2, NB(2)] * b'(2) = -2 < 0, \quad \Rightarrow \quad k = 2$

$\mathbf{2°}$ $a'[2, NB(2)] = -1$, and there is one positive coefficient,
$a'[2, NF(2)] = 4$

$$\min_j \left(\frac{c'_j}{|\,a'_{kj}\,|} \right) = \min \left(\frac{1}{|\,4\,|} \right) = \frac{1}{4} \quad \Rightarrow \quad l = 2, \ s = 2$$

$\mathbf{3°}$ After normalization:

$$
\begin{array}{|ccccc|c|}
\hline
0 & 1 & -1 & 0 & 0 & -1 \\
0 & 1 & 0 & -1/4 & 0 & 1/2 \\
0 & 0 & 0 & 0 & 1 & 1/2 \\
1 & 1 & 0 & 0 & 0 & 7/2 \\
\hline
0 & 1 & 0 & 0 & 0 & -7/2 \\
\hline
\end{array}
$$

(with $\rightarrow$ at the second row, and $\uparrow$ under the second column)

$\mathbf{4°}$ After subtractions:

$$
\begin{array}{|ccccc|c|}
\hline
0 & 0 & -1 & 1/4 & 0 & -3/2 \\
0 & 1 & 0 & -1/4 & 0 & 1/2 \\
0 & 0 & 0 & 0 & 1 & 1/2 \\
1 & 0 & 0 & 1/4 & 0 & 3 \\
\hline
0 & 0 & 0 & 1/4 & 0 & -4 \\
\hline
\end{array}
$$

NB: 3, 2, 5, 1 NF: $_1\,6$, $_2\,4$

$$BS = (3, 1/2, 3/2, 0, 1/2) = BFS = BOS, \quad 4 = z_{\min}$$

Step 3.

$\mathbf{1°}$ All $a'[i, NB(i)] * b'(i) \geq 0, \quad \Rightarrow \quad$ the end

Chapter 5.

The Revised Simplex Method

This modification of the standard simplex method is sometimes called the improved simplex method. It takes this name from the fact that it reduces the number of operations to a necessary minimum. This minimum is almost evident: ($1°$) to determine what nonbasic variable to insert into the basis we only need to check $(n-m)$ coefficients for nonbasic variables in the objective function row; ($2°$) having decided to insert, say, a variable x_s, we only need to know m numbers in the column a'_s and m numbers in the right hand side b' in order to decide which basic variable to delete from the basis. Thus to select the pivot in the standard simplex method we need to have only $(n+m)$ numbers available.

The similar feature has the dual simplex method: ($1°$) to determine what basic variable to delete from the basis we only need m numbers in the right hand side b' while selecting index k for one of these numbers; ($2°$) having decided to delete, say, a variable $x_b, b = NB(k)$, we only need to know $(n-m)$ numbers in the k-th row and $(n-m)$ numbers in the objective function row of the tableau in order to decide which of $(n-m)$ nonbasic variables to insert into the basis. Thus to select the pivot in the dual simplex method we need to have only $2n-m$ numbers available.

Clearly, it would be more efficient if we could first compute just the numbers needed for Action $1°$ above and then perform this action, then compute only the numbers needed for Action $2°$ and then perform it. By doing so, we could minimize the amount of work needed to determine what variables to exchange in the basis. As for Actions $3°$ and $4°$ constituting Gauss-Jordan elimination, they could be avoided at all if we could exchange the variables so as to be able to repeat Actions $1°$ and $2°$ at the next step.

A way of implementing so modified Actions $1°$ and $2°$ without Actions $3°$ and $4°$ at each step is the *revised simplex method*. The basic idea of it consists in possibility to obtain each canonical form of a problem independently of others canonical forms directly from the original standard form of the problem. To implement this idea we need: (1) to keep the original standard form of the

problem original standard form of the problem unchanged over the whole course of solution; (2) to use the so-called *simplex-multipliers* π being the coefficients for direct transition from the original standard form to the current canonical form for a basis; and (3) to use the so-called *inverse basis B^{-1}* being $m \times m$-matrix which makes it possible to obtain all the numbers needed for Actions 1° and 2° above.

5.1. The simplex multipliers

Given the original form of a standard LP problem:

$$Ax = b, \qquad x \in R^m, \qquad b \in R^m, \qquad m < n$$

$$rank(A) = m, \qquad x \geq 0, \qquad c^T x = z \to \min$$

The current step of simplex calculations is determined by choosing a set of m elements from vector x to become the basic variables, x^B, while the rest $n - m$ elements appear to be the nonbasic (free) variables, x^F. Let such a choice be made, so that we have $x = (x^B, x^F)$.

Remark 5.1.1. *The assumption that it is precisely the first m elements of x that formed x^B, makes no loss of generality and is taken only as a matter of convenience in presenting the material.*

A choice of x^B means that the corresponding m columns of matrix A are selected to form a linearly independent set of vectors, viz. a basis in R^m. This set of columns, which are ordered according to how elements of x appear in x^B, is called the "current basis" and considered as a matrix B. It is quite clear that elements of c, the objective function coefficients, turn out divided into two groups, too: c_B for x^B and c_F for x^F.

Under the above assumption, the original form of LP problem looks as follows:

$$\begin{bmatrix} B & | & R \\ -- & | & -- \\ c_B^T & | & c_F^T \end{bmatrix} \begin{bmatrix} x^B \\ --- \\ x^F \end{bmatrix} = \begin{bmatrix} b \\ --- \\ z \to \min \end{bmatrix} \tag{5.1}$$

where matrix R collects the columns used as factors by nonbasic variables in the system of constraints:

$$Ax = Bx^B + Rx^F = b \tag{5.2}$$

The current step of the simplex method requires the canonical form of problem for the chosen basis, instead of the original form. As it is described in Section 3.1, this can

be done in three stages: (1) premultiplying the constraints (5.2) by the inverse basis B^{-1} in order to have the constraints in the form

$$\begin{bmatrix} I & | & A' \end{bmatrix} \begin{bmatrix} x^B \\ -\,-\,- \\ x^F \end{bmatrix} = b', \quad A' = B^{-1}R, \quad b' = B^{-1}b$$

(2) premultiplying the last equation by c_B^T to obtain

$$\begin{bmatrix} c_B^T & | & c_B^T A' \end{bmatrix} \begin{bmatrix} x^B \\ -\,-\,- \\ x^F \end{bmatrix} = c_B^T b'$$

and (3) subtracting this equation from the last row of (5.1) to represent the objective function in the form

$$\begin{bmatrix} O^T & | & c'^T \end{bmatrix} \begin{bmatrix} x^B \\ -\,-\,- \\ x^F \end{bmatrix} = z - z_0 \to \min, \quad c'^T = c_F^T - c_B^T A' \tag{5.3}$$

in order, by doing so, to clarify whether it is possible or not to minimize further z (by increasing one of nonbasic variables chosen from x^F) with reference to the value $z = z_0$, $z_0 = c_B^T b'$, which has yet been gained at the current step, viz. at the current basic solution $BS = (x^B = b', x^F = 0)$.

However, the same result can be reached by outwardly different, but entirely identical, means: premultiplying (5.2) by a matrix-row of coefficients $\pi^T = [\pi_1, \pi_2, \ldots, \pi_m]$ and then adding the product to the last row of (5.1) to obtain

$$\begin{bmatrix} (\pi^T B + c_B^T) & | & (\pi^T R + c_F^T) \end{bmatrix} \begin{bmatrix} x^B \\ -\,-\,- \\ x^F \end{bmatrix} = \pi^T b + z \to \min$$

where the condition $\pi^T B + c_B^T = 0^T$ is imposed. Coefficients π determined from this condition, $\pi^T = -c_B^T B^{-1}$, are refered to as the **simplex multipliers**. They serve the same purpose: to represent the objective function in the canonical form (5.3). Therefore, $c'^T = \pi^T R + c_F^T$, $z_0 = -\pi^T b$.

Both expressions $\pi^T B + c_B^T$ and $\pi^T R + c_F^T$ determine by themselves the updated coefficient values for x^B and x^F in the objective function. These expressions are absolutely of the same updating rule, namely: to update the i-th coefficient (for variable x_i) it is necessary to: (1) take a_i, the i-th column of matrix A describing the *original* system of constraints $Ax = b$; (2) calculate the scalar (dot) product $(\pi, a_i) = \pi^T a_i$; and (3) add (π, a_i) to c_i, the i-th coefficient which was given for x_i in the *original* expression $c^T x = z$ for the objective function z. Obviously, to operate like this one needs to have the simplex-multipliers π.

They are obtained as follows. (1) Take the next, in turn, linearly independent set of m columns of matrix A arranged in matrix B according to their original index numbers

$j_1, j_2, \ldots j_m$, i.e., so that $b_i = a_{j_i}, i = 1, 2, \ldots, m$. (Note that the choice of these columns to be included in B means the choice of variables $x_{j_i}, i = 1, 2, \ldots, m$, to be included in the set of basic variables x^B). (2) Premultiply B by the row-matrix π^T, (3) add the product to coefficients $c_B^T = (c_{j_1}, c_{j_2}, \ldots c_{j_m})$ given for the corresponding variables in the original objective function form $c^T x = z$, and (4) require that the sum be a zero row-matrix. This is just the requirement having the form of linear equations $\pi^T B = -c_B^T$ from which the sought-for simplex multipliers π^T can be found.

Every time they are unique because $rank B = m$, and every time they are found, as well as the updated coefficients $c'^T = \pi^T R + c_F^T$ for nonbasic variables x^F, on the basis of the **original** columns of matrix A and of the **original** coefficients c_B^T and c_F^T, which describe correspondingly, the constraints $Ax = b$ and the objective function $c^T x = c_B^T x^B + c_F^T x^F = z$ in the standard LP problem statement.

One case of the above coefficients updating deserves to be emphasized. If column a_j of the *original* matrix A is precisely a unit vector e_q, where e_q is defined to be the $m \times 1$ column vector whose q-th element is unity and all other elements are zero, and if the corresponding coefficient c_j is zero, then: (1) while including the column a_j into basis B, we have $\pi_q = 0$; (2) while including the column a_j into non-basis matrix R, we have $c'_q = \pi_q$. Thus, any computer memory location intended to hold the given and then the updated objective function coefficient c'_j for that variable x_j which originally appears in the problem with a unit vector of constraints e_q and with the zero value of the objective function coefficient $(c_j = 0)$, in actuality holds nothing but the corresponding simplex multiplier π_q, with $\pi_q = 0$ if the corresponding variable x_j has currently turned out a variable included into the basis set x^B.

5.2. The inverse basis

As mentioned in Sections 3.3 and 4.1, the simplex method comprises the Gauss-Jordan elimination, the same as in linear algebraic equations solution, which is preceded by the special pivoting actions 1° and 2°. In the standard simplex method (*cf.* Section 3.3), first is found the pivot column (action 1°) and then the pivot row (action 2°), while in the dual simplex method (*cf.* Section 4.1) the pivoting order is reverse: first is found the pivot row (action 1°) and then the pivot column (action 2°). For Gauss-Jordan elimination, the pivoting order does not matter. Substantial is that the pivot element has been determined in order to execute one in turn elimination step. From the point of view of solving equations, every such step implies elimination of the pivot variable from all equations excepting the pivot equation where it remains with coefficient 1.

Let us consider the Gauss-Jordan elimination at the current step t from the viewpoint of basis inversion in the simplex method. We make use of elementary matrices of two types. The first one is a diagonal $m \times m$ elementary matrix D_k whose all the main diagonal elements are units except the k-th diagonal element, which must not be zero. If one needs to normalize the k-th pivot row of the *current matrix A_t* through division this row by the current pivot element a'_{ks}, then the result can be written shortly as $D_{ks}^{-1} A_t$

where D_{ks} denotes matrix D_k with its k-th diagonal element being a'_{ks} (taken from the s-th column of A_t). By doing so, variable x_s occurs to be prepared for elimination from all the rest equations and, at the same time, for entering the basis set x^B in place of variable $x_r, r = NB(k)$, which departs x^B in order to enter x^F.

The second elementary matrix is the total column $m \times m$ matrix T_k^c whose non-zero elements are located only at the main diagonal and in the k-th column, all the main diagonal elements being units. If the k-th column of T_k^c is formed so that its k-th (diagonal) element is unit and other elements are taken correspondingly from the s-th column of A_t, then such a matrix will be denoted as T_{ks}^c. The inverse matrix $(T_{ks}^c)^{-1}$ equals T_{ks}^c with the sign changed for all the non-diagonal elements of the k-th column. With these two matrices, it is convenient to express the result of elimination by the Gauss-Jordan method at step t as the product $(T_{ks}^c)^{-1} D_{ks}^{-1} A_t$.

When solving an ordinary set of algebraic equations, $Ax = b$ with the $m \times m$ matrix A, numbers k and s coincide and run all their values in the natural order: $k = s = 1, 2, \ldots, m$. Then, after m steps of elimination we have

$$(T_{mm}^c)^{-1} D_{mm}^{-1} \ldots (T_{22}^c)^{-1} D_{22}^{-1} (T_{11}^c)^{-1} D_{11}^{-1} A = I$$

hence

$$A^{-1} = (T_{mm}^c)^{-1} D_{mm}^{-1} \ldots (T_{22}^c)^{-1} D_{22}^{-1} (T_{11}^c)^{-1} D_{11}^{-1} I$$

This reflects the well-known fact that the Gauss-Jordan elimination, when applied to identity matrix I, yields the inverse matrix A^{-1}. This procedure, in its recurrent form, can be written as

$$A_t^{-1} = (T_{tt}^c)^{-1} D_{tt}^{-1} A_{t-1}^{-1}; \quad A_0^{-1} = I, \quad t = 1, 2, \ldots, m, \tag{5.4}$$

where t is the procedure step number and $A_m^{-1} = A^{-1}$.

A feature of the simplex method is that we need to reformulate the original LP problem in terms of equalities in order to be able to seek a basic solution to the underdetermined set of m equations $Ax = b$ by the Gauss-Jordan elimination. We obtain this set by introducing one new (*slack*) variable in each inequality and one new (*artificial*) variable in each equality given in the original set of constraints. After that the total number of variables becomes $n, n > m$. In any case, we can assume that all such variables have been introduced with coefficient 1 (as described in Sections 3.3, 3.4 and 3.6). Thus, $m \times n$ matrix A of constraints can be considered as consisting of two blocks, $A = [\,G \mid I\,]$ in the *initial state*, where G denotes the left-hand block and I is $m \times m$ identity matrix corresponding to the introduced variables.

Analogously to (5.4), the Gauss-Jordan procedure, existing in the simplex method, can be written as follows

$$B_t^{-1} = (T_{ks}^c)^{-1} D_{ks}^{-1} B_{t-1}^{-1}, \quad B_0^{-1} = I, \quad t = 1, 2, \ldots, N$$

when considered in application to the right hand side I of matrix A, where t is a simplex method step number; B_t is a basis at step t; B_t^{-1} is the inverse basis at step t; B_0^{-1} is

the inverse basis at step 0 (in the initial state); and N is the total number of steps of the simplex procedure. In the last expression, numbers k and s correspond to the pivot row and pivot column of the current matrix A_t. They are determined by the rules of the standard or dual simplex method at each t, so it would be well to mark them by subscript t (we omit this subscript only for simplicity).

Therefore, a matrix corresponding to the right hand block I of matrix A in the initial state, which has its own column numbering $k = 1, 2, \ldots, m$, becomes equal to the inverse basis B_t^{-1} as the result of the t-step Gauss-Jordan procedure. The difference between B_t^{-1} and B_{t-1}^{-1} is determined by the following: basis B_t is the preceding basis B_{t-1} with the k-th column, $k \in \{1, 2, \ldots, m\}$, replaced by the s-th column, $s \in \{1, 2, \ldots, n\}$, of the whole original matrix A. Namely such is the role of factors $(T_{ks}^c)^{-1} D_{ks}^{-1}$ in the above expression for the current inverse basis B_t^{-1} via the preceding inverse basis B_{t-1}^{-1}.

Of course, in reality no multiplication of B_{t-1}^{-1} by these factors is done to obtain B_t^{-1}. Obviously, affecting B_{t-1}^{-1} by D_{ks}^{-1} and then by $(T_{ks}^c)^{-1}$ reduces to the following.

Let elements of B^{-1} be denoted via β_{ij}, $i, j = 1, 2, \ldots, m$, i.e.

$$
B^{-1} = \left[\begin{array}{ccc} \beta_{11} & \cdots & \beta_{1m} \\ \cdots & \cdots & \cdots \\ \beta_{m1} & \cdots & \beta_{mm} \end{array} \right]
$$

At $t = 0$ we have $B = I$, the right hand block of the original matrix A. Before step t, $t = 1, 2, \ldots, N$, we have $B = B_{t-1}^{-1}$. After step t, $t = 1, 2, \ldots, N$, matrix B^{-1} must contain B_t^{-1}. Conversion from B_{t-1}^{-1} to B_t^{-1}, that is the inverse basis update, is effected by two actions:

(1) The action of matrix D_{ks}^{-1}:

$$
\text{for} \quad j = 1, 2, \ldots, m \quad \text{compute} \quad \beta_{kj}^+ = \beta_{kj}/a'_{ks} \tag{5.5}
$$

(2) The action of matrix $(T_{ks}^c)^{-1}$:

$$
\text{for} \quad i = 1, 2, \ldots, m, \ i \neq k \quad \text{and} \quad j = 1, 2, \ldots, m
$$

$$
\text{compute} \quad \beta_{ij}^+ = \beta_{ij} - a'_{is}\beta_{kj}^+ \tag{5.6}
$$

Here and below the sign "+" is used to denote the updated values; a'_{ks} is the pivot element, and a'_{is} are any other elements of the pivot column a'_s.

The pivot column is determined by

$$
a'_s = B^{-1}a_s
$$

where $B^{-1} = B_t^{-1}$. This formula is part of the more general expression $A' = B^{-1}R$ in the description of any step of the simplex method in Section 5.1.

