Nonlinear Programming Methods.S2
Quadratic Programming

A linearly constrained optimization problem with a quadratic objective function is called a quadratic program (QP). Because of its many applications, quadratic programming is often viewed as a discipline in and of itself. More importantly, though, it forms the basis of several general nonlinear programming algorithms. We begin this section by examining the Karush-Kuhn-Tucker conditions for the QP and see that they turn out to be a set of linear equalities and complementarity constraints. Much like in separable programming, a modified version of the simplex algorithm can be used to find solutions.

Problem Statement

The general quadratic program can be written as

$$\text{Minimize } f(\mathbf{x}) = \mathbf{cx} + \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{Q}\,\mathbf{x}$$

$$\text{subject to } \mathbf{Ax} \le \mathbf{b} \text{ and } \mathbf{x} \ge \mathbf{0}$$

where $\mathbf{c}$ is an n-dimensional row vector describing the coefficients of the linear terms in the objective function, and $\mathbf{Q}$ is an $(n \times n)$ symmetric matrix describing the coefficients of the quadratic terms. If a constant term exists it is dropped from the model. As in linear programming, the decision variables are denoted by the n-dimensional column vector $\mathbf{x}$, and the constraints are defined by an $(m \times n)$ $\mathbf{A}$ matrix and an m-dimensional column vector $\mathbf{b}$ of right-hand-side coefficients. We assume that a feasible solution exists and that the constraint region is bounded.

When the objective function $f(\mathbf{x})$ is strictly convex for all feasible points the problem has a unique local minimum which is also the global minimum. A sufficient condition to guarantee strictly convexity is for $\mathbf{Q}$ to be positive definite.

Karush-Kuhn-Tucker Conditions

We now specialize the general first-order necessary conditions given in Section 11.3 to the quadratic program. These conditions are sufficient for a global minimum when $\mathbf{Q}$ is positive definite; otherwise, the most we can say is that they are necessary.

Excluding the nonnegativity conditions, the Lagrangian function for the quadratic program is

$$L(\mathbf{x}, \boldsymbol{\mu}) = \mathbf{cx} + \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{Q}\,\mathbf{x} + \boldsymbol{\mu}(\mathbf{Ax} - \mathbf{b}),$$

where μ is an m-dimensional row vector. The Karush-Kuhn-Tucker conditions for a local minimum are given as follows.

$$\frac{\partial L}{\partial x_j} \geq 0, \; j = 1,\ldots,n \qquad \mathbf{c} + \mathbf{x}^{\mathrm{T}}\mathbf{Q} + \boldsymbol{\mu}\mathbf{A} \geq \mathbf{0} \tag{12a}$$

$$\frac{\partial L}{\partial \mu_i} \leq 0, \; i = 1,\ldots,m \qquad \mathbf{Ax} - \mathbf{b} \leq \mathbf{0} \tag{12b}$$

$$x_j \frac{\partial L}{\partial x_j} = 0, \; j = 1,\ldots,n \qquad \mathbf{x}^{\mathrm{T}}(\mathbf{c}^{\mathrm{T}} + \mathbf{Qx} + \mathbf{A}^{\mathrm{T}}\boldsymbol{\mu}) = 0 \tag{12c}$$

$$\mu_i g_i(\mathbf{x}) = 0, \; i = 1,\ldots,m \qquad \boldsymbol{\mu}(\mathbf{Ax} - \mathbf{b}) = 0 \tag{12d}$$

$$x_j \geq 0, \; j = 1,\ldots,n \qquad \mathbf{x} \geq \mathbf{0} \tag{12e}$$

$$\mu_i \geq 0, \; i = 1,\ldots,m \qquad \boldsymbol{\mu} \geq \mathbf{0} \tag{12f}$$

To put (12a) – (12f) into a more manageable form we introduce nonnegative surplus variables $\mathbf{y} \quad {}^{n}$ to the inequalities in (12a) and nonnegative slack variables $\mathbf{v} \quad {}^{m}$ to the inequalities in (12b) to obtain the equations

$$\mathbf{c}^{\mathrm{T}} + \mathbf{Qx} + \mathbf{A}^{\mathrm{T}}\boldsymbol{\mu}^{\mathrm{T}} - \mathbf{y} = \mathbf{0} \text{ and } \mathbf{Ax} - \mathbf{b} + \mathbf{v} = \mathbf{0}.$$

The KKT conditions can now be written with the constants moved to the right-hand side.

$$\mathbf{Qx} + \mathbf{A}^{\mathrm{T}}\boldsymbol{\mu}^{\mathrm{T}} - \mathbf{y} = -\mathbf{c}^{\mathrm{T}} \tag{13a}$$

$$\mathbf{Ax} + \mathbf{v} = \mathbf{b} \tag{13b}$$

$$\mathbf{x} \geq \mathbf{0}, \boldsymbol{\mu} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}, \mathbf{v} \geq \mathbf{0} \tag{13c}$$

$$\mathbf{y}^{\mathrm{T}}\mathbf{x} = 0, \boldsymbol{\mu}\mathbf{v} = 0 \tag{13d}$$

The first two expressions are linear equalities, the third restricts all the variables to be nonnegative, and the fourth prescribes complementary slackness.