5.3. Simplex multipliers sequential update

In Section 5.1, simplex multipliers are defined by expression $\pi^T = -c_B^T B^{-1}$ where the original columns of matrix A are used to form B^{-1} and the original coefficients of the objective function $z = c^T x$ are used to form c_B^T, the coefficient set for x^B, the set of basic variables, at each step of the simplex method. At the next step, the belonging to which is marked by "+", we have

$$(\pi^+)^T = -(c_B^+)^T (B^+)^{-1}$$

From this, the j-th element of the row $(\pi^+)^T$ equals

$$\pi_j^+ \overset{(1)}{=} -\sum_{i=1}^{m} c_{NB^+(i)} \beta_{ij}^+ \overset{(2)}{=} -\sum_{i=1,(i\neq k)}^{m} c_{NB^+(i)} \beta_{ij}^+ - c_{NB^+(k)} \beta_{kj}^+$$

$$\overset{(3)}{=} -\sum_{i=1,(i\neq k)}^{m} c_{NB^+(i)} \left(\beta_{ij} - a_{is}' \beta_{kj}^+\right) - c_{NB^+(k)} \beta_{kj}^+$$

$$\overset{(4)}{=} -\sum_{i=1}^{m} c_{NB(i)} \left(\beta_{ij} - a_{is}' \beta_{kj}^+\right) - c_{NB^+(k)} \beta_{kj}^+$$

$$\overset{(5)}{=} -\sum_{i=1}^{m} c_{NB^+(i)} \beta_{ij} + \sum_{i=1}^{m} c_{NB(i)} a_{is}' \beta_{kj}^+ - c_{NB^+(k)} \beta_{kj}^+$$

$$\overset{(6)}{=} \pi_j - \beta_{kj}^+ \left(c_s - \sum_{i=1}^{m} c_{NB(i)} a_{is}'\right) \overset{(7)}{=} \pi_i - \beta_{kj}^+ c_s'$$

In these transformations, the first and the second equalities are evident, the third is obtained from (5.6) and also it takes into account that only one variable (with number $NB(k)$) is departing the basis when changing from $NB(i)$ to $NB^+(i), i = 1, 2, \ldots, m$. The fourth equality is true because of (5.5). The sixth and seventh equalities are obtained from two facts: $NB^+(k) = s$, s is the number of the entering variable, and

$$c_s' = c_s - \sum_{i=1}^{m} c_{NB(i)} a_{is}'$$

which counts the s-th element of the general formula $c'^T = c_F^T - c_B^T A'$ given in Section 5.1 to update those coefficients that were in set c_F before the next step marked here by "+". Indeed, in the k-th position of row vector c_{B+}^T, in the capacity of c_k^+, now is c_s, $s = NB^+(k)$. This is the original coefficient for that variable x_s which has entered the set x^B at step "+" and which was in set x^F before the step and then was selected as the pivot to be included in x^B in exchange for variable x_r, $r = NB(k)$. Hence,

$$\pi_j^+ = \pi_j - \beta_{kj}^+ c_s'$$

is the sequential updating rule for the simplex-multipliers, $j = 1, 2, \ldots, m$.

5.4. Algorithm of the revised simplex method

Let us summarize material of Sections 5.1 to 5.3 as applied to the conventional (non-dual) simplex method. The initial form of standard LP problem

$$
\begin{bmatrix}
G & | & I \\
-- & | & -- \\
c^T & | & O^T
\end{bmatrix}
x =
\begin{bmatrix}
b \\
-- \\
z
\end{bmatrix}, \qquad b \geq 0
$$

implies that I is $m \times m$ identity matrix for basic variables x^B, x^B having not appeared in the objective function $c^T x^F = z \rightarrow \min$. Usually, these x^B are slack variables. Notation G stands for the rest columns of matrix $A = [\, G \mid I \,]$ in the constraints $Ax = b$; these are columns for nonbasic variables x^F within vector $x = (x^F, x^B)$ of variables; b is the right hand side of constraints. From the initial form of the problem, the following data have to be stored for the algorithm: A, b, c^T. The sub-row O^T will be used to store the simplex miltipliers π^T. To store the inverse basis B^{-1} we need the additional $m \times m$ matrix.

Before each step t of the algorithm, $t = 1, 2, \ldots, N$, there are known the current values π^T, B^{-1}, b' and z; these values are to be updated. Before the first step they are: $\pi^T = O^T$, $B^{-1} = I$, $b' = b$, $z = 0$. From the initial data the following values are to be used: G and c^T to obtain the current coefficients $c'^T = \pi^T B + c^T$; and matrix A to find the current pivot column $a'_s = B^{-1} a_s$ where a_s is the s-th column of the initial matrix A.

Step t.

1° Obtain $c'^T = \pi^T B + c^T$. Look through $NF(l)$ at the compound row (c'^T, π^T) as at a whole, corresponding to the bottom row of the initial form of the problem, in order to find if it contains negative elements or not. If "not", then the solution has been found; the end. If "yes", then fix the number l and, correspondingly, number $s = NB(l)$ of the nonbasic variable x_s which has the most negative coefficient c'_s and which will be introduced into basis at this step.

2° Obtain $a'_s = B^{-1} a_s$. Compare all the corresponding elements of columns b' and a'_s so that to find number k such that

$$
\frac{b'}{a'_{ks}} = \min_{i, a'_{is} > 0} \left(\frac{b'_i}{a'_{is}} \right)
$$

By doing so, it has been found number $r = NB(k)$ of the variable x_r which will be excluded from basis. Besides, there has been found the pivot a'_{ks} among elements of column a'_s.

3° Update all the values needed for the next step:

(a) pointers: $NB(k) \leftrightarrow NF(l)$.

(b) elements of the basic values column:

$$
b_k^+ = b'_k / a'_{ks}; \quad b_i^+ = b'_i - a'_{is} b_k^+ \quad (i \neq k)
$$

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(c) objective function: $z^+ = z - c'_s b^+_k$

Remark: *In reality, notations z and z^+ stand here for the objective function with the sign changed, i.e. for $-z$. It must be kept in mind while displaying results.*

(d) inverse basis:

$$\beta^+_{kj} = \beta_{kj}/a'_{ks}; \quad \beta^+_{ij} = \beta_{ij} - a'_{is}\beta^+_{kj} \quad (i \neq k)$$

(e) simplex miltipliers: $\pi^{+T} = \pi^T - \beta^+_k c'_s$.

Remark 5.4.1.

1. *In the last expression β^+_k stands for the k-th row of the updated inverse basis B^{-1} composed of elements $\beta^+_{kj}, j = 1, 2, \ldots, m$.*

2. *The updated values have been marked by superscript "+" in items (b) to (e) only for convenience sake. In computer programs, one should use the assignment statement. For example, instead item (c) we write $z := z - c'_s b_k$.*

Example 5.4.1. For $x_1, x_2, x_3 \geq 0$ solve the problem

$$
\begin{array}{rrrrrrl}
x_1 & + & 2x_2 & + & 5x_3 & \leq & 45 \\
2x_1 & + & 3x_2 & + & 3x_3 & \leq & 60 \\
x_1 & + & x_2 & + & 2x_3 & \leq & 27 \\
-9x_1 & - & 10x_2 & - & 15x_3 & = & z \quad \to \quad \min
\end{array}
$$

Step 0. (The initial form of the problem).

	A					b	NB		NF	
1	2	5	1	0	0	45	1 | 4		1 | 1	
2	3	3	0	1	0	60	2 | 5		2 | 2	
1	1	2	0	0	1	27	3 | 6		3 | 3	
−9	−10	−15	0	0	0	0				

$$BS = (0, 0, 0, 45, 60, 27) = BFS, \quad 0 = z$$

Step 1.

1° $c'^T = \pi^T G + c^T$; $c'^T = c^T = (-9, -10, -15)$. In view of the fact that in the initial state $\pi^T = (0, 0, 0)$, this operation can be omitted considering that $c'^T = c^T$.

$$(c'^T, \pi^T) = (-9, -10, -15, 0, 0, 0) \Rightarrow l = 3, \ s = 3, \ c'_s = -15$$

2° $a'_s = B^{-1}a_s$; $a'_s = a_s = a_3 = \boxed{\begin{array}{c|c|c} \overset{1}{3} & \overset{2}{5} & \overset{3}{2} \end{array}}$.   Since $B^{-1} = I$, in the initial state, this operation can also be omitted at step 1 considering $a'_s = a_s$.

$$\min\left(\frac{45}{5}, \frac{60}{3}, \frac{27}{2}\right) = \frac{45}{5} = 9 \quad \Rightarrow \quad k = 1, \quad a'_{ks} = 5$$

3° **(a)** $NB = \boxed{\begin{array}{c|c|c} \overset{1}{3} & \overset{2}{5} & \overset{3}{6} \end{array}}$, $NF = \boxed{\begin{array}{c|c|c} \overset{1}{1} & \overset{2}{2} & \overset{3}{4} \end{array}}$

(b) $b' = \boxed{\begin{array}{c|c|c} \overset{1}{9} & \overset{2}{33} & \overset{3}{9} \end{array}}$

(c) $-z = 135$

(d) $B^{-1} = \begin{array}{|ccc|} \hline 1/5 & 0 & 0 \\ -3/5 & 1 & 0 \\ -2/5 & 0 & 1 \\ \hline \end{array}$

(e) $\pi^T = \boxed{\begin{array}{c|c|c} 3 & 0 & 0 \end{array}} = \boxed{\begin{array}{c|c|c} 0 & 0 & 0 \end{array}} - \boxed{\begin{array}{c|c|c} 1/5 & 0 & 0 \end{array}} * \boxed{-15}$

Step 2.

$$\overset{c'^T}{\underset{\underset{\uparrow 1 \quad\; 2 \quad\; 3}{}}{\boxed{\begin{array}{c|c|c} -6 & -4 & 0 \end{array}}}} = \overset{\pi^T}{\underset{4 \quad\; 5 \quad\; 6}{\boxed{\begin{array}{c|c|c} 3 & 0 & 0 \end{array}}}} * \overset{G}{\begin{array}{|ccc|} \hline 1 & 2 & 5 \\ 2 & 3 & 3 \\ 1 & 1 & 2 \\ \hline \end{array}} + \overset{c^T}{\boxed{\begin{array}{c|c|c} -9 & -10 & -15 \end{array}}}$$

$l = 1,\quad s = 1,\quad c'_s = -6$

2° $a'_s = \begin{array}{|c|} \hline 1/5 \\ 7/5 \\ 3/5 \\ \hline \end{array} = \begin{array}{|ccc|} \hline 1/5 & 0 & 0 \\ -3/5 & 1 & 0 \\ -2/5 & 0 & 1 \\ \hline \end{array} * \begin{array}{|c|} \hline 1 \\ 2 \\ 1 \\ \hline \end{array}$

$$\min\left(\frac{9}{1/5}, \frac{33}{7/5}, \frac{9}{3/5}\right) = \frac{9}{3/5} = 15 \quad \Rightarrow \quad k = 3, \quad a'_{ks} = 3/5$$

3° **(a)** $NB = \boxed{\begin{array}{c|c|c} \overset{1}{3} & \overset{2}{5} & \overset{3}{1} \end{array}}$, $NF = \boxed{\begin{array}{c|c|c} \overset{1}{6} & \overset{2}{2} & \overset{3}{4} \end{array}}$

(b) $b' = \boxed{\begin{array}{c|c|c} \overset{1}{6} & \overset{2}{12} & \overset{3}{15} \end{array}}$

(c) $-z = 225$

(d) $B^{-1} = \begin{array}{|ccc|} \hline 1/3 & 0 & -1/3 \\ 1/3 & 1 & -7/3 \\ -2/3 & 0 & 5/3 \\ \hline \end{array}$

(e) $\pi^T = \boxed{-1 \mid 0 \mid 10} = \boxed{3 \mid 0 \mid 0} - \boxed{-2/3 \mid 0 \mid 5/3} * \boxed{-6}$

Step 3.

$$G$$

$$
\mathbf{1°} \quad
\underset{\substack{1 \quad \uparrow2 \quad 3}}{\overset{c'^T}{\boxed{0 \mid -2 \mid 0}}}
=
\underset{\substack{4 \quad 5 \quad 6}}{\overset{\pi^T}{\boxed{-1 \mid 0 \mid 10}}}
*
\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 3 \\ 1 & 1 & 2 \end{bmatrix}
+
\overset{c^T}{\boxed{-9 \mid -10 \mid -15}}
$$

$$l = 2, \quad s = 2, \quad c'_s = -2$$

$$
\mathbf{2°} \quad a'_s =
\begin{bmatrix} 1/3 \\ 4/3 \\ 1/3 \end{bmatrix}
=
\begin{bmatrix} 1/3 & 0 & -1/3 \\ 1/3 & 1 & -7/3 \\ -2/3 & 0 & 5/3 \end{bmatrix}
*
\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}
$$

$$
\min\left(\frac{6}{1/3}, \frac{12}{4/3}, \frac{15}{1/3}\right) = \frac{12}{4/3} = 9 \quad \Rightarrow \quad k = 2, \quad a'_{ks} = 4/3
$$

$$
\mathbf{3°} \quad \textbf{(a)} \quad NB = \underset{\substack{1 \quad 2 \quad 3}}{\boxed{3 \mid 2 \mid 1}}, \qquad NF = \underset{\substack{1 \quad 2 \quad 3}}{\boxed{6 \mid 5 \mid 4}}
$$

$$
\textbf{(b)} \quad b' = \underset{\substack{1 \quad 2 \quad 3}}{\boxed{3 \mid 9 \mid 12}}
$$

(c) $-z = 243$

$$
\textbf{(d)} \quad B^{-1} =
\begin{bmatrix} 1/4 & -1/4 & 1/4 \\ 1/4 & 3/4 & -7/4 \\ -3/4 & -1/4 & 9/4 \end{bmatrix}
$$

(e) $\pi^T = \boxed{-1/2 \mid 3/2 \mid 13/2} = \boxed{-1 \mid 0 \mid 10} - \boxed{1/4 \mid 3/4 \mid -7/4} * \boxed{-2}$

Step 4.

$$G$$

$$
\mathbf{1°} \quad
\underset{\substack{1 \quad 2 \quad 3}}{\overset{c'^T}{\boxed{0 \mid 0 \mid 0}}}
=
\underset{\substack{\uparrow4 \quad 5 \quad 6}}{\overset{\pi^T}{\boxed{-1/2 \mid 3/2 \mid 13/2}}}
*
\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 3 \\ 1 & 1 & 2 \end{bmatrix}
+
\overset{c^T}{\boxed{-9 \mid -10 \mid -15}}
$$

$$l = 3, \quad s = 4, \quad c'_s = -1/2$$

$$
\mathbf{2°} \quad a'_s =
\begin{bmatrix} 1/4 \\ 1/4 \\ -3/4 \end{bmatrix}
=
\begin{bmatrix} 1/4 & -1/4 & 1/4 \\ 1/4 & 3/4 & -7/4 \\ -3/4 & -1/4 & 9/4 \end{bmatrix}
*
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\min\left(\frac{3}{1/4}, \frac{9}{1/4}, \infty\right) = \frac{3}{1/4} = 12 \quad \Rightarrow \quad k = 1, \quad a'_{ks} = 1/4
$$

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3° **(a)** $NB = \boxed{\begin{array}{c|c|c} \overset{1}{4} & \overset{2}{2} & \overset{3}{1} \end{array}}$, $\qquad NF = \boxed{\begin{array}{c|c|c} \overset{1}{6} & \overset{2}{5} & \overset{3}{3} \end{array}}$

(b) $b' = \boxed{\begin{array}{c|c|c} \overset{1}{12} & \overset{2}{6} & \overset{3}{21} \end{array}}$

(c) $-z = 249$

(d) $B^{-1} = \begin{array}{|rrr|} \hline 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 3 \\ \hline \end{array}$

(e) $\pi^T = \boxed{\begin{array}{c|c|c} 0 & 1 & 7 \end{array}} = \boxed{\begin{array}{c|c|c} -1/2 & 3/2 & 13/2 \end{array}} - \boxed{\begin{array}{c|c|c} 1 & -1 & 1 \end{array}} * \boxed{-1/2}$

Step 5.

$$
\mathbf{1°} \quad \overset{c'^T}{\boxed{\begin{array}{c|c|c} \underset{1}{0} & \underset{2}{0} & \underset{3}{2} \end{array}}} = \overset{\pi^T}{\boxed{\begin{array}{c|c|c} \underset{4}{0} & \underset{5}{1} & \underset{6}{7} \end{array}}} * \overset{G}{\begin{array}{|ccc|} \hline 1 & 2 & 5 \\ 2 & 3 & 3 \\ 1 & 1 & 2 \\ \hline \end{array}} + \overset{c^T}{\boxed{\begin{array}{c|c|c} -9 & -10 & -15 \end{array}}}
$$

There are no negative elements with the numbers from NF within the compound row (c'^T, π^T); the end. The basic solution, in this case the optimal one, can be read out from b' through NB subject to the condition that nonbasic variables with the numbers from NF are zero:

$$BOS = (21, 6, 0, 12, 0, 0); \quad z_{\min} = -249$$

Along the same lines, basic solutions can be read out at the completion of each step.

5.5.   Algorithm of the revised dual simplex method

Let us formulate the dual version of the revised simplex method building upon the main concepts of Sections 5.1 to 5.3. Outwardly, the problem has the same initial form as in Section 5.4:

$$
\left[\begin{array}{c|c} G & I \\ \hline c^T & O^T \end{array} \right] x = \left[\begin{array}{c} b \\ \hline z \end{array} \right], \quad c \ge 0
$$

The distinctive property allowing to launch the dual simplex method is that $c \ge 0$ and b may contain elements of different signs, whereas the property allowing to initiate the conventional simplex method looks quite the reverse: $b \ge 0$ and c may contain elements of different signs.

Before each step t, $t = 1, 2, \ldots, N$, there are known π^T, B^{-1}, b' and z. Conditions $\pi^T \ge 0$ and $c' \ge 0$ are evidences for further applying the algorithm. Before step 1 we

have: $b' = b$, $\pi^T = 0^T$, $B^{-1} = I$, $c' = c$, $z = 0$. From the initial data we use the following values: matrix G to find the left part of the k-th (the pivot) row $a_k'^T = \beta_k^T G$ where β_k^T is the k-th row of B^{-1}; matrix G to obtain the updated coefficients $c'^T = \pi^T G + c^T$ where c^T are the initial coefficients; the whole matrix $A = [\, G \mid I \,]$ to find the updated s-th (the pivot) column $a_s' = B^{-1} a_s$ where a_s is the s-th column of the initial matrix A.

Step t.

1° Look at column b' to find negative elements. If there is not any, the solution has been found; the end. If any, determine munber k of the most negative element of b'.

2° Obtain $a_k'^T = \beta_k^T G$. Look through $NF(l)$ at the k-th compound row $(a_k'^T, \beta_k^T)$ as at a whole in order to select only coefficients for nonbasic variables from the k-th constraint. Among them, select only negative coefficients $a_{kj}' < 0$, in order to fix the ordinal number l and the index number $s = NF(l)$ of the pivot column from the condition

$$\min_j \left(\frac{(c'^T, \pi^T)_j}{|\, a_{kj}' \,|} \right) = \frac{(c'^T, \pi^T)_s}{|\, a_{ks}' \,|}$$

Also, find $c_s' = (c'^T, \pi^T)_s$.
Remark: $(c'^T, \pi^T)_j$ denotes the j-th element of the compound row (c'^T, π^T) taken as a whole.

3° Obtain the pivot column $a_s' = B^{-1} a_s$ and the pivot element a_{ks}' in it.

4° Update all the values needed for the next step:

(a) pointers: $NB(k) \leftrightarrow NF(l)$.

(b) elements of the basic values column:

$$b_k^+ = b_k'/a_{ks}'; \quad b_i^+ = b_i' - a_{is}' b_k^+ \quad (i \neq k)$$

(c) objective function: $z^+ = z - c_s' b_k^+$.
Remark: *It must be kept in mind that here notations z and z^+ stand for the objective function with the sign changed.*

(d) inverse basis: $\beta_{kj}^+ = \beta_{kj}/a_{ks}'; \quad \beta_{ij}^+ = \beta_{ij} - a_{is}' \beta_{kj}^+ \quad (i \neq k)$.

(e) simplex multipliers: $\pi^{+T} = \pi^T - \beta_k^T c_s'; \quad \pi := \pi^+$.

(f) coefficients $c'^T = \pi^T G + c^T$.

Remark 5.5.1.

1. *Superscript "+" is used in items (b) to (e) only for simplicity; in computer programs the assignment statment should be used instead, for example, $b_k' := b_k'/a_{ks}'$, $z := z - c_s' b_k^+$ and so on.*

2. Item (c) does compute the objective function with the sign changed, i.e. $-z$.

By way of illustration let us consider the same problems as in Section 4.2.

Example 5.5.1. Find $x_1, x_2 \geq 0$ subject to

$$
\begin{array}{rcrcl}
2x_1 & + & 3x_2 & \geq & 6 \\
2x_1 & + & 6x_2 & \geq & 9 \\
2x_1 & + & 2x_2 & \leq & 7 \\
x_1 & + & 4x_2 & = & z \quad \rightarrow \quad \min
\end{array}
$$

Step 0. (The initial form of the problem).

		A			b	NB		NF	
-2	-3	1	0	0	-6	1	3	1	1
-2	-6	0	1	0	-9	2	4	2	2
2	2	0	0	1	7	3	5		
1	4	0	0	0	0				

$$ BS = (0, 0, -6, -9, 7) \neq BFS, \quad 0 = z $$

Step 1.

$$\mathbf{1^\circ} \quad b'^T = \begin{array}{|c|c|c|} \hline -6 & -9 & 7 \\ \hline \end{array} \quad \Rightarrow \quad k = 2$$

(with column labels 1, 2, 3)

$$\mathbf{2^\circ} \quad \underset{1 \quad 2}{\underbrace{\begin{array}{|c|c|} \hline -2 & -6 \\ \hline \end{array}}_{a_k'^T}} = \underset{3 \quad 4 \quad 5}{\underbrace{\begin{array}{|c|c|c|} \hline 0 & 1 & 0 \\ \hline \end{array}}_{\beta_k^T}} * \underbrace{\begin{array}{|cc|} \hline -2 & -3 \\ -2 & -6 \\ 2 & 2 \\ \hline \end{array}}_{G}$$

$$\min \left(\frac{1}{|-2|}, \frac{4}{|-6|} \right) = \frac{1}{2} \quad \Rightarrow \quad l = 1, \quad s = 1, \quad c_s' = 1$$

$\mathbf{3^\circ}$ $a_s' = B^{-1} a_s = a_s = (-2, -2, 2)^T; \quad a_{ks}' = -2$
  Remark: *At step 1 one can not premultiply a_s by B^{-1} if before this step $B^{-1} = I$.*

$\mathbf{4^\circ}$ Updating:

(a) $NB = \begin{array}{|c|c|c|} \hline 3 & 1 & 5 \\ \hline \end{array}$, $\quad NF = \begin{array}{|c|c|} \hline 4 & 2 \\ \hline \end{array}$

(with column labels 1, 2, 3 and 1, 2)

(b) $b'^T = \begin{array}{|c|c|c|} \hline 3 & 9/2 & -2 \\ \hline \end{array}$

(with column labels 1, 2, 3)

(c) $-z = -9/2$

(d) $B^{-1} = \begin{array}{|ccc|} \hline 1 & -1 & 0 \\ 0 & -1/2 & 0 \\ 0 & 1 & 1 \\ \hline \end{array}$

(e) $\pi^T = \boxed{\begin{array}{c|c|c} 0 & 1/2 & 0 \end{array}} = \boxed{\begin{array}{c|c|c} 0 & 0 & 0 \end{array}} - \boxed{\begin{array}{c|c|c} 0 & -1/2 & 0 \end{array}} * \boxed{-1}$

(f) $\underset{1 \quad 2}{\underset{c'^T}{\boxed{\begin{array}{c|c} 0 & 1 \end{array}}}} = \underset{3 \quad 4 \quad 5}{\underset{\pi^T}{\boxed{\begin{array}{c|c|c} 0 & 1/2 & 0 \end{array}}}} * \overset{G}{\begin{array}{|cc|} \hline -2 & -3 \\ -2 & -6 \\ 2 & 2 \\ \hline \end{array}} + \underset{c^T}{\boxed{\begin{array}{c|c} 1 & 4 \end{array}}}$

Step 2.

1° Look at b' in the above $\Rightarrow$ $k = 3$

2° $\underset{1 \quad 2}{\underset{a_k'^T}{\boxed{\begin{array}{c|c} 0 & -4 \end{array}}}} = \underset{3 \quad 4 \quad 5}{\underset{\beta_k^T}{\boxed{\begin{array}{c|c|c} 0 & 1 & 1 \end{array}}}} * \overset{G}{\begin{array}{|cc|} \hline -2 & -3 \\ -2 & -6 \\ 2 & 2 \\ \hline \end{array}}$

$$\min\left(\frac{1}{|-4|}\right) = \frac{1}{4} \quad \Rightarrow \quad l = 2, \quad s = 2, \quad c_s' = 1$$

3° $a_s' = \begin{array}{|ccc|} \hline 1 & -1 & 0 \\ 0 & -1/2 & 0 \\ 0 & 1 & 1 \\ \hline \end{array} * \begin{array}{|c|} \hline -3 \\ -6 \\ 2 \\ \hline \end{array} = \begin{array}{|c|} \hline 3 \\ 3 \\ -4 \\ \hline \end{array}; \qquad a_{ks}' = -4$

4° Updating:

(a) $NB = \underset{1 \quad 2 \quad 3}{\boxed{\begin{array}{c|c|c} 3 & 1 & 2 \end{array}}}, \qquad NF = \underset{1 \quad 2}{\boxed{\begin{array}{c|c} 4 & 5 \end{array}}}$

(b) $b'^T = \underset{1 \quad 2 \quad 3}{\boxed{\begin{array}{c|c|c} 3/2 & 3 & 1/2 \end{array}}}$

(c) $-z = -5$

(d) $B^{-1} = \begin{array}{|ccc|} \hline 1 & -1/4 & 3/4 \\ 0 & 1/4 & 3/4 \\ 0 & -1/4 & -1/4 \\ \hline \end{array}$

(e) $\pi^T = \boxed{\begin{array}{c|c|c} 0 & 3/4 & 1/4 \end{array}} = \boxed{\begin{array}{c|c|c} 0 & 1/2 & 0 \end{array}} - \boxed{\begin{array}{c|c|c} 0 & -1/4 & -1/4 \end{array}} * \boxed{1}$

(f) $\underset{1 \quad 2}{\underset{c'^T}{\boxed{\begin{array}{c|c} 0 & 0 \end{array}}}} = \underset{3 \quad 4 \quad 5}{\underset{\pi^T}{\boxed{\begin{array}{c|c|c} 0 & 3/4 & 1/4 \end{array}}}} * \overset{G}{\begin{array}{|cc|} \hline -2 & -3 \\ -2 & -6 \\ 2 & 2 \\ \hline \end{array}} + \underset{c^T}{\boxed{\begin{array}{c|c} 1 & 4 \end{array}}}$

Step 3.

1° Look at b' in the above $\Rightarrow$ the end

$BFS = (3,\ 1/2\ ,\ 3/2\ ,\ 0,0);\quad z_{min} = 5$

Remark 5.5.2. *To accelerate coming to "the end" it is advisable to insert the checking vector b', analogously to item 1°, immediately following item 4°c.*

Example 5.5.2. Find $x_1, x_2 \geq 0$ subject to

$$
\begin{array}{rcrcll}
2x_1 & + & 3x_2 & \geq & 6 & \\
2x_1 & + & 6x_2 & \geq & 9 & \\
x_1 & + & x_2 & \leq & 4 & \\
2x_1 & + & 2x_2 & = & 7 & \\
x_1 & + & 2x_2 & = & z & \rightarrow \quad \min
\end{array}
$$

Step 0. (The initial form of the problem).

$$
\begin{array}{c}
\quad A \qquad\qquad\qquad b \qquad\quad NB \qquad NF
\end{array}
$$

-2	-3	1	0	0	0	0	-6	1 $\,$ 3	1 $\,$ 1
-2	-6	0	1	0	0	0	-9	2 $\,$ 4	2 $\,$ 2
1	1	0	0	1	0	0	4	3 $\,$ 5	
2	2	0	0	0	1	0	7	4 $\,$ 6	
-2	-2	0	0	0	0	1	-7	5 $\,$ 7	
1	2	0	0	0	0	0	0		

$$BS = (0, 0, -6, -9, 4, 7, -7) \neq BFS, \quad 0 = z$$

Step 1.

1°
$$
\begin{array}{c}
\quad 1 \quad\ 2 \quad\ 3 \quad 4 \quad\ 5 \\
b'^{T} = \boxed{\ -6\ \mid\ -9\ \mid\ 4\ \mid\ 7\ \mid\ -7\ } \quad \Rightarrow \quad k = 2
\end{array}
$$

$$
\begin{array}{c}
G \\
\boxed{\begin{array}{rr} -2 & -3 \\ -2 & -6 \\ 1 & 1 \\ 2 & 2 \\ -2 & -2 \end{array}}
\end{array}
$$

2°
$$
\underset{\substack{1 \quad\ 2}}{\underbrace{a_k'^{T}}_{\boxed{-2 \mid -6}}} = \underset{\substack{3 \quad 4 \quad 5 \quad 6 \quad 7}}{\underbrace{\beta_k^{T}}_{\boxed{0 \mid 1 \mid 0 \mid 0 \mid 0}}} * G
$$

$$\min\left(\frac{1}{|-2|}, \frac{2}{|-6|}\right) = \frac{1}{3} \quad \Rightarrow \quad l = 2, \quad s = 2 \quad c_s' = 2$$

3° $a_s' = B^{-1}a_s = a_s = (-3, -6, 1, 2, -2)^T;\quad a_{ks}' = -6$

See the remark in the preceding example, Step 1, item 3°.

4° Updating:

(a) $NB = \begin{array}{|ccccc|}\hline 3 & 2 & 5 & 6 & 7 \\\hline\end{array}$, $\qquad NF = \begin{array}{|cc|}\hline 1 & 4 \\\hline\end{array}$

with column indices $1\ 2\ 3\ 4\ 5$ over NB and $1\ 2$ over NF.

(b) $b'^T = \begin{array}{|ccccc|}\hline -3/2 & 3/2 & 5/2 & 4 & -4 \\\hline\end{array}$

with column indices $1\ 2\ 3\ 4\ 5$.

(c) $-z = -3$

(d) $B^{-1} = \begin{array}{|ccccc|}\hline 1 & -1/2 & 0 & 0 & 0 \\ 0 & -1/6 & 0 & 0 & 0 \\ 0 & 1/6 & 1 & 0 & 0 \\ 0 & 1/3 & 0 & 1 & 0 \\ 0 & -1/3 & 0 & 0 & 1 \\\hline\end{array}$

(e) $\pi^T = \begin{array}{|ccccc|}\hline 0 & 1/3 & 0 & 0 & 0 \\\hline\end{array} = \begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & 0 \\\hline\end{array} - \begin{array}{|ccccc|}\hline 0 & -1/6 & 0 & 0 & 0 \\\hline\end{array} * \begin{array}{|c|}\hline 2 \\\hline\end{array}$

(f)
$$\underset{\substack{c'^T}}{\begin{array}{|cc|}\hline 1/3 & 0 \\\hline\end{array}} = \underset{\pi^T}{\begin{array}{|ccccc|}\hline 0 & 1/3 & 0 & 0 & 0 \\\hline\end{array}} * \underset{G}{\begin{array}{|cc|}\hline -2 & -3 \\ -2 & -6 \\ 1 & 1 \\ 2 & 2 \\ -2 & -2 \\\hline\end{array}} + \underset{c^T}{\begin{array}{|cc|}\hline 1 & 2 \\\hline\end{array}}$$

with column indices $1\ 2$ under c'^T and $3\ 4\ 5\ 6\ 7$ under π^T.

Step 2.

1° Check b' in the above $\quad\Rightarrow\quad k = 5$

2°
$$\underset{\substack{a_k'^T}}{\begin{array}{|cc|}\hline -4/3 & 0 \\\hline\end{array}} = \underset{\beta_k^T}{\begin{array}{|ccccc|}\hline 0 & -1/3 & 0 & 0 & 1 \\\hline\end{array}} * \underset{G}{\begin{array}{|cc|}\hline -2 & -3 \\ -2 & -6 \\ 1 & 1 \\ 2 & 2 \\ -2 & -2 \\\hline\end{array}}$$

with column indices $1\ 2$ under $a_k'^T$ and $3\ 4\ 5\ 6\ 7$ under β_k^T.

$$\min\left(\frac{1/3}{|-4/3|}, \frac{1/3}{|-1/3|}\right) = \frac{1}{4} \quad\Rightarrow\quad l = 1, \quad s = 1 \quad c_s' = 1/3$$

3° $a_s' = B^{-1}a_s = (-1, 1/3, 2/3, 4/3, -4/3)^T; \quad a_{ks}' = -4/3$

4° Updating:

(a) $NB = \begin{array}{|ccccc|}\hline 3 & 2 & 5 & 6 & 1 \\\hline\end{array}$, $\qquad NF = \begin{array}{|cc|}\hline 7 & 4 \\\hline\end{array}$

with column indices $1\ 2\ 3\ 4\ 5$ over NB and $1\ 2$ over NF.

(b) $b'^T = \begin{array}{|ccccc|}\hline 3/2 & 1/2 & 1/2 & 0 & 3 \\\hline\end{array}$

with column indices $1\ 2\ 3\ 4\ 5$.

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(c) $-z = -4$

(d), (e), (f) are omitted, see the remark at the end of the preceding example
$\Rightarrow$ the end.
$$BFS = (3, 1/2, 3/2, 0, 1/2, 0, 0); \quad z_{min} = 4$$

Remark 5.5.3. *The revised simplex method offers some advantages over the conventional simplex method. The first one is in reduced calculations. The second is the possibility of each step calculations to be checked by the relation $B^{-1}B = I$. One can see that $B^{-1} = [\ \beta_{ij}\ ]$ is the inverse of $B(i, j) = A[i, NB(j)]$. For example, after Step 3 of Example 5.4.1 we have the following checking relation*

$$\begin{pmatrix} 1/4 & -1/4 & 1/4 \\ 1/4 & 3/4 & -7/4 \\ -3/4 & -1/4 & 9/4 \end{pmatrix} * \begin{pmatrix} 5 & 2 & 1 \\ 3 & 3 & 2 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where the second factor, matrix B, is obtained from the initial matrix A by extracting its columns in order specified by pointer $NB = \begin{array}{|c|c|c|} \hline 3 & 2 & 1 \\ \hline \end{array}$.

5.6. Algorithm of the revised simplex method with artificial variables

Algorithm of Section 5.4 assumes that the initial form of problem makes the starting BFS obvious. The same assumption has been made in Sections 3.2 and 3.3. However, it often does not hold when some constraints are not changed from "greater or equal to" type to "less or equal to" type through premuptiplying by -1, or some constraints are of "equal to" type. As a precautionary measure againt this situation, artificial variables and artificial objective function are used as it is shown in Section 3.6. Now we show this practice as applied to the revised simplex method drawing the analogy to Section 5.4.

By the starting point of the algorithm the initial form of problem must look as follows:

$$\begin{bmatrix} G & | & I \\ --- & | & ---- \\ c^T & | & 0^T = \pi^T \\ -- & | & ---- \\ d^T & | & 0^T = \sigma^T \end{bmatrix} x = \begin{bmatrix} b \\ -- \\ z \\ -- \\ w \end{bmatrix}, \quad b \geq 0$$

This means that $x = (x^F, x^B)$ and $BFS = (b, 0)$. For the main objective function z we have the coefficients c^T and simplex multipliers π^T (initially, the latter are zero). For the artificial objective function w we have the coefficients d^T and simplex multipliers σ^T (initially, σ^T are zero, too).

Two stages are to be executed. Stage I is for minimizing w to generate a BFS by the beginning of Stage II. Stage II is for minimizing z; here both artificial variables and artificial objective function w have to be deleted from the problem. At Stage I, all actions

are executed in the same manner as described in Section 5.4, but simultaneously with the two pairs of coefficients: (c^T, π^T) and (d^T, σ^T). In doing so, the governing factor for Stage I is pair (d^T, σ^T) because namely these data are associated with the artificial objective function w to be minimized at this stage. Acting at Stage II are only (c^T, π^T) and z, being reduced to a minimum.

There is no need for a detailed description of the algorithm because, in view of the above peculiarities of Stages I and II, it would be completely similar to the algorithm of Section 5.4. It will suffice to give an example.