Solving for the Optimum

The simplex algorithm can be used to solve (13a) – (13d) by treating the complementary slackness conditions (13d) implicitly with a restricted basis entry rule. The procedure for setting up the linear programming model follows.

- Let the structural constraints be Eqs. (13a) and (13b) defined by the KKT conditions.
- If any of the right-hand-side values are negative, multiply the corresponding equation by –1.
- Add an artificial variable to each equation.
- Let the objective function be the sum of the artificial variables.
- Put the resultant problem into simplex form.

The goal is to find the solution to the linear program that minimizes the sum of the artificial variables with the additional requirement that the complementarity slackness conditions be satisfied at each iteration. If the sum is zero, the solution will satisfy (13a) – (13d). To accommodate (13d), the rule for selecting the entering variable must be modified with the following relationships in mind.

x_j and y_j are complementary for $j = 1, \ldots, n$

μ_i and v_i are complementary for $i = 1, \ldots, m$

The entering variable will be the one whose reduced cost is most negative provided that its complementary variable is not in the basis or would leave the basis on the same iteration. At the conclusion of the algorithm, the vector $\mathbf{x}$ defines the optimal solution and the vector $\boldsymbol{\mu}$ defines the optimal dual variables.

This approach has been shown to work well when the objective function is positive definite, and requires computational effort comparable to a linear programming problem with $m + n$ constraints, where m is the number of constraints and n is the number of variables in the QP. Positive semi-definite forms of the objective function, though, can present computational difficulties. Van De Panne (1975) presents an extensive discussion of the conditions that will yield a global optimum even when $f(\mathbf{x})$ is not positive definite. The simplest practical approach to overcome any difficulties caused by semi-definiteness is to add a small constant to each of the diagonal elements of $\mathbf{Q}$ in such a way that the modified $\mathbf{Q}$ matrix becomes positive definite. Although the resultant solution will not be exact, the difference will be insignificant if the alterations are kept small.

Example 14

Solve the following problem.

Minimize $f(\mathbf{x}) = -8x_1 - 16x_2 + x_1^2 + 4x_2^2$

subject to $x_1 + x_2 \leq 5$, $x_1 \leq 3$, $x_1 \geq 0$, $x_2 \geq 0$

Solution: The data and variable definitions are given below. As can be seen, the **Q** matrix is positive definite so the KKT conditions are necessary and sufficient for a global optimum.

$$\mathbf{c}^T = \begin{bmatrix} -8 \\ -16 \end{bmatrix} \quad \mathbf{Q} = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\mathbf{x} = (x_1, x_2),\ \mathbf{y} = (y_1, y_2),\ \boldsymbol{\mu} = (\mu_1, \mu_2),\ \mathbf{v} = (v_1, v_2)$$

The linear constraints (13a) and (13b) take the following form.

$$\begin{aligned} 2x_1 + \mu_1 + \mu_2 - y_1 &= 8 \\ 8x_2 + \mu_1 - y_2 &= 16 \\ x_1 + x_2 + v_1 &= 5 \\ x_1 + v_2 &= 3 \end{aligned}$$

To create the appropriate linear program, we add artificial variables to each constraint and minimize their sum.

Minimize $a_1 + a_2 + a_3 + a_4$

subject to
$$\begin{aligned} 2x_1 + \mu_1 + \mu_2 - y_1 + a_1 &= 8 \\ 8x_2 + \mu_1 - y_2 + a_2 &= 16 \\ x_1 + x_2 + v_1 + a_3 &= 5 \\ x_1 + v_2 + a_4 &= 3 \end{aligned}$$

all variables ≥ 0 and complementarity conditions

Applying the modified simplex technique to this example, yields the sequence of iterations given in Table 7. The optimal solution to the original problem is $(x_1^*, x_2^*) = (3, 2)$.

Table 7. Simplex iterations for QP example

Iteration	Basic variables	Solution	Objective value	Entering variable	Leaving variable
1	(a_1, a_2, a_3, a_4)	(8, 16, 5, 3)	32	x_2	a_2
2	(a_1, x_2, a_3, a_4)	(8, 2, 3, 3)	14	x_1	a_3
3	(a_1, x_2, x_1, a_4)	(2, 2, 3, 0)	2	μ_1	a_4
4	(a_1, x_2, x_1, μ_1)	(2, 2, 3, 0)	2	μ_1	a_1
5	(μ_2, x_2, x_1, μ_1)	(2, 2, 3, 0)	0	—	—