Example 5.6.1. For $x_1, x_2 \geq 0$ solve the problem

$$
\begin{array}{rcrcll}
2x_1 & + & 2x_2 & \leq & 7 & \\
2x_1 & + & 3x_2 & \geq & 6 & \\
2x_1 & + & 6x_2 & \geq & 9 & \\
x_1 & + & 4x_2 & = & z & \to \quad \min
\end{array}
$$

Let us compare the solution with examples of Sections 4.2 and 5.5. There the same problem was solved, however, without artificial variables. This fact sent us in application not the conventional but the dual simplex method (Section 4.2) or the revised dual simplex method (Section 5.5).

Solution. The initial form of problem is as follows:

2	2	0	0	1	0	0	7
2	3	−1	0	0	1	0	6
2	6	0	−1	0	0	1	9
1	4	0	0	0	0	0	0
0	0	0	0	0	1	1	0

First, we need to change from the initial form to the correct form of problem in order to have $\sigma^T = 0^T$. In the above, we have $\sigma^T = \boxed{0\ 1\ 1}$ and this means that $w = x_6 + x_7$. To convert to $\sigma^T = \boxed{0\ 0\ 0}$, we have to do the so-called preparatory subtraction. Obviously, it means the elimination of artificial variables x_6 and x_7 from the above (the initial) form of w. In any case, it is made by subtraction of the appropriate rows from the bottom row of the tableau. The appropriate rows are those where the artificial variables, here x_6 and x_7, appear as basic variables with coefficient 1.

Step 0. (After subtraction of the second and third rows from the bottom row).

$$A$$

1	2	3	4	5	6	7		NB	NF	
2	2	0	0	1	0	0	7	5	1	1
2	3	−1	0	0	1	0	6	6	2	2
2	6	0	−1	0	0	1	9	7	3	3
1	4	0	0	0	0	0	0		4	4
−4	−9	1	1	0	0	0	−15			

$$BS = (0,0,0,0,7,6,9) = BFS, \qquad 0 = -z, \qquad -15 = -w$$

Now the problem has the correct form and both z and w have the correct values: $z = 0, w = 15$.

Stage I. (Following the algorithm of Section 5.4 and using pair (d^T, σ^T) as the governing one and (c^T, π^T) as the concomitant pair).

Step 1.

1° $d'^T = \sigma^T G + d^T; \quad c'^T = \pi^T G + c^T$

Because at the very beginning we have $\sigma^T = 0$ and $\pi^T = 0$, then $d'^T = d^T$ and $c'^T = c^T$, here: $d'T = \boxed{-4 \ -9 \ 1 \ 1}$, $c'^T = \boxed{1 \ 4 \ 0 \ 0}$

$(d'^T, \sigma^T) = (-4, -9, 1, 1, 0, 0, 0) \quad \Rightarrow \quad l = 2, s = 2, d'_s = -9, c'_s = 4$

2° $a'_s = B^{-1} a_s = a_s = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$, because now $B = I$. We have

$b' = \begin{bmatrix} 7 \\ 6 \\ 9 \end{bmatrix}, \quad \min(7/2, 6/3, 9/6) = 9/6 \quad \Rightarrow \quad k = 3, \quad a'_{ks} = 6$

3°

NB	NF	b^+
5	1	4
6	7	3/2
2	3	3/2
	4	

$z^+ = z - c'_s b^+_k = 0 - 4 * 3/2 = -6$

$w^+ = w - d'_s b^+_k = -15 + 9 * 3/2 = -3/2$

B^{-1}
1	0	$-1/3$
0	1	$-1/2$
0	0	$1/6$

$\pi^{+T} = \boxed{0 \ \ 0 \ \ -2/3} = \pi^T \boxed{0 \ \ 0 \ \ 0} - \beta^T_k \boxed{0 \ \ 0 \ \ 1/6} * c'_s \boxed{4}$

$\sigma^{+T} = \boxed{0 \ \ 0 \ \ 3/2} = \sigma^T \boxed{0 \ \ 0 \ \ 0} - \beta^T_k \boxed{0 \ \ 0 \ \ 1/6} * d'_s \boxed{-9}$

$BS = (0, 3/2, 0, 0, 4, 3/2, 0) = BFS; \quad z = 6, \quad w = 3/2$

Remark: *The last two equations determine the true value z and w. It is necessary to distinguish them from z, z^+ and w, w^+ used in action 3° at every step as the calculation values whose sign is always opposite to the sign of true values z and w.*

Step 2.

1°

d'^T
$\boxed{-1 \ \ 0 \ \ 1 \ \ -1/2}$ $= \sigma^T \boxed{0 \ \ 0 \ \ 3/2}$ $G \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 3 & -1 & 0 \\ 2 & 6 & 0 & -1 \end{bmatrix} + d^T \boxed{-4 \ \ -9 \ \ 1 \ \ 1}$

positions: $\underline{1} \ \ 2 \ \ \underline{3} \ \ \underline{4}$; $5 \ \ 6 \ \ \underline{7}$

c'^T
$\boxed{-1/3 \ \ 0 \ \ 0 \ \ 2/3}$ $= \pi^T \boxed{0 \ \ 0 \ \ -2/3}$ $G \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 3 & -1 & 0 \\ 2 & 6 & 0 & -1 \end{bmatrix} + c^T \boxed{1 \ \ 4 \ \ 0 \ \ 0}$

positions: $\underline{1} \ \ 2 \ \ \underline{3} \ \ \underline{4}$; $5 \ \ 6 \ \ \underline{7}$

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Look at (d'^T, σ^T) through $NF = (1, 7, 3, 4)$ (see the underlined numbers) in order to find the most negative element. We find $l = 1, s = 1, d'_s = -1$. Look at (c'^T, π^T) through NF. By $s = 1$ we find $c'_s = -1/3$.

$$\mathbf{2°} \quad a'_s = B^{-1}a_s = \begin{vmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1/6 \end{vmatrix} \begin{vmatrix} 2 \\ 2 \\ 2 \end{vmatrix} = \begin{vmatrix} 4/3 \\ 1 \\ 1/3 \end{vmatrix}; \quad b' = \begin{vmatrix} 4 \\ 3/2 \\ 3/2 \end{vmatrix}$$

$$\min\left(\frac{4}{4/3}, \frac{3/2}{1}, \frac{3/2}{1/3}\right) \quad \Rightarrow \quad k = 2, \quad a'_{ks} = 1$$

$\mathbf{3°}$

NB	NF	b^+
5	6	2
1	7	3/2
2	3	1
	4	

$$z^+ = z - c'_s b^+_k = -11/2$$
$$w^+ = w - d'_s b^+_k = 0$$

Appearing $w := w^+ = 0$ is the intermediate, preliminary indication of successful completing the first stage, and now it is clear that

$$BS = (3/2, 1, 0, 0, 2, 0, 0) = BFS, \quad z = 11/2$$

In order to check the final indication and change to Stage II, we continue the calculations.

$$\begin{array}{c} B^{-1} \\ \begin{vmatrix} 1 & -4/3 & 1/3 \\ 0 & 1 & -1/2 \\ 0 & -1/3 & 1/3 \end{vmatrix} \end{array}$$

$$\begin{array}{ccccc} \pi^T & := & \pi^T & - & \beta^T_k & * c'_s \\ \boxed{0 \quad 1/3 \quad -5/6} & & \boxed{0 \quad 0 \quad -2/3} & & \boxed{0 \quad 1 \quad -1/2} & \boxed{-1/3} \end{array}$$

$$\begin{array}{ccccc} \sigma^T & := & \sigma^T & - & \beta^T_k & * d'_s \\ \boxed{0 \quad 1 \quad 1} & & \boxed{0 \quad 0 \quad 3/2} & & \boxed{0 \quad 1 \quad -1/2} & \boxed{-1} \end{array}$$

Remark: *As already noted, the revised simplex method allows a check of calculations to be made by the relation:*

$$\begin{array}{ccc} B^{-1} & B & I \\ \begin{vmatrix} 1 & -4/3 & 1/3 \\ 0 & 1 & -1/2 \\ 0 & -1/3 & 1/3 \end{vmatrix} * \begin{vmatrix} 1 & 2 & 2 \\ 0 & 2 & 3 \\ 0 & 2 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \end{array}$$

In doing this, matrix B should be formed by extracting columns from the initial matrix A according to numbers stored in NB. At this point we have $NB = (5, 1, 2)$.

Step 3.

$$
\mathbf{1^\circ}\quad
\underset{\substack{1\ \ 2\ \ \underline{3}\ \ 4}}{\underbrace{\boxed{0\ \ 0\ \ 0\ \ 0}}^{d'^T}}
=
\underset{\substack{5\ \ \underline{6}\ \ \underline{7}}}{\underbrace{\boxed{0\ \ 1\ \ 1}}^{\sigma^T}}
\overset{G}{\begin{vmatrix} 2 & 2 & 0 & 0 \\ 2 & 3 & -1 & 0 \\ 2 & 6 & 0 & -1 \end{vmatrix}}
+
\underbrace{\boxed{-4\ \ -9\ \ 1\ \ 1}}^{d^T}
$$

$$
\underset{\substack{1\ \ 2\ \ \ \ \underline{3}\ \ \ \ 4}}{\underbrace{\boxed{0\ \ 0\ \ -1/3\ \ 5/6}}^{c'^T}}
=
\underset{\substack{5\ \ \ 6\ \ \ \ 7}}{\underbrace{\boxed{0\ \ 1/3\ \ -5/6}}^{\pi^T}}
\overset{G}{\begin{vmatrix} 2 & 2 & 0 & 0 \\ 2 & 3 & -1 & 0 \\ 2 & 6 & 0 & -1 \end{vmatrix}}
+
\underbrace{\boxed{1\ \ 4\ \ 0\ \ 0}}^{c^T}
$$

Look at elements of (d'^T, σ^T) through $NF = (6,7,3,4)$. Among them, there no negatives and, as above noted, $w = 0$. This is the final indication of successful completing the first stage. In what follows, Stage II, we need neither artificial variables x_6, x_7 nor artificial objective function w. Also, we do not need (d'^T, σ^T), so we switch to viewing (c'^T, π^T) through $NF = (6,7,3,4)$ where numbers 6 and 7 will be ignored.

Stage II. (Continuing Step 3, action 1°) As a result of viewing (c'^T, π^T) through $NF = (6,7,3,4)$ with numbers 6 and 7 ignored, we have $l = 3, s = 3, c'_s = -1/3$.

$$
\mathbf{2^\circ}\quad a'_s = B^{-1}a_s =
\begin{vmatrix} 1 & -4/3 & 1/3 \\ 0 & 1 & -1/2 \\ 0 & -1/3 & 1/3 \end{vmatrix}
\begin{vmatrix} 0 \\ -1 \\ 0 \end{vmatrix}
=
\begin{vmatrix} 4/3 \\ -1 \\ 1/3 \end{vmatrix} ;
\quad b' =
\begin{vmatrix} 2 \\ 3/2 \\ 1 \end{vmatrix}
$$

Remark: *Matrix B^{-1} needs to be calculated as a whole in order to find a'_s even though artificial variables are needless at Stage II. The same pertains equally to π^T that will be seen below at Step 4, 1°.*

$$
\min\left(\frac{2}{4/3}, \infty, \frac{1}{1/3}\right) \quad \Rightarrow \quad k = 1, \quad a'_{ks} = 4/3
$$

$$
\mathbf{3^\circ}\quad
\overset{NB}{\begin{vmatrix} 3 \\ 1 \\ 2 \end{vmatrix}}
\overset{NF}{\begin{vmatrix} 6 \\ 7 \\ 5 \\ 4 \end{vmatrix}}
\overset{b^+}{\begin{vmatrix} 3/2 \\ 3 \\ 1/2 \end{vmatrix}}
\quad z^+ = z - c'_s b^+_k = -11/2 - (-1/3)(3/2) = -5
$$

$$
\overset{B^{-1}}{\begin{vmatrix} 3/4 & -1 & 1/4 \\ 3/4 & 0 & -1/4 \\ -1/4 & 0 & 1/4 \end{vmatrix}}
\qquad
\overset{\pi^T}{\boxed{1/4\ \ 0\ \ -3/4}} := \overset{\pi^T}{\boxed{0\ \ 1/3\ \ -5/6}} -
$$
$$
- \underset{\beta^T_k}{\boxed{3/4\ \ -1\ \ 1/4}} * \underset{c'_s}{\boxed{-1/3}}
$$

$BS = (3, 1/2, 3/2, 0, 0) = BOS, \quad z_{\min} = 5.$

However, the fact that the basic optimal solution (BOS) has been found and has the above form, will be formally seen only on the next step.

Step 4.

$$
\mathbf{1°}\quad \underset{\substack{1\quad 2\quad 3\quad \underline{4}}}{\underbrace{\boxed{\begin{array}{cccc} 0 & 0 & 0 & 3/4 \end{array}}}_{c'^T}} = \underset{\substack{5\quad\ 6\quad\ \ 7}}{\underbrace{\boxed{\begin{array}{ccc} 1/4 & 0 & -3/4 \end{array}}}_{\pi^T}}\ \underbrace{\begin{array}{|cccc|} \hline 2 & 2 & 0 & 0 \\ 2 & 3 & -1 & 0 \\ 2 & 6 & 0 & -1 \\ \hline \end{array}}_{G} + \underbrace{\boxed{\begin{array}{cccc} 1 & 4 & 0 & 0 \end{array}}}_{c^T}
$$

Through $NF = (6, 7, 5, 4)$ in which numbers 6 and 7 are ignored, we look at (c'^T, π^T) and discover that there are no negative values among numbers $(3/4, 1/4)$. This indicates the work is now complete and we have achieved the basic optimal solution.

Remark 5.6.1. *It is necessary to calculate c'^T not in part but completely. Because of it, we must have the complete π^T, together with elements number 6 and 7, although these are numbers of the ignored artificial variables. In other words, these numbers are being ignored only while looking at row (c'^T, π^T) through NF in action $1°$ of every step in the sequel to the algorithm.*

5.7. The revised dual simplex method with artificial variables

In Section 4.3 we considered the dual simplex method without correct form of basis for such a case when among constraints there is at least one of "equal to" type and only by this fact it is stipulated that artificial variables and artificial objective function should be effected at Stage I of the conventional simplex method. In this case the problem is such that the necessity to use the dual simplex method is detected at Stage II. This situation was considered as applied to the non-revised algorithm. Now we take the same example as in Section 4.3 and show how the revised algorithm works. In so doing, we will see that the algorithm of Section 5.6 works at Stage I and the algorithm of Section 5.5 works at Stage II.

Example 5.7.1. For $x_1, x_2 \geq 0$ solve the problem:

$$
\begin{array}{rcrcrcrcl}
2x_1 & + & 3x_2 & & & \geq & 6 \\
2x_1 & + & 6x_2 & & & \geq & 9 \\
x_1 & + & x_2 & & & \leq & 4 \\
& & 2x_2 & + & 2x_3 & = & 7 \\
x_1 & + & 2x_2 & & & = & z & \rightarrow & \min
\end{array}
$$

All the revised algorithms use the correct form of basis. It means that basis B must be initially formed by columns of identity matrix. Because of it, inequalities of "$\geq$" type should be premultiplied by -1 to change them to "$\leq$" type. In view of this, the initial form of the problem is characterized by the following tableau:

Step 0.

$$\begin{array}{cccccc|c}
-2 & -3 & 1 & 0 & 0 & 0 & -6 \\
-2 & -6 & 0 & 1 & 0 & 0 & -9 \\
1 & 1 & 0 & 0 & 1 & 0 & 4 \\
2 & 2 & 0 & 0 & 0 & 1 & 7 \\
\hline
1 & 2 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{array}$$

Obviously, variable x_6 is artificial and the bottom row has stored the coefficients and value of artificial objective function w. By the time being, x_6 is not in basic set of variables x^B with reference to w. To make x_6 basic, we eliminate x_6 from the bottom row by the evident subtracting the fourth row from the bottom row. Having done this preparatory subtraction, we obtain the correct initial state:

$$\begin{array}{cc|cccc|c}
 & & & & & & & NB & & NF \\
-2 & -3 & 1 & 0 & 0 & 0 & -6 & & 3 & & _1\boxed{1} \\
-2 & -6 & 0 & 1 & 0 & 0 & -9 & & 4 & & _2\boxed{2} \\
1 & 1 & 0 & 0 & 1 & 0 & 4 & & 5 & & \\
2 & 2 & 0 & 0 & 0 & 1 & 7 & & 6 & & \\
\hline
1 & 2 & 0 & 0 & 0 & 0 & 0 & & & & \\
-2 & -2 & 0 & 0 & 0 & 0 & -7 & & & & \\
\end{array}$$

$$BS = (0, 0, -6, -9, 4, 7) \neq BFS, \quad 0 = z, \ 7 = w$$

Now the initial state of the problem is similar to that one specified in Section 5.6 with the only difference that inequality $b \geq 0$ does not hold, i.e., $BS \neq BFS$.

Stage I. (Minimizing the artificial objective function w) By the bottom row we recognize that the conventional simplex method (Section 5.4 and Section 5.6) should be applied.

Step 1.

1° $d'^T = \sigma^T G + d^T$; $c'^T = \pi^T G + c^T$. Because $\sigma^T = 0$ and $\pi^T = 0$, then $d'^T = d^T$, $c'^T = c^T$, $d'^T = (-2, -2)$, $c'^T = (1, 2)$
$(d'^T, \sigma^T) = (-2, -2, 0, 0, 0, 0, 0) \quad \Rightarrow l = 1, \ s = 1, \ d'_s = -2, \ c'_s = 1$

2° $a'_s = B^{-1} a_s = a_s = (-2, -2, 1, 2)^T$, $b' = (-6, -9, 4, 7)^T$
$\min(\infty, \infty, 4/1, 7/2) = 7/2 \quad \Rightarrow k = 4, \ a'_{ks} = 2$

3°
$$\begin{array}{ccc}
NB & NF & b^+ \\
\boxed{\begin{array}{c} 3 \\ 4 \\ 5 \\ 1 \end{array}} & \boxed{\begin{array}{c} 6 \\ 2 \end{array}} & \boxed{\begin{array}{c} 1 \\ -2 \\ 1/2 \\ 7/2 \end{array}}
\end{array}$$

$z^+ = z - c'_s b_k^+ = -7/2; \quad b' := b^+$
$w^+ = w - d'_s b_k^+ = 0$

$$
\begin{array}{l}
B^{-1} \\
\begin{array}{|cccc|}
\hline
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & -1/2 \\
0 & 0 & 0 & 1/2 \\
\hline
\end{array}
\end{array}
\qquad
\begin{array}{l}
\pi^T \\
\boxed{\begin{array}{cccc} 0 & 0 & 0 & -1/2 \end{array}}
\end{array}
\begin{array}{c} := \\ \\ \end{array}
\begin{array}{l}
\pi^T \\
\boxed{\begin{array}{cccc} 0 & 0 & 0 & 0 \end{array}}
\end{array}
\begin{array}{c} - \\ \\ \end{array}
\begin{array}{l}
\beta_k^T \\
\boxed{\begin{array}{cccc} 0 & 0 & 0 & 1/2 \end{array}}
\end{array}
\begin{array}{l}
* c_s' \\
\boxed{1}
\end{array}
$$

$$
\begin{array}{l}
\sigma^T \\
\boxed{\begin{array}{cccc} 0 & 0 & 0 & 1 \end{array}}
\end{array}
\begin{array}{c} := \\ \end{array}
\begin{array}{l}
\sigma^T \\
\boxed{\begin{array}{cccc} 0 & 0 & 0 & 0 \end{array}}
\end{array}
\begin{array}{c} - \\ \end{array}
\begin{array}{l}
\beta_k^T \\
\boxed{\begin{array}{cccc} 0 & 0 & 0 & 1/2 \end{array}}
\end{array}
\begin{array}{l}
* d_s' \\
\boxed{-2}
\end{array}
$$

$$BS = (7/2, 0, 1, -2, 1/2, 0) \neq BFS, \quad z = 7/2, \ w = 0$$

Step 2.

$$
1° \quad
\underset{d'^T}{\boxed{\begin{array}{cc} 0 & 0 \end{array}}}
=
\underset{\sigma^T}{\boxed{\begin{array}{cccc} 0 & 0 & 0 & 1 \end{array}}}
*
\underset{G}{\begin{array}{|cc|}
\hline
-2 & -3 \\
-2 & -6 \\
1 & 1 \\
2 & 2 \\
\hline
\end{array}}
+
\underset{d^T}{\boxed{\begin{array}{cc} -2 & -2 \end{array}}}
$$

$$
\underset{c'^T}{\boxed{\begin{array}{cc} 0 & 1 \end{array}}}
=
\underset{\pi^T}{\boxed{\begin{array}{cccc} 0 & 0 & 0 & -1/2 \end{array}}}
*
\underset{G}{\begin{array}{|cc|}
\hline
-2 & -3 \\
-2 & -6 \\
1 & 1 \\
2 & 2 \\
\hline
\end{array}}
+
\underset{c^T}{\boxed{\begin{array}{cc} 1 & 2 \end{array}}}
$$

Look at elements of (d'^T, σ^T) through $NF = (6, 2)$. Because there are no negative elements and $w = 0$, then Stage I has been successfully completed. Further, while looking at row (c'^T, π^T), we ignore the sixth element corresponding to the artificial variable x_6, i.e. we look at (c'^T, π^T) only through the remained, the second, element of NF: $l = 2$, $NF(l) = 2$.

Stage II. (Pursueing Step 2, action 1°)
Looking at (c'^T, π^T) through NF testifies that from this point on the dual simplex method must be used (see Remark 4.2.1), and here we use the revised version of the method (see Section 5.5).

$$
1° \quad b'^T =
\begin{array}{cccc}
\overset{1}{} & \overset{2}{} & \overset{3}{} & \overset{4}{} \\
\end{array}
\boxed{\begin{array}{c|c|c|c} -1 & -2 & 1/2 & 7/2 \end{array}}
$$

$$
2° \quad
\underset{a_k'^T}{\boxed{\begin{array}{c|c} 0 & -4 \end{array}}}
=
\underset{\beta_k^T}{\boxed{\begin{array}{cccc} 0 & 1 & 0 & 1 \end{array}}}
*
\underset{G}{\begin{array}{|cc|}
\hline
-2 & -3 \\
-2 & -6 \\
1 & 1 \\
2 & 2 \\
\hline
\end{array}}
\ ; \ a_k'[NF(2)] = -4 < 0
$$

$$
\min_{j = NF(2)} \left(\frac{(c'^T, \pi^T)}{|\,a_{kj}'\,|} \right) = \min \left(\frac{1}{|-4|} \right) = \frac{1}{4} \quad \Rightarrow l = 2, s = 2, c_s' = 1
$$

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3° $a'_s = \begin{array}{|cccc|} \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 1/2 \\ \hline \end{array} * \begin{array}{|c|} \hline -3 \\ -6 \\ 1 \\ 2 \\ \hline \end{array} = \begin{array}{|c|} \hline -1 \\ -4 \\ 0 \\ 1 \\ \hline \end{array} \; ; \quad a'_{ks} = -4$

4° Updating:

    (a) $NB = \begin{array}{|c|c|c|c|} \hline 3 & 2 & 5 & 1 \\ \hline \end{array}$, $\qquad NF = \begin{array}{|c|c|} \hline 6 & 4 \\ \hline \end{array}$

    (the top indices of NB are $1\;2\;3\;4$; of NF are $1\;2$)

    (b) $b'^T = \begin{array}{|c|c|c|c|} \hline 3/2 & 1/2 & 1/2 & 3 \\ \hline \end{array}$

    (top indices $1\;2\;3\;4$)

    (c) $z^+ = -7/2 - 1 * 1/2 = -4$

    (d) $B^{-1} = \begin{array}{|cccc|} \hline 1 & -1/4 & 0 & 3/4 \\ 0 & -1/4 & 0 & -1/4 \\ 0 & 0 & 1 & -1/2 \\ 0 & 1/4 & 0 & 3/4 \\ \hline \end{array}$

    (e) $\underset{\pi^T}{\begin{array}{|cccc|} \hline 0 & 1/4 & 0 & -1/4 \\ \hline \end{array}} := \underset{\pi^T}{\begin{array}{|cccc|} \hline 0 & 0 & 0 & -1/2 \\ \hline \end{array}} - \underset{\beta_k^T}{\begin{array}{|cccc|} \hline 0 & -1/4 & 0 & -1/4 \\ \hline \end{array}} * \underset{c'_s}{\begin{array}{|c|} \hline 1 \\ \hline \end{array}}$

    (f) $\underset{c'^T}{\begin{array}{|cc|} \hline 0 & 0 \\ \hline \end{array}} := \underset{\pi^T}{\begin{array}{|cccc|} \hline 0 & 1/4 & 0 & -1/4 \\ \hline \end{array}} * \underset{G}{\begin{array}{|cc|} \hline -2 & -3 \\ -2 & -6 \\ 1 & 1 \\ 2 & 2 \\ \hline \end{array}} + \underset{c^T}{\begin{array}{|cc|} \hline 1 & 2 \\ \hline \end{array}}$

    (bottom indices under c'^T: $1\;2$; under π^T: $3\;4\;5\;(6)$)

Step 3.

  1° See b' in the above $\Rightarrow$ the end.

    BS=(3, 1/2, 3/2, 0, 1/2)=BFS=BOS, $\quad z_{\min} = 4$

5.8. Adding a constraint in the revised simplex method

In Chapter 4 we considered how to continue the solution when a constraint is added to the problem that just have been solved. Now we address to the same question when applied to the revised simplex method.

    Let us cosider Example 5.4.1 and continue the solution asking: "What if we add one more constraint ?"

    Let the added constraint be

$$x_1 + x_2 + x_3 \le 15$$

At this point (see the end of Section 5.4) we have

$$
\begin{array}{ccccc}
G & B^{-1} & b' & NB & NF
\end{array}
$$

G			B^{-1}			b'	NB		NF	
1	2	5	1	−1	1	12	1	4	1	6
2	3	3	0	1	−2	6	2	2	2	5
1	1	2	0	−1	3	21	3	1	3	3
1	2	3								

c'^T			π^T			$-z$
0	0	2	0	1	7	249
1	2	3	4	5	6	

$$A$$

1	2	5	1	0	0	45
2	3	3	0	1	0	60
1	1	2	0	0	1	27
−9	−10	−15	0	0	0	0

Adding the constraint $x_1 + x_2 + x_3 \le 15$ entails the following modifications in these data:

G			B^{-1}				b'	NB		NF	
1	2	5	1	−1	1	0	12	1	4	1	6
2	3	3	0	1	−2	0	6	2	2	2	5
1	1	2	0	−1	3	0	21	3	1	3	3
1	1	1	0	0	0	1	15	4	(7)		
1	2	3									

c'^T			π^T				$-z$
0	0	2	0	1	7	0	249
1	2	3	4	5	6	7	

$$A$$

1	2	5	1	0	0	0	45
2	3	3	0	1	0	0	60
1	1	2	0	0	1	0	27
1	1	1	0	0	0	1	15
−9	−10	−15	0	0	0	0	0

Artificial variable x_7 has been introduced to represent the added (the fourth) constraint in the form:

$$x_1 + x_2 + x_3 + x_7 = 15, \quad x_7 \ge 0$$

However, we must make x_7 the only basic variable in the fourth constraint in order to represent the problem in the canonical form for a basis. It means that we have to eliminate all the other basic variables appearing in that row. Through $NB = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline 4 & 2 & 1 & (7) \end{array}$ we see that we have to eliminate x_2 and x_1. This correcting elimination will change only the bottom row of B^{-1} and the bottom element of b'.

Step 0. (The correcting elimination)

Look through NB at the added (the bottom) row of G. For this, we scan in succession elements of $NB(i) : i = 1, 2, \ldots, m$, and look at elements of the bottom row of G in locations whose numbers are $NB(i)$ (m denotes the initial number of constraints, here $m = 3$). If such an element exists and it is not zero, then the bottom rows of B^{-1} and b' are modified. Sequentially, it looks as follows:

$i = 1, NB(1) = 4 \quad \Rightarrow$ The 4-th element does not exist in the bottom row of G $\Rightarrow$ we continue.

$i = 2, NB(2) = 4 \quad \Rightarrow G(m+1, 2)$ exists and is not zero $\quad \Rightarrow$ we subtract the i-th (the 2nd) row of B^{-1} (and b') taken with coefficient $G(m+1, NB(i))$, from the bottom row. Doing so, we have

$$
\begin{array}{cccc}
\multicolumn{4}{c}{B^{-1}} \\
\hline
1 & -1 & 1 & 0 \\
0 & 1 & -2 & 0 \\
0 & -1 & 3 & 0 \\
0 & -1 & 2 & 1 \\
\end{array}
\qquad
\begin{array}{c}
b' \\
\hline
12 \\
6 \\
21 \\
9 \\
\end{array}
$$

$i = 3, NB(3) = 1 \quad \Rightarrow G(m+1, 1)$ exists and is not zero $\quad \Rightarrow$ we subtract the i-th (the 3rd) row of B^{-1} (and b') taken with coefficient $G(4, NB(i))$, from the bottom row. Thus we obtain

$$
\begin{array}{cccc}
\multicolumn{4}{c}{B^{-1}} \\
\hline
1 & -1 & 1 & 0 \\
0 & 1 & -2 & 0 \\
0 & -1 & 3 & 0 \\
0 & 0 & -1 & 1 \\
\end{array}
\qquad
\begin{array}{c}
b' \\
\hline
12 \\
6 \\
21 \\
-12 \\
\end{array}
\qquad
\begin{array}{cc}
\multicolumn{2}{c}{NB} \\
\hline
1 & 4 \\
2 & 2 \\
3 & 1 \\
4 & 7 \\
\end{array}
$$

At this point the correcting elimination has been completed; it means that the parentheses in the 4-th row of NB can be taken off: variable x_7 is, from now on, basic in the canonical form for a basis. We have

$$BS = (21, 6, 0, 12, 0, 0, -12) \neq BFS, \quad z = -249$$

Elements of (c'^T, π^T) cannot be negative, so we check elements of b' and owing to -12 we see that the dual simplex method should start, in our example, the revised version of it.

Step 1. (The revised dual simplex method works, see Section 5.5)

1° $k = 4$

2°
$$
\underbrace{\begin{array}{|ccc|}
\hline
0 & 0 & -1 \\
\hline
\end{array}}_{a_k'^T}
=
\underbrace{\begin{array}{|cccc|}
\hline
0 & 0 & -1 & 1 \\
\hline
\end{array}}_{\beta_k^T}
*
\begin{array}{c}
G \\
\hline
\begin{array}{|ccc|}
1 & 2 & 5 \\
2 & 3 & 3 \\
1 & 1 & 2 \\
1 & 1 & 1 \\
\end{array}
\end{array}
$$

with the column labels $1, 2, \underline{3}$ under $a_k'^T$ and $4, \underline{5}, \underline{6}, 7$ under β_k^T.

Looking at (c'^T, π^T) and $(a_k'^T, \beta_k^T)$ through $NF = \begin{array}{|ccc|} \hline 6 & 5 & 3 \\ \hline \end{array}$ (with column labels 1 2 3 above) (see also underlined numbers in the above), we find

$$\min\left(\frac{7}{|-1|}, \infty, \frac{2}{|-1|}\right) = 2 \quad \Rightarrow l = 3, s = NF(l) = 3, c_s' = 2$$

3°
$$
\begin{array}{c}
a_s' \\
\boxed{\begin{array}{c} 4 \\ -1 \\ 3 \\ -1 \end{array}}
\end{array}
=
\begin{array}{c}
B^{-1} \\
\boxed{\begin{array}{cccc} 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & -1 & 1 \end{array}}
\end{array}
*
\begin{array}{c}
a_s \\
\boxed{\phantom{\begin{array}{c} 4 \\ -1 \\ 3 \\ -1 \end{array}}}
\end{array}
, \quad a_{ks}' = -1
$$

4° **(a)** $NB = \begin{array}{|cccc|} \hline 4 & 2 & 1 & 3 \\ \hline \end{array}$ (labels 1 2 3 4), $\qquad NF = \begin{array}{|ccc|} \hline 6 & 5 & 7 \\ \hline \end{array}$ (labels 1 2 3)

(b) $b'^T = \begin{array}{|cccc|} \hline -36 & 18 & -15 & 12 \\ \hline \end{array}$ (labels 1 2 3 4)

(c) $-z = 225$

$BS = (-15, 18, 12, -36, 0, 0, 0) \neq BFS$

(d)
$$
\begin{array}{c}
B^{-1} \\
\boxed{\begin{array}{cccc} 1 & -1 & -3 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}}
\end{array}
$$

(e) $\begin{array}{c} \pi^T \\ \boxed{\begin{array}{cccc} 0 & 1 & 5 & 2 \end{array}} \end{array} := \begin{array}{c} \pi^T \\ \boxed{\begin{array}{cccc} 0 & 1 & 7 & 0 \end{array}} \end{array} - \begin{array}{c} \beta_k^T \\ \boxed{\begin{array}{cccc} 0 & 0 & 1 & -1 \end{array}} \end{array} * \begin{array}{c} c_s' \\ \boxed{2} \end{array}$

(f) $\begin{array}{c} c'^T \\ \boxed{\begin{array}{ccc} 0 & 0 & 0 \end{array}} \\ \begin{array}{ccc} \underline{1} & \underline{2} & \underline{3} \end{array} \end{array} := \begin{array}{c} \pi^T \\ \boxed{\begin{array}{cccc} 0 & 1 & 5 & 2 \end{array}} \\ \begin{array}{cccc} \underline{4} & \underline{5} & \underline{6} & \underline{7} \end{array} \end{array} * \begin{array}{c} G \\ \boxed{\begin{array}{ccc} 1 & 2 & 5 \\ 2 & 3 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{array}} \end{array} + \begin{array}{c} c^T \\ \boxed{\begin{array}{ccc} -9 & -10 & -15 \end{array}} \end{array}$

Step 2.

1° $k = 1$

2° $\begin{array}{c} a_k'^T \\ \boxed{\begin{array}{ccc} 0 & 0 & 0 \end{array}} \\ \begin{array}{ccc} \underline{1} & \underline{2} & \underline{3} \end{array} \end{array} = \begin{array}{c} \beta_k^T \\ \boxed{\begin{array}{cccc} 0 & -1 & -3 & 4 \end{array}} \\ \begin{array}{cccc} \underline{4} & \underline{5} & \underline{6} & \underline{7} \end{array} \end{array} * \begin{array}{c} G \\ \boxed{\begin{array}{ccc} 1 & 2 & 5 \\ 2 & 3 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{array}} \end{array}$

Looking at (c'^T, π^T) and $(a_k'^T, \beta_k^T)$ through $NF = \begin{array}{|ccc|} \hline 6 & 5 & 7 \\ \hline \end{array}$ (labels 1 2 3) (see underlined numbers

in the above), we find

$$\min\left(\frac{5}{\mid -3\mid}, \frac{1}{\mid -1\mid}, \infty\right) = 1 \quad \Rightarrow l = 2, s = 5, c'_s = 1$$

3°

$$
\begin{array}{c}
a'_s \\
\boxed{\begin{array}{c} -1 \\ 1 \\ -1 \\ 0 \end{array}}
\end{array}
=
\begin{array}{c}
B^{-1} \\
\boxed{\begin{array}{cccc} 1 & -1 & -3 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}}
\end{array}
*
\begin{array}{c}
a_s \\
\boxed{\begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \end{array}}
\end{array}
, \quad a'_{ks} = -1
$$

4° **(a)** $NB = \begin{array}{cccc} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 & \scriptstyle 4 \\ \boxed{5}\boxed{2}\boxed{1}\boxed{3} \end{array}$, $\qquad NF = \begin{array}{ccc} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 \\ \boxed{6}\boxed{4}\boxed{7} \end{array}$

(b) $b'^T = \begin{array}{cccc} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 & \scriptstyle 4 \\ \boxed{36}\boxed{-18}\boxed{21}\boxed{12} \end{array}$

(c) $-z = 189$

$BS = (21, -18, 12, 0, 36, 0, 0) \neq BFS$

(d)

$$
\begin{array}{c}
B^{-1} \\
\boxed{\begin{array}{cccc} -1 & 1 & 3 & -4 \\ 1 & 0 & -4 & 3 \\ -1 & 0 & 3 & -1 \\ 0 & 0 & 1 & -1 \end{array}}
\end{array}
$$

(e) $\begin{array}{c} \pi^T \\ \boxed{1}\boxed{0}\boxed{2}\boxed{6} \end{array} := \begin{array}{c} \pi^T \\ \boxed{0}\boxed{1}\boxed{5}\boxed{2} \end{array} - \begin{array}{c} \beta_k^T \\ \boxed{-1}\boxed{1}\boxed{3}\boxed{-4} \end{array} * \begin{array}{c} c'_s \\ \boxed{1} \end{array}$

(f) $\begin{array}{c} c'^T \\ \boxed{0}\boxed{0}\boxed{0} \\ \scriptstyle 1 \quad \scriptstyle 2 \quad \scriptstyle 3 \end{array} := \begin{array}{c} \pi^T \\ \boxed{1}\boxed{0}\boxed{2}\boxed{6} \\ \scriptstyle \underline{4} \quad \scriptstyle 5 \quad \scriptstyle \underline{6} \quad \scriptstyle \underline{7} \end{array} * \begin{array}{c} G \\ \boxed{\begin{array}{ccc} 1 & 2 & 5 \\ 2 & 3 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{array}} \end{array} + \begin{array}{c} c^T \\ \boxed{-9}\boxed{-10}\boxed{-15} \end{array}$

Step 3.

1° $k = 2$

2° $\begin{array}{c} a_k'^T \\ \boxed{0}\boxed{1}\boxed{0} \\ \scriptstyle 1 \quad \scriptstyle 2 \quad \scriptstyle 3 \end{array} = \begin{array}{c} \beta_k^T \\ \boxed{1}\boxed{0}\boxed{-4}\boxed{3} \\ \scriptstyle 4 \quad \scriptstyle 5 \quad \scriptstyle 6 \quad \scriptstyle 7 \end{array} * \begin{array}{c} G \\ \boxed{\begin{array}{ccc} 1 & 2 & 5 \\ 2 & 3 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{array}} \end{array}$

Looking at (c'^T, π^T) and $(a_k'^T, \beta_k^T)$ through $NF = \begin{array}{ccc} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 \\ \boxed{6 \quad 4 \quad 7} \end{array}$ (see underlined numbers

in the above), we find

$$\min\left(\frac{2}{|-4|}, \infty, \infty\right) = \frac{1}{2} \quad \Rightarrow l = 1, s = 6, c'_s = 2$$

3°

$$\begin{array}{c}a'_s\\\begin{array}{|c|}\hline 3\\ -4\\ 3\\ 1\\\hline\end{array}\end{array} = \begin{array}{c}B^{-1}\\\begin{array}{|rrrr|}\hline -1 & 1 & 3 & -4\\ 1 & 0 & -4 & 3\\ -1 & 0 & 3 & -1\\ 0 & 0 & 1 & -1\\\hline\end{array}\end{array} * \begin{array}{c}a_s\\\begin{array}{|c|}\hline 0\\ 0\\ 1\\ 0\\\hline\end{array}\end{array}, \quad a'_{ks} = -4$$

4° **(a)** $NB = \begin{array}{cccc}1&2&3&4\\\hline 5&6&1&3\\\hline\end{array}$, $\qquad NF = \begin{array}{ccc}1&2&3\\\hline 2&4&7\\\hline\end{array}$

(b) $b'^T = \begin{array}{cccc}1&2&3&4\\\hline 45/2 & 9/2 & 15/2 & 15/2\\\hline\end{array}$

(c) $-z = 180$

$BS = (15/2, 0, 15/2, 0, 45/2, 9/2, 0) = BFS$

(d) $\begin{array}{c}B^{-1}\\\begin{array}{|rrrr|}\hline -1/4 & 1 & 0 & -7/4\\ -1/4 & 0 & 1 & -3/4\\ -1/4 & 0 & 0 & 5/4\\ -1/4 & 0 & 0 & -1/4\\\hline\end{array}\end{array}$

(e) $\begin{array}{c}\pi^T\\\boxed{\begin{array}{c|c|c|c}3/2&0&0&15/2\end{array}}\end{array} := \begin{array}{c}\pi^T\\\boxed{\begin{array}{c|c|c|c}1&0&2&6\end{array}}\end{array} - \begin{array}{c}\beta_k^T\\\boxed{\begin{array}{c|c|c|c}-1/4&0&1&-3/4\end{array}}\end{array} * \begin{array}{c}c'_s\\\boxed{2}\end{array}$

(f) $\begin{array}{c}c'^T\\\boxed{\begin{array}{c|c|c}0&1/2&0\end{array}}\end{array} := \begin{array}{c}\pi^T\\\boxed{\begin{array}{c|c|c|c}3/2&0&0&15/2\end{array}}\end{array} * \begin{array}{c}G\\\begin{array}{|ccc|}\hline 1&2&5\\ 2&3&3\\ 1&1&2\\ 1&1&1\\\hline\end{array}\end{array} + \begin{array}{c}c^T\\\boxed{\begin{array}{c|c|c}-9&-10&-15\end{array}}\end{array}$

Step 4.

1° There are no negative elements in b' $\quad \Rightarrow$ the end.

$BOS = (15/2, 0, 15/2, 0, 45/2, 9/2, 0), \quad z_{\min} = -180$

Chapter 6.

Special Cases

All the above examples and theorems were for nondegenerate problems where we had one and only one solution and no special difficulties arose. The only complication was considered in Section 3.6 where we had no obvious initial basic feasible solution. To cope with this difficulty, we had to solve an auxiliary LP problem with some artificial variables and the artificial objective function. There, in Remark 3.6.4, we mentioned another difficulty: the possibility of contradictory constraints.

In practice, there may arise some other special situations that also complicate matters: degenerate basis, nonlimited feasible region, and nonunique optimal solution. This chapter illustrates such cases because each computer program must be so powerful as to be able to cope with any special situation.

6.1. Contradictory constraints

This means that the feasible region does not exist.

Example 6.1.1. For x_1, $x_2 \geq 0$ solve the problem:

$$
\begin{array}{rrrrcl}
x_1 & + & x_2 & \geq & 3 \\
x_1 & + & 2x_2 & \leq & 2 \\
-x_1 & - & x_2 & = & z & \rightarrow & \min
\end{array}
$$

Step 0.

$$
\rightarrow
\begin{array}{|rrrr|r|}
\hline
-1 & -1 & 1 & 0 & -3 \\
1 & 2 & 0 & 1 & 2 \\
\hline
-1 & -1 & 0 & 0 & 0 \\
\hline
\end{array}
\qquad
\begin{array}{c}
NB \\
\boxed{\begin{array}{c} 3 \\ 4 \end{array}}
\end{array}
\qquad
\begin{array}{c}
NF \\
\boxed{\begin{array}{c} 1 \\ 2 \end{array}}
\end{array}
$$

$$
\uparrow
$$

$$
BS = (0, 0, -3, 2) \neq BFS, \quad z = 0
$$

Step 1.

1° $s=1$

2° $\min(\infty, 2/1) = 2 \quad \Rightarrow \quad k = 2$

3° After normalization: the tableau does not change.

4° After subtractions:

						NB		NF
$\rightarrow$	0	1	1	1	-1	3		4
	1	2	0	1	2	1		2
	0	1	0	1	2			

$$BS = (2, 0, -1, 0) \neq BFS, \quad z = -2$$

There are met the conditions to start the dual simplex method.

Step 2.

1° $k=1$

2° There are no negative numbers in the first four elements of the k-th row. Therefore there is no basic feasible solution to this problem. The end.

Let us show another possible way to build the process of finding the solution in this example.

In attempting to find a basic feasible solution, we introduce an artificial variable x_5 and try to solve, for the first, the auxiliary problem: $w = x_5$, $w \to \min$.

Step 0. Having input the given data, we start from the tableau where the third and fourth columns are for the slack variables x_3 and x_4, and the fifth column is for the artificial variable x_5:

1	1	-1	0	1	3
1	2	0	1	0	2
-1	-1	0	0	0	0
0	0	0	0	1	0

Then we do the preparatory subtraction in order to present the starting tableau in the canonical form with respect to $x^B = (x_4, x_5)$ and $w = x_5$. Subtracting the first row from the bottom row yields

								NB		NF
	1	1	-1	0	1	3		5		1
$\rightarrow$	1	2	0	1	0	2		4		2
	-1	-1	0	0	0	0				3
	-1	-1	1	0	0	-3				
	$\uparrow$									

$$BS = (0, 0, 0, 2, 3) = BFS, \quad z = 0, \quad w = 3$$

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This is the correct initial state, and we are ready to proceed. We begin with Stage I:

Step 1.

1° $l=1$, $s=1$

2° $\min(\frac{3}{1}, \frac{2}{1}) = 2 \qquad \Rightarrow \qquad k = 2$

3° After normalization: the tableau does not change.

4° After subtractions:

							NB		NF
0	−1	−1	−1	1	1		5		4
1	2	0	1	0	2		1		2
0	1	0	1	0	2				3
0	1	1	1	0	−1				

$$BS = (2, 0, 0, 0, 1) = BFS, \quad z = -2, \quad w = 1$$

Step 2.

1° There are no negative numbers in the first five elements of the last row, so the minimum for w has been reached: $w_{\min} = 1$. We have come to case (2) preceding Example 3.6.1: Stage II can not start. This is the case of contradictory constraints. From a slightly different point of view (see Remark 3.6.4), the reason for the difficulty is that the BFS obtained at item 4° (above) contains one of the artificial variables, namely x_5.

6.2. Degenerate basis

Let us recall the formal definition that LP problem $Ax = b$, $c^T x = z \to \min$, is *degenerate* if there exists at least one $m \times m$ submatrix selected from the $m \times (n + 1)$ augmented matrix $[A, b]$ that is singular. One can easily see that the following fact is true.

Theorem 6.2.1. *If a problem is nondegenerate, then the basic variables in **any** basic solution are precisely the nonzero values.*

So, if one encounters a basic solution, — no matter feasible or not, — with at least one zero variable, it is a good marker of problem degeneracy. The fact of degeneracy is dangerous only when we encounter such a basic solution, in other words if we get a zero among elements of the right hand side b' of constraints $A'x = b'$. If then we choose a pivot row to be the row with the zero right hand side, then evidently the objective function remains unchanged, in which case the procedure can cycle indefinitely. It does

not matter whether we have encountered or not a zero in the right hand side of constraints, this must be considered a real possibility in the computing procedure. The procedure must be designed in such a way that to be able to get out of any indefinite cycling.

Let us consider some examples.

Example 6.2.1. For $x_1, x_2 \geq 0$ solve the problem

$$
\begin{array}{rrrcl}
-2x_1 & + & x_2 & \leq & 2 \\
-x_1 & + & 2x_2 & \leq & 4 \\
x_1 & + & x_2 & \leq & 8 \\
x_1 & - & 3x_2 & = & z \rightarrow \min
\end{array}
$$

We start by using the standard simplex method.

Step 0.

<table>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>NB</td><td colspan="2">NF</td></tr>
<tr><td>−2</td><td>1</td><td>1</td><td>0</td><td>0</td><td>2</td><td>3</td><td>1</td><td>1</td></tr>
<tr><td>−1</td><td>2</td><td>0</td><td>1</td><td>0</td><td>4</td><td>4</td><td>2</td><td>2</td></tr>
<tr><td>1</td><td>1</td><td>0</td><td>0</td><td>1</td><td>8</td><td>5</td><td></td><td></td></tr>
<tr><td>1</td><td>−3</td><td>0</td><td>0</td><td>0</td><td>0</td><td></td><td></td><td></td></tr>
</table>

$$BS = (0,0,2,4,8) = BFS, \quad z = 0$$

Step 1.

1° $l = 2$, $s = 2$

2° $\min\left(\frac{2}{1}, \frac{4}{2}, \frac{8}{1}\right) = \min(2, 2, 8)$

Let us arbitrary choose $k = 1$ (although there is another variant $k = 2$).

3° After normalization: the tableau remains unchanged.

4° After subtractions:

<table>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td>NB</td><td colspan="2">NF</td></tr>
<tr><td>−2</td><td>1</td><td>1</td><td>0</td><td>0</td><td>2</td><td>2</td><td>1</td><td>1</td></tr>
<tr><td>3</td><td>0</td><td>−2</td><td>1</td><td>0</td><td>0</td><td>4</td><td>2</td><td>3</td></tr>
<tr><td>3</td><td>0</td><td>−1</td><td>0</td><td>1</td><td>6</td><td>5</td><td></td><td></td></tr>
<tr><td>−5</td><td>0</td><td>3</td><td>0</td><td>0</td><td>6</td><td></td><td></td><td></td></tr>
</table>

$\uparrow$

$$BS = (0, 2, 0, \underline{0}, 6) = BFS, \quad z = -6$$

The basic variable $x_4 = 0$ (this value is scored under in the last expression).

Step 2.

1° $l = 1$, $s = 1$.

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2° $\min(\infty, -\infty, 6/3)$. In compliance with Section 3.4 and Table 3.4.1, the situation marked by $-\infty$ ("prohibited") requires to reconsider the choice of s. Because at this step we have no another choice in the matter, we return to the preceding Step 1. There we have a possibility to make another choice: $k = 2$. As this Step 2 will be resumed again after the changed Step 1 (marked below as Step 1'), we shall defer our discussion until then.

Step 1'.

1° $l = 2$, $s = 2$

2° $\min\left(\frac{2}{1}, \frac{4}{2}, \frac{8}{1}\right) = \min(2, 2, 8)$. Now let us choose $k = 2$.

3° After normalization: **4°** After subtractions:

-2	1	1	0	0	2
$-1/2$	1	0	$1/2$	0	2
1	1	0	0	1	8
1	-3	0	0	0	0

$\uparrow$

$-3/2$	0	1	$-1/2$	0	0
$-1/2$	1	0	$1/2$	0	2
$3/2$	0	0	$-1/2$	1	6
$-1/2$	0	0	$3/2$	0	6

$\uparrow$

NB	NF
3	1 $\boxed{1}$
2	2 $\boxed{4}$
5	

$$BS = (0, 2, \underline{0}, 0, 6) = BFS, \quad z = -6$$

The basic variable $x_3 = 0$ (scored under in the last expression).

Step 2'.

1° $l = 1$, $s = 1$

2° $\min\left(\infty, \infty, \frac{6}{3/2}\right) \quad \Rightarrow \quad k = 3$

3° After normalization: **4°** After subtractions:

$-3/2$	0	1	$-1/2$	0	0
$-1/2$	1	0	$1/2$	0	2
1	0	0	$-1/3$	$2/3$	4
$-1/2$	0	0	$3/2$	0	6

$\uparrow$

0	0	1	-1	1	6
0	1	0	$1/3$	$1/3$	4
1	0	0	$-1/3$	$2/3$	4
0	0	0	$4/3$	$1/3$	8

NB	NF
3	1 $\boxed{5}$
2	2 $\boxed{4}$
1	

$$BS = (4, 4, 6, 0, 0) = BOS, \quad z_{\min} = -8$$

As it is seen, the above response to situation "prohibited" (marked by $-\infty$) has given the possibility to finish the solution process successfully. Let us see what will be if the possibility is ignored by resuming Step 2.

Step 2.

$1°$ $l = 1$, $s = 1$

$2°$ $\min(\infty, -\infty, 6/3)$. This time let us choose $k = 2$

$3°$ After normalization: $4°$ After subtractions:

$$\to \begin{array}{ccccc|c} -2 & 1 & 1 & 0 & 0 & 2 \\ 1 & 0 & -2/3 & 1/3 & 0 & 0 \\ 3 & 0 & -1 & 0 & 1 & 6 \\ \hline -5 & 0 & 3 & 0 & 0 & 6 \end{array} \qquad \to \begin{array}{ccccc|c} 0 & 1 & -1/3 & 2/3 & 0 & 2 \\ 1 & 0 & -2/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 6 \\ \hline 0 & 0 & -1/3 & 5/3 & 0 & 6 \end{array} \quad \begin{array}{c} NB \\ \hline 2 \\ 1 \\ 5 \end{array} \quad \begin{array}{c} NF \\ \hline 4 \\ 3 \end{array}$$

$$BS = (\underline{0}, 2, 0, 0, 6) = BFS, \quad z = -6$$

The basic variable $x_1 = 0$ (scored under in the above).

Step 3.

$1°$ $l = 2$, $s = 3$

$2°$ $\min(\infty, \infty, 6/1)$. Let us choose $k = 2$, contrary to the rules of Table 3.4.1.

$3°$ After normalization: $4°$ After subtractions:

$$\to \begin{array}{ccccc|c} 0 & 1 & -1/3 & 2/3 & 0 & 2 \\ -3/2 & 0 & 1 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 6 \\ \hline 0 & 0 & -1/3 & 5/3 & 0 & 6 \end{array} \qquad \to \begin{array}{ccccc|c} -1/2 & 1 & 0 & 1/2 & 0 & 2 \\ -3/2 & 0 & 1 & -1/2 & 0 & 0 \\ 3/2 & 0 & 0 & -1/2 & 1 & 6 \\ \hline -1/2 & 0 & 0 & 3/2 & 0 & 6 \end{array} \quad \begin{array}{c} NB \\ \hline 2 \\ 3 \\ 5 \end{array} \quad \begin{array}{c} NF \\ \hline 4 \\ 1 \end{array}$$

$$BS = (0, 2, \underline{0}, 0, 6) = BFS, \quad z = -6$$

The basic variable $x_3 = 0$ (scored under in the above).

Step 4.

$1°$ $l = 2$, $s = 1$

$2°$ $\min\left(\infty, \infty, \frac{6}{3/2}\right)$. Let us choose $k = 2$, contrary to the rules of Table 3.4.1.

3° After normalization: **4°** After subtractions:

							NB	NF
$-1/2$	1	0	$1/2$	0	2			
1	0	$-2/3$	$1/3$	0	0			
$3/2$	0	0	$-1/2$	1	6			
$-1/2$	0	0	$3/2$	0	6			

0	1	$-1/3$	$2/3$	0	2	2	4
1	0	$-2/3$	$1/3$	0	0	1	3
0	0	1	-1	1	6	5	
0	0	$-1/3$	$5/3$	0	6		

$$BS = (\underline{0}, 2, 0, 0, 6) = BFS, \quad z = -6$$

The basic variable $x_1 = 0$ (scored under in the above).

Hence we have returned to the state of Step 2 and we shall repeat Steps 2 and 3 indefinitely. This is the case of **cycling.** The objective reason of cycling is that several hyperplanes representing different constraints go through one and the same point in the space of variables $x = (x_1, \ldots, x_n)$. If this point lies in the feasible region, two or more vertices of the feasible hyperpolyhedron coincide. This situation takes place in the above example where three several lines meet at point $(x_1, x_2) = (0, 2)$:

(1) $x_1 = 0$
(2) $x_3 = 0$, i.e., $-2x_1 + x_2 = 2$
(3) $x_4 = 0$, i.e., $-x_1 + 2x_2 = 4$

Hence three vertices coincide: the intersection of lines (1) and (2), the intersection of lines (1) and (3), and the intersection of lines (2) and (3). Clearly, the objective function has one and the same value, $z = -6$, at these vertices. We change from one vertex to the other remaining actually at one and the same point if we act as in the above Steps 3 and 4. These changes are shown in Fig. 6.2.1.

As it is seen, unsuccessful are actions at Steps 2, 3 and 4. Common for them is that the key row selected for normalization, its number $k = 2$, has the zero value $b'_k = 0$ for the basic variable of the preceding step: for Step 2 the equality $b'_k = 0$ means $x_4 = 0$; for Step 3 the equality $b'_k = 0$ means $x_1 = 0$; and for Step 4 the equality $b'_k = 0$ means $x_3 = 0$.

The basis in which at least one basic variable equals zero is called the **degenarate basis**. Hence namely it is a potential cause of cycling, and this cause becomes actual if one acts in such a manner.

The computer program must be so powerful as to avoid the possible cycling. Dantzig suggests to slightly change the right hand sides of constraints by adding the small values ε^i where $i = 1, 2, \ldots, m$ and i is the constraint number [1], [2]. Due to the adding, the primarily coincided vertices are being "pulled apart", but then one should let ε be equal zero in the solution found.

Another way to avoid cycling has been shown yet in this example: Step 1' and then Step 2' (see Fig 6.2.1). Note that in itself Step 1 leading to state S_1 does not contain any unsuccessful actions, but Step 2 leads us up the cycle $S_2 \leftrightarrow S_3$ (see Fig. 6.2.1). Therefore one can avoid getting into the cycle if after Step 1 (and state S_1) one will act as follows:

Step 1'

S_0

$z = 0$

S_1

$x_4 = 0$
$z = -6$

Step 2

S_2

$x_1 = 0$
$z = -6$

Step 3

S_3

$x_3 = 0$
$z = -6$

Step 2'

S_*

BOS
$z = -8$

Step 4

S_4

Step 2"

$z = -16$

Step 3"
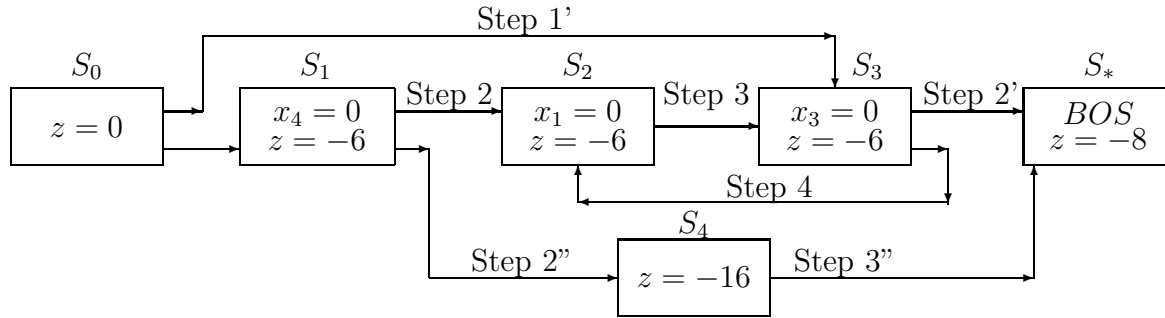
Fig. 6.2.1. Graph of the solution with possible cycling: S_0 is the initial state; S_* is the optimal state; S_1, S_2, S_3 are states in one of three coinciding vertices where one of basic variables, correspondingly x_4, x_1 or x_3, is zero; S_4 is the state out of feasible region.

Step 2".

1° $l = 1$, $s = 1$

2° $\min(\infty, -\infty, 6/3)$. Choose $k = 3$.

3° After normalization: **4°** After subtractions:

$$
\rightarrow
\begin{array}{|ccccc|c|}
-2 & 1 & 1 & 0 & 0 & 2 \\
3 & 0 & -2 & 1 & 0 & 0 \\
1 & 0 & -1/3 & 0 & 1/3 & 2 \\
-5 & 0 & 3 & 0 & 0 & 6 \\
\end{array}
\qquad \rightarrow \qquad
\begin{array}{|ccccc|c|}
0 & 1 & 1/3 & 0 & 2/3 & 6 \\
0 & 0 & -1 & 1 & -1 & -6 \\
1 & 0 & -1/3 & 0 & 1/3 & 2 \\
0 & 0 & 4/3 & 0 & 5/3 & 16 \\
\end{array}
$$

NB
$\begin{array}{c} 2 \\ 4 \\ 1 \end{array}$

NF
$\begin{array}{cc} 1 & 5 \\ 2 & 3 \end{array}$

$$BS = (2, 6, 0, -6, 0) \neq BFS, \quad z = -16 < z_{\min}$$

By doing so, we get out of the feasible region into the point of intersection of two lines: $-2x_1 + x_2 = 2$ (as $x_3 = 0$) and $x_1 + x_2 = 8$ (as $x_5 = 0$). We are able to return from there and, by so doing, to track down the BOS (basic optimal solution) but only by the dual simplex method. (The dual method applicability conditions are being recognized as fulfilled along the bottom row and the right hand column of the tableau.)

Step 3".

1° $k = 2$

2° $\min\left(\frac{4/3}{|-1|}, \frac{5/3}{|-1|}\right) = 4/3 \qquad \Rightarrow \qquad l = 2, s = 3$

3° After normalization: **4°** After subtractions:

0	1	1/3	0	2/3	6
0	0	1	−1	1	6
1	0	−1/3	0	1/3	2
0	0	4/3	0	5/3	16

$\rightarrow$ (arrow ↑ under the third column of the left table)

							NB		NF
0	1	0	1/3	1/3	4		2		5
0	0	1	−1	1	6		3		4
1	0	0	−1/3	2/3	4		1		
0	0	0	4/3	1/3	8				

$$BS = (4, 4, 6, 0, 0) = BOS, \quad z_{\min} = -8$$

In conclusion, note that even if we found ourselves in a cycle (in the example, states $S_2 \leftrightarrow S_3$), it is possible to get out from there if we act properly. In the example, it is evident that state S_3 after Step 3 is identical to the state obtained after Step 1'. Hence if we execute Step 2' after Step 3, then the exit from the cycle is guaranteed. Just the same, we can reach the final (optimal) state S_* after state S_2, too, if we do well, i.e. if we do not permit such a hoice of k for which $b'_k = 0$, for example:

Step 3' (after Step 2).

  1° $l = 2$, $s = 3$

  2° $\min(\infty, \infty, 6/1)$. $\Rightarrow$ $k = 3$

  3° After normalization: **4°** After subtractions:

0	1	−1/3	2/3	0	2
1	0	−2/3	1/3	0	0
0	0	1	−1	1	6
0	0	−1/3	5/3	0	6

$\rightarrow$ (arrow ↑ under the third column of the left table)

							NB		NF
0	1	0	1/3	1/3	4		2		4
1	0	0	−1/3	2/3	4		1		5
0	0	1	−1	1	6		3		
0	0	0	4/3	1/3	8				

$$BS = (4, 4, 6, 0, 0) = BOS, \quad z = -8 = z_{\min}$$

Exercise 6.1.1. E.M.L. Beale has invented another example with possible cycling [3]

$$
\begin{aligned}
1/4 x_1 &- 8 x_2 &- &x_3 &+ 9 x_4 &\leq 0, \\
1/2 x_2 &- 12 x_2 &- 1/2 &x_3 &+ 3 x_4 &\leq 0, \\
& & &x_3 & &\leq 1, \\
-3/4 x_1 &+ 20 x_2 &- 1/2 &x_3 &+ 6 x_4 &= z \rightarrow \min
\end{aligned}
$$

Number the constraints in the same order and choose pairs (k, s) step by step as follows: (1, 1); (2, 2); (1, 3); (2, 4); (1, 5); and (2, 6). Make sure that after Step 6 the problem returns to the initial state S_0 that was before Step 1 (see Fig. 6.2.2). In doing so, state S_0 has a pair of basic variables, x_5 and x_6, both equal zero. Further, as the above mentioned

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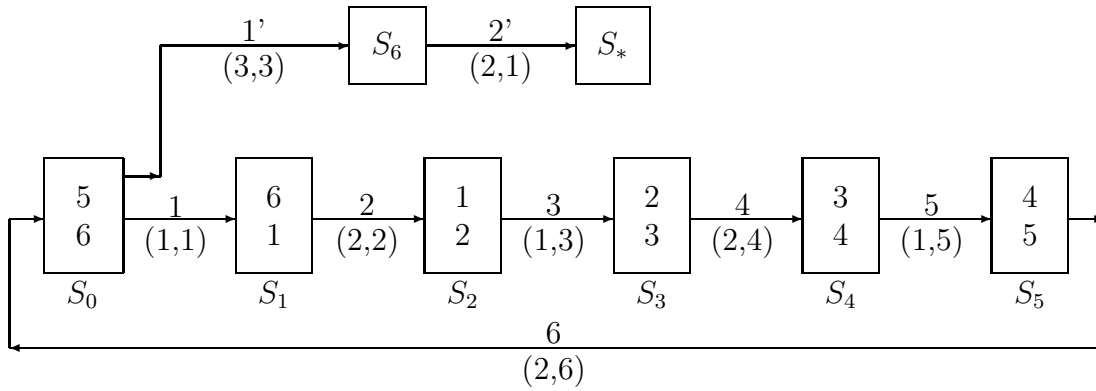
Fig. 6.2.2. Cycling and the exit from cycling in the problem by E.M.L. Beale

steps are executed, the pairs of zero basic variables appear according to Fig. 6.2.2 (where the numbers of zero basic variables are shown within the boxes). If however one chooses the pairs (k, s) in another way, namely: at Step 1' $(k, s) = (3, 3)$; and at Step 2' $(k, s) = (2, 1)$, then the appearing states S_6 and S_* have no zero basic variables, and S_* provides the sought (optimal) solution: $BOS = (1, 0, 1, 0, 3/4, 0, 0)$ and $z_{\min} = -5/4$.

6.3. Unbounded feasible region

This is the situation in Theorem 3.2.1 (ii) (a) showing that the objective function can be made as small as desired and hence no solution exists.

Example 6.3.1. For x_1, $x_2 \geq 0$ solve the problem:

$$
\begin{array}{rrrcll}
2x_1 & + & x_2 & \geq & 7 & \\
x_1 & + & x_2 & \geq & 4 & \\
x_1 & + & 3x_2 & \geq & 9 & \\
-2x_1 & - & 3x_2 & = & z & \rightarrow \quad \min
\end{array}
$$

Step 0.

2	1	−1	0	0	1	0	0	7
1	1	0	−1	0	0	1	0	4
1	3	0	0	−1	0	0	1	9
−2	−3	0	0	0	0	0	0	0
0	0	0	0	0	1	1	1	0

The last row is introduced for the auxiliary problem $w = x_6 + x_7 + x_8 \rightarrow \min$ with the artificial variables x_6, x_7, x_8 and artificial objective function w. At this step we do the preparatory subtraction to represent the tableau in the canonical form for a basis:

2	1	−1	0	0	1	0	0	7
1	1	0	−1	0	0	1	0	4
1	3	0	0	−1	0	0	1	9
−2	−3	0	0	0	0	0	0	0
−4	−5	1	1	1	0	0	0	−20

$\rightarrow$ (arrow pointing at top-left block)

NB
6
7
8

NF
1
2
3
4
5

$$BS = (0,0,0,0,0,7,4,9) = BFS, \quad z = 0, \quad w = 20$$

Stage I (solving the auxiliary LP problem).

Step 1.

1° $l = 2$, $s = 2$

2° $\min(\frac{7}{1}, \frac{4}{1}, \frac{9}{3}) = 3 \qquad \Rightarrow \qquad k = 3$

3° After normalization:

2	1	−1	0	0	1	0	0	7
1	1	0	−1	0	0	1	0	4
1/3	1	0	0	−1/3	0	0	1/3	3
−2	−3	0	0	0	0	0	0	0
−4	−5	1	1	1	0	0	0	−20

$\rightarrow$

4° After subtractions:

5/3	0	−1	0	1/3	1	0	−1/3	4
2/3	0	0	−1	1/3	0	1	−1/3	1
1/3	1	0	0	−1/3	0	0	1/3	3
−1	0	0	0	−1	0	0	1	9
−7/3	0	1	1	−2/3	0	0	5/3	−5

$\rightarrow$

NB
6
7
2

NF
1
8
3
4
5

$$BS = (0,3,0,0,0,4,1,0) \quad z = -9, \quad w = 5$$

Step 2.

1° $l = 1$, $s = 1$

2° $\min(\frac{4}{5/3}, \frac{1}{2/3}, \frac{3}{1/3}) = \frac{3}{2} \qquad \Rightarrow \qquad k = 2$

3° After normalization:

$$
\rightarrow
\begin{array}{|ccccccccc|}
\hline
5/3 & 0 & -1 & 0 & 1/3 & 1 & 0 & -1/3 & 4 \\
1 & 0 & 0 & -3/2 & 1/2 & 0 & 3/2 & -1/2 & 3/2 \\
1/3 & 1 & 0 & 0 & -1/3 & 0 & 0 & 1/3 & 3 \\
\hline
-1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 9 \\
\hline
-7/3 & 0 & 1 & 1 & -2/3 & 0 & 0 & 5/3 & -5 \\
\hline
\end{array}
$$

$$\uparrow$$

4° After subtractions:

$$
\begin{array}{cc}
& NB \quad\quad NF
\end{array}
$$

$$
\rightarrow
\begin{array}{|cccccccc|c|}
\hline
0 & 0 & -1 & 5/2 & -1/2 & 1 & -5/2 & 1/2 & 3/2 \\
1 & 0 & 0 & -3/2 & 1/2 & 0 & 3/2 & -1/2 & 3/2 \\
0 & 1 & 0 & 1/2 & -1/2 & 0 & -1/2 & 1/2 & 5/2 \\
\hline
0 & 0 & 0 & -3/2 & -1/2 & 0 & 3/2 & 1/2 & 21/2 \\
\hline
0 & 0 & 1 & -5/2 & 1/2 & 0 & 7/2 & 1/3 & -3/2 \\
\hline
\end{array}
\quad
\begin{array}{|c|}
\hline 6 \\ 1 \\ 2 \\ \hline
\end{array}
\quad
\begin{array}{|c|}
\hline 7 \\ 8 \\ 3 \\ 4 \\ 5 \\ \hline
\end{array}
$$

$$\uparrow$$

$$
BS = \left(\frac{3}{2}, \frac{5}{2}, 0, 0, 0, \frac{3}{2}, 0, 0\right) \quad z = -21/2, \quad w = 3/2
$$

Step 3.

1° $l = 4,\ s = 4$

2° $\min(\frac{3/2}{5/2}, \infty, \frac{5/2}{1/2}) = \frac{3}{5} \qquad \Rightarrow \quad k = 1$

3° After normalization:

$$
\rightarrow
\begin{array}{|cccccccc|c|}
\hline
0 & 0 & -2/5 & 1 & -1/5 & 2/5 & -1 & 1/5 & 3/5 \\
1 & 0 & 0 & -3/2 & 1/2 & 0 & 3/2 & -1/2 & 3/2 \\
0 & 1 & 0 & 1/2 & -1/2 & 0 & -1/2 & 1/2 & 5/2 \\
\hline
0 & 0 & 0 & -3/2 & -1/2 & 0 & 3/2 & 1/2 & 21/2 \\
\hline
0 & 0 & 1 & -5/2 & 1/2 & 0 & 7/2 & 1/2 & -3/2 \\
\hline
\end{array}
$$

$$\uparrow$$

4° After subtractions:

$$
\begin{array}{cc}
NB \quad\quad NF
\end{array}
$$

$$
\begin{array}{|ccccccccc|}
\hline
0 & 1 & -2/5 & 1 & -1/5 & 2/5 & -1 & 1/5 & 3/5 \\
1 & 0 & -3/5 & 0 & 1/5 & 3/5 & 0 & -1/5 & 12/5 \\
0 & 1 & 1/5 & 0 & -2/5 & -1/5 & 0 & 2/5 & 11/5 \\
\hline
0 & 0 & -3/5 & 0 & -4/5 & 3/5 & 0 & 4/5 & 57/5 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\hline
\end{array}
\quad
\begin{array}{|c|}
\hline 4 \\ 1 \\ 2 \\ \hline
\end{array}
\quad
\begin{array}{|c|}
\hline 7 \\ 8 \\ 3 \\ 6 \\ 5 \\ \hline
\end{array}
$$

$$
BS = \left(\frac{12}{5}, \frac{11}{5}, 0, \frac{3}{5}, 0, 0, 0, 0\right) \quad z = -57/5, \quad w = 0
$$

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Stage I has been successfully completed, and the artificial variables x_6, x_7, x_8, as well as the artificial objective function w, should be deleted. As a result, we begin the next stage, Stage II, with the following tableau:

$$
\begin{array}{|ccccc|c|} \hline
0 & 0 & -2/5 & 1 & -1/5 & 3/5 \\
1 & 0 & -3/5 & 0 & 1/5 & 12/5 \\
0 & 1 & 1/5 & 0 & -2/5 & 11/5 \\ \hline
0 & 0 & -3/5 & 0 & -4/5 & 57/5 \\ \hline
\end{array}
\qquad
\begin{array}{c}
NB \\ \boxed{\begin{array}{c} 4 \\ 1 \\ 2 \end{array}}
\end{array}
\qquad
\begin{array}{c}
NF \\ \boxed{\begin{array}{c} 3 \\ 5 \end{array}}
\end{array}
$$

with $\rightarrow$ on the second row and $\uparrow$ below the fifth column.

Stage II (solving the given LP problem).

Step 4.

1° $l = 2$, $s = 5$

2° $\min(\infty, \frac{12/5}{1/5}, \infty) = 12 \qquad \Rightarrow \qquad k = 2$

3° After normalization:

$$
\begin{array}{|ccccc|c|} \hline
0 & 0 & -2/5 & 1 & -1/5 & 3/5 \\
5 & 0 & -3 & 0 & 1 & 12 \\
0 & 1 & 1/5 & 0 & -2/5 & 11/5 \\ \hline
0 & 0 & -3/5 & 0 & -4/5 & 57/5 \\ \hline
\end{array}
$$

with $\rightarrow$ on the second row and $\uparrow$ below the fifth column.

4° After subtractions:

$$
\begin{array}{|ccccc|c|} \hline
1 & 0 & -1 & 1 & 0 & 3 \\
5 & 0 & -3 & 0 & 1 & 12 \\
2 & 1 & -1 & 0 & 0 & 7 \\ \hline
4 & 0 & -3 & 0 & 0 & 21 \\ \hline
\end{array}
\qquad
\begin{array}{c}
NB \\ \boxed{\begin{array}{c} 4 \\ 5 \\ 2 \end{array}}
\end{array}
\qquad
\begin{array}{c}
NF \\ \boxed{\begin{array}{c} 3 \\ 1 \end{array}}
\end{array}
$$

$$BS = (0, 7, 0, 3, 12) \quad z = -21$$

Step 5.

1° $l = 1$, $s = 3$

2° $\min(\infty, \infty, \infty) = \infty \qquad \Rightarrow \qquad$ the end

The appearance of indicators "∞" for all constraints evidences the unbounded feasible region.

If one continue from this point with an arbitrary choice of the pivot in the third column, then the situation of all "∞" will repeat and the indefinite cycling is possible. To show this, let us consider another (more simple in calculation) example.

Example 6.3.2. For x_1, $x_2 \geq 0$ solve the problem:

$$
\begin{array}{rcrcll}
x_1 & - & x_2 & \geq & 1 & \\
 & & x_2 & \leq & 2 & \\
-x_1 & - & x_2 & = & z & \rightarrow \quad \min
\end{array}
$$

Step 0.

$$
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{|rrrr|r|}
\hline
1 & -1 & -1 & 0 & 1 \\
0 & 1 & 0 & 1 & 2 \\
\hline
-1 & -1 & 0 & 0 & 0 \\
\hline
\end{array}
\qquad
\begin{array}{cc}
NB & NF \\
\boxed{\begin{array}{c} 3 \\ 4 \end{array}} & \boxed{\begin{array}{c} 1 \\ 2 \end{array}}
\end{array}
$$

$$BS = (0, 0, -1, 2) \neq BFS, \quad z = 0$$

Step 1.

1° $s = 1$

2° $\min(\infty, \infty) = \infty \qquad \Rightarrow \quad k = 1$

3° After normalization: the tableau does not change.

4° After subtractions:

$$
\begin{array}{c}
NB \quad NF \\
\rightarrow
\begin{array}{|rrrr|r|}
\hline
1 & -1 & -1 & 0 & 1 \\
0 & 1 & 0 & 1 & 2 \\
\hline
0 & -2 & -1 & 0 & 1 \\
\hline
\end{array}
\quad
\boxed{\begin{array}{c} 1 \\ 4 \end{array}} \quad \boxed{\begin{array}{c} 3 \\ 2 \end{array}}
\end{array}
$$

$$BS = (1, 0, 0, 2) = BFS, \quad z = -1$$

Step 2.

1° $s = 2$

2° $\min(\infty, 2/1), \qquad \Rightarrow \quad k = 2$

3° After normalization: the tableau does not change.

4° After subtractions:

$$
\begin{array}{c}
NB \quad NF \\
\rightarrow
\begin{array}{|rrrr|r|}
\hline
1 & 0 & -1 & 1 & 3 \\
0 & 1 & 0 & 1 & 2 \\
\hline
0 & 0 & -1 & 2 & 5 \\
\hline
\end{array}
\quad
\boxed{\begin{array}{c} 1 \\ 2 \end{array}} \quad \boxed{\begin{array}{c} 3 \\ 4 \end{array}}
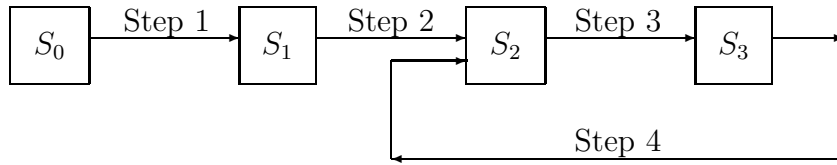\end{array}
$$

$$BS = (3, 2, 0, 0) = BFS, \quad z = -5$$

Fig. 6.3.1. Cycling in Example 6.3.2.

Step 3.

$1°$ $s = 3$

$2°$ $\min(\infty, \infty),$ $\Rightarrow$ $k = 1$

$3°$ After normalization: $\qquad\qquad\qquad$ $4°$ After subtractions:

$$\begin{array}{c} NB \quad NF \end{array}$$

$$\rightarrow \begin{array}{|cccc|c|} \hline -1 & 0 & 1 & -1 & -3 \\ 0 & 1 & 0 & 1 & 2 \\ \hline 0 & 0 & -1 & 2 & 5 \\ \hline \end{array} \qquad \rightarrow \begin{array}{|cccc|c|} \hline -1 & 0 & 1 & -1 & -3 \\ 0 & 1 & 0 & 1 & 2 \\ \hline -1 & 0 & 0 & 1 & 2 \\ \hline \end{array} \begin{array}{|c|} \hline 3 \\ 2 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ 4 \\ \hline \end{array}$$

$$\qquad\qquad\uparrow \qquad\qquad\qquad\qquad\qquad \uparrow$$

$$BS = (0, 2, -3, 0) \neq BFS, \quad z = -2$$

Step 4.

$1°$ $s = 1$

$2°$ $\min(\infty, \infty),$ $\Rightarrow$ $k = 1$

$3°$ After normalization: $\qquad\qquad\qquad$ $4°$ After subtractions:

$$\begin{array}{c} NB \quad NF \end{array}$$

$$\rightarrow \begin{array}{|cccc|c|} \hline 1 & 0 & -1 & 1 & 3 \\ 0 & 1 & 0 & 1 & 2 \\ \hline -1 & 0 & 0 & 1 & 2 \\ \hline \end{array} \qquad \rightarrow \begin{array}{|cccc|c|} \hline 1 & 0 & -1 & 1 & 3 \\ 0 & 1 & 0 & 1 & 2 \\ \hline 0 & 0 & -1 & 2 & 5 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ 2 \\ \hline \end{array} \begin{array}{|c|} \hline 3 \\ 4 \\ \hline \end{array}$$

$$\quad\uparrow$$

$$BS = (3, 2, 0, 0) = BFS, \quad z = -5$$

The indefinite cycling is evident and shown in Fig. 6.3.1. To avoid cycling, one should stop when indicators "∞" appear for all constraints.

6.4. Nonunique optimal solutions

Example 6.4.1. For x_1, $x_2 \geq 0$ solve the problem:

$$
\begin{array}{rcrcl}
2x_1 & + & x_2 & \leq & 7 \\
x_1 & + & x_2 & \leq & 4 \\
x_1 & + & 3x_2 & \leq & 9 \\
-x_1 & - & x_2 & = & z \quad \rightarrow \quad \min
\end{array}
$$

Step 0.

$$
\begin{array}{|ccccc|c|}
\hline
2 & 1 & 1 & 0 & 0 & 7 \\
1 & 1 & 0 & 1 & 0 & 4 \\
1 & 3 & 0 & 0 & 1 & 9 \\
\hline
-1 & -1 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
$$

NB : 3, 4, 5 $\quad NF$: 1, 2

$$BS = (0, 0, 7, 4, 9) = BFS, \quad z = 0$$

Step 1.

1° $l = 2$, $s = 2$

2° $\min(7/1, 4/1, 9/3) = 3 \quad \Rightarrow \quad k = 3$

3° After normalization: **4°** After subtractions:

$$
\begin{array}{|ccccc|c|}
\hline
2 & 1 & 1 & 0 & 0 & 7 \\
1 & 1 & 0 & 1 & 0 & 4 \\
1/3 & 1 & 0 & 0 & 1/3 & 3 \\
\hline
-1 & -1 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
\quad \rightarrow \quad
\begin{array}{|ccccc|c|}
\hline
5/3 & 0 & 1 & 0 & -1/3 & 4 \\
2/3 & 0 & 0 & 1 & -1/3 & 1 \\
1/3 & 1 & 0 & 0 & 1/3 & 3 \\
\hline
-2/3 & 0 & 0 & 0 & 1/3 & 3 \\
\hline
\end{array}
$$

NB : 3, 4, 2 $\quad NF$: 1, 5

$$BS = (0, 3, 4, 1, 0), \quad z = -3$$

Step 2.

1° $l = 1$, $s = 1$

2° $\min\left(\frac{4}{5/3}, \frac{1}{2/3}, \frac{3}{1/3}\right) = \frac{3}{2} \quad \Rightarrow \quad k = 2$

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3° After normalization:

5/3	0	1	0	−1/3	4
1	0	0	3/2	−1/2	3/2
1/3	1	0	0	1/3	3
−2/3	0	0	0	1/3	3

$\rightarrow$ above; the ↑ under column 1.

4° After subtractions:

0	0	1	−5/2	1/2	3/2	3	5
1	0	0	3/2	−1/2	3/2	1	4
0	1	0	−1/2	1/2	5/2	2	
0	0	0	1	0	4		

$$BS = (3/2, 5/2, 3/2, 0, 0), \quad z = -4$$

This is the situation in Theorem 3.2.1 (iii) detected by a zero in the last row for a nonbasic variable, here x_5, when no negative numbers exist in the elements of this row. Here by pivoting on a nonnegative element of the fifth column, we can obtain a second feasible solution from the last tableau without changing the value of z, evidently due to the above mentioned zero.

Step 3.

1° $l = 2$, $s = 5$

2° $\min(\frac{3/2}{1/2}, \infty, \frac{5/2}{1/2}) = 3 \quad \Rightarrow \quad k = 1$

3° After normalization:

0	0	2	−5	1	3
1	0	0	3/2	−1/2	3/2
0	1	0	−1/2	1/2	5/2
0	0	0	1	0	4

4° After subtractions:

						NB	NF
0	0	2	−5	1	3	5	4
1	0	1	−1	0	3	1	3
0	1	−1	2	0	1	2	
0	0	0	1	0	4		

$$BS = (3, 1, 0, 0, 3), \quad z = -4$$

The situation is repeated, now with x_3. If we proceed, we have:

Step 4.

1° $l = 2$, $s = 3$

2° $\min(\frac{3}{2}, \frac{3}{1}, \infty) = 3/2 \quad \Rightarrow \quad k = 1$

3° After normalization:

0	0	1	−5/2	1/2	3/2
1	0	1	−1	0	3
0	1	−1	2	0	1
0	0	0	1	0	4

4° After subtractions:

						NB	NF
0	0	1	−5/2	1/2	3/2	3	4
1	0	0	3/2	−1/2	3/2	1	5
0	1	0	−1/2	1/2	5/2	2	
0	0	0	1	0	4		

$$BS = (3/2, 5/2, 3/2, 0, 0), \quad z = -4$$

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This is identical to the state after Step 2, so one can see the cycling between Step 3 and Step 4. Naturally, there is no need to continue. The basic optimal solution is given by the convex span of the solutions obtained within the cycle. Here we have

$$BOS = \left(3 - \frac{3}{2}\theta,\ 1 + \frac{3}{2}\theta,\ \frac{3}{2}\theta,\ 0,\ 3 - 3\theta\right)$$

where $0 \leq \theta \leq 1$.

Remark 6.4.1. *From the above, one can see that the cycling is possible not only for degenerate problems.*

Chapter 7.

Case Study Assignments

All the offered assignments are based on the above considered algorithms of the simplex method. We have ten basic algorithms numbered by figures from 0 to 9. Corresponding variants are listed below:

0. The standard simplex method with the obvious initial basic solution and with the correct form of a basis. Permissible for programming constraints are "$\geq$" and "$\leq$". Before inputing a problem, constraints "$\geq$" should be replaced by "$\leq$" using a premultiplier -1. The algorithm is described in Section 3.4.

1. The standard simplex method with the obvious initial basic solution and without the correct form of a basis. Permissible for programming are constraints "$\geq$" and "$\leq$". This algorithm is described in Section 3.5.

2. The standard simplex method with generation the basic feasible solution by using artificial variables and artificial objective function, with the correct form of a basis. This algorithm is described in Section 3.6. It is subclassified into the following 5 assignment variants marked by the second figure in the variant number:

   2.1. Permissible for programming are constraints "$\geq$". Before inputing a problem, constraints "$\leq$" should be replaced by "$\geq$" using a premultiplier -1.

   2.2. Permissible for programming are constraints "$\geq$" and "$\leq$".

   2.3. Permissible for programming are constraints "$=$" and "$\leq$". Constraints "$\geq$" should be replaced by "$\leq$" using a premultiplier -1 before inputing a problem into computer.

   2.4. Permissible for programming are constraints "$=$" and "$\geq$". Constraints "$\leq$" should be replaced by "$\geq$" using a premultiplier -1 before inputing a problem into computer.

   2.5. Permissible for programming are constraints "$=$", "$\geq$" and "$\leq$".

3. The dual simplex method with the obvious initial basic solution and with the correct form of a basis. Permissible for programming are the same constraints as in variant 0. This algorithm is described in Section 4.1.

4. The dual simplex method with the obvious initial basic solution and without the correct form of a basis. Permissible for programming are the same constraints as in variant 1. This algorithm is described in Section 4.2.

5. The dual simplex method with artificial variables and artificial objective function (for constraints "=" only) used to generate a basic feasible solution without the correct form for a basis. This algorithm is described in Section 4.3. It is classified into the following 3 assignment variants marked by the second figure in the variant number:

5.1. Permissible for programming are constraints "=" and "≤". Constraints "≥" should be replaced by "≤" using a premultiplier −1 before inputing a problem into computer.

5.2. Permissible for programming are constraints "=" and "≥". Constraints "≤" should be replaced by "≥" using a premultiplier −1 before inputing a problem into computer.

5.3. Permissible for programming are constraints "=", "≥" and "≤".

6. The revised (conventional) simplex method with the obvious basic solution. Permissible for programming are the same constraints as in variant 0. This algorithm is described in Section 5.4.

7. The revised (conventional) simplex method with generation the basic feasible solution by using artificial variables and artificial objective function. This algorithm is described in Section 5.6. It is classified into the following 5 assignment variants marked by the second figure in the variant number:

7.1. Corresponds to 2.1.

7.2. Corresponds to 2.2.

7.3. Corresponds to 2.3.

7.4. Corresponds to 2.4.

7.5. Corresponds to 2.5.

8. The revised dual simplex method with the obvious initial basic solution. Permissible for programming are the same constraints as in variant 0. This algorithm is described in Section 5.5.

9. The revised dual simplex method with artificial variables and artificial objective function (for constraints "=" only) used to generate a basic (non-feasible) solution. This algorithm is described in Section 5.7. Permissible for programming are the same constraints as in variant 5.1.

Remark 7.1.2. *Variants 6 to 9 (the revised simplex method) require the correct form of a basis: amongst the coefficients for basic variables there are "+1"s only and no "−1"s.*

Thus we have the following 20 assignment variants of the first level of complexity (level 1):

0	1	2.1	2.2	2.3	2.4	2.5	3	4	5.1
5.2	5.3	6	7.1	7.2	7.3	7.4	7.5	8	9

The next 50 variants will be obtained by placing a requirement on the program design to implement "What if ..." mode of operation. This means that, having solved a problem within any variant of the first level of complexity, the program must suggest to continue the solution by introducing one more constraint. The type of this additional constraint is identified by the third figure added to any of the above 20 variant numbers according to the following table:

$\leq$	$\geq$	$=$
1	2	3

From these 50 variants the following 30 variants are rated in the advanced complexity class (level 2):

0.1	2.1.1	2.3.1	3.1	5.1.1
0.2	2.1.2	2.3.2	3.2	5.1.2
0.3	2.1.3	2.4.1	3.3	5.2.1
1.1	2.2.1	2.4.2	4.1	5.2.2
1.2	2.2.2	2.5.1	4.2	5.3.1
1.3	2.2.3	2.5.2	4.3	5.3.2

The remaining 20 variants of 50 fall into the heightened complexity class (level 3):

6.1	7.1.2	7.2.3	7.4.2	8.2
6.2	7.1.3	7.3.1	7.5.1	8.3
6.3	7.2.1	7.3.2	7.5.2	9.1
7.1.1	7.2.2	7.4.1	8.1	9.2

Therefore, we have 70 variants in total: 20 variants of the first (basic) level of complexity, 30 variants of the second (advanced) level of complexity, and 20 variants of the third (heightened) level of complexity.

The level of complexity chosen by a student may be brought into correlation with the grade he/she has a clame on in the result of this case study: satisfactory, good or excelent. Because variants of advanced and heightened complexity levels are derivative of basic level variants, the student can change to higher level during his/her work in order to improve his/her rating.

The computing program should have the convenient interface (a user's menu), demonstrate student's individual work and make full use of computer resources and programming language potentialities for savings in computer time and memory. It is recommended to prepare in advance some demonstrational problems to show the computing program in action.

Conclusion

In this book, we have solved the problem to provide students with the complete and carefully adjusted set of possible case study assignments for linear programming by the simplex method. All the case study variants differ not in the input data but in the algorithms ought to be implemented in a student-designed software.

The book contains not only the necessary theoretical facts but also all the grounds for each variant of the simplex method, suggestions on how to implement the calculations and a number of illustrative examples. All this makes the book self-sufficient, not requiring to address the other books, while performing the individual case study project. Students are given the possibility to develop their own projects selected from the set of 70 different variants.

References

[1] Dantzig, G.B. 'Maximisation of a linear function of variables subject to linear inequalities', in *Activity Analysis of Production and Allocation* (Edited by T.C. Koopmans), Wiley, New York, 1951.

[2] Dantzig, G.B. *Linear Programming and Extensions* Princeton University Press, New Jersey, 1963.

[3] Beale, E.M.L. Cycling in the Dual Simplex Algorithm, *Nav. Res. Logistics Quart.*, 2, 269-276, 1955.

[4] Dantzig, G.B. and Orchard-Hays, W. 'Alternate Algorithm for the Revised Simplex Method Using Product Form for the Inverse', *Rand Corp. Rept.*, RM-1268, 1953.

[5] Orchard-Hays, W.*Advanced Linear Programming Computing Techniques*, McGraw-Hill, New York, 1968.

[6] Bunday, B.D. *Basic Linear Programming*, Edward Arnold, Bradford, 1984.