

Benders' Method

Paul A Jensen

The mixed integer programming model has some variables, \mathbf{x} , identified as real variables and some variables, \mathbf{y} , identified as integer variables. Except for the integrality requirements on \mathbf{y} , the model has the linear programming format.

Model P : The original model.

$$\begin{aligned} Z^* = \text{Max} \quad & \mathbf{c}_1\mathbf{x} + \mathbf{c}_2\mathbf{y} \\ \text{Subject to:} \quad & \\ & \mathbf{A}_1\mathbf{x} + \mathbf{A}_2\mathbf{y} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0} \text{ and integer.} \end{aligned}$$

In the following we will obtain several related formulations and each will be given a letter designation. We call the original model P . The various vectors and matrices shown in the formulation have size and orientation appropriate to the vectors \mathbf{x} , \mathbf{y} , and \mathbf{b} .

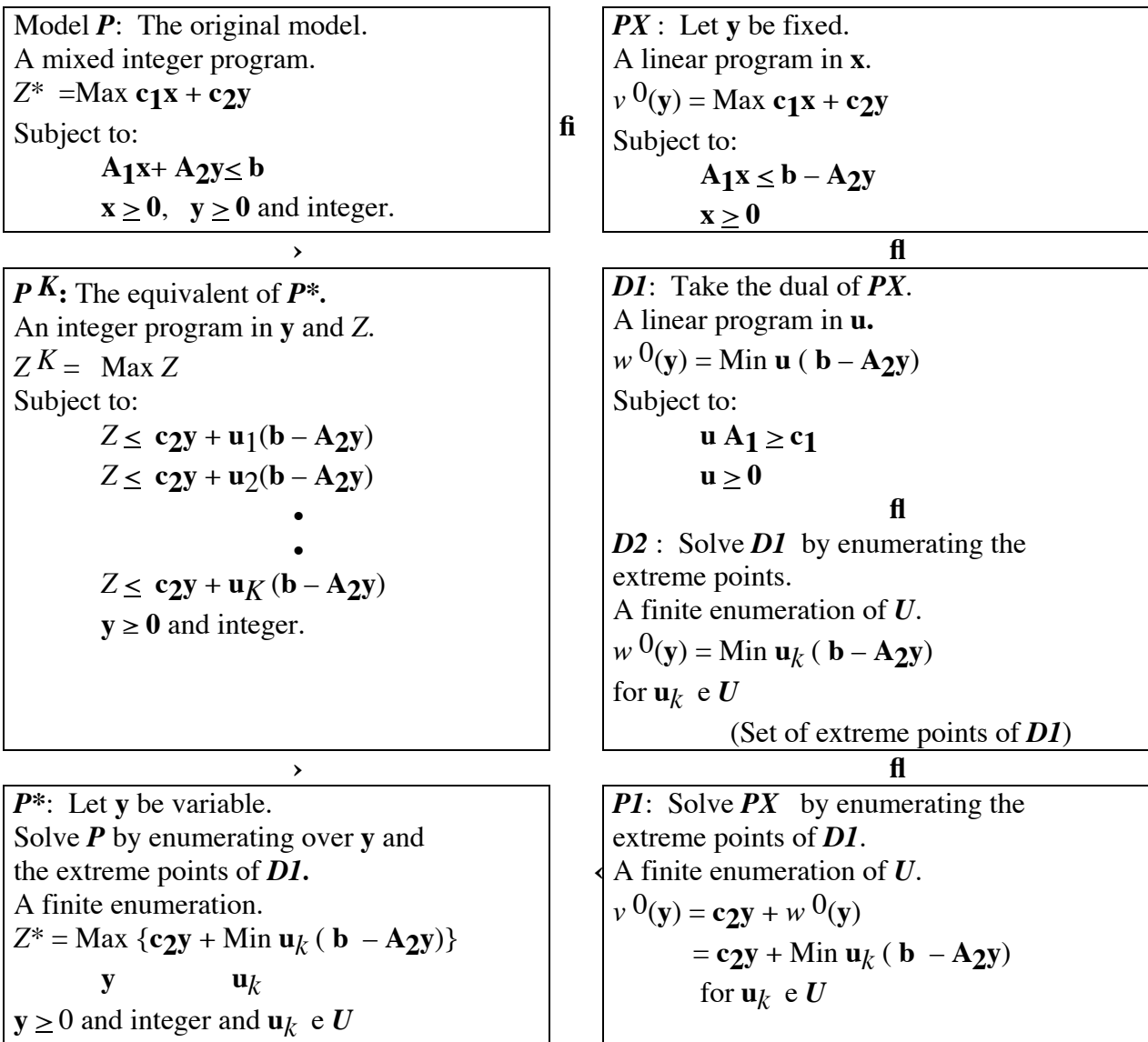
Benders' method decomposes this model in such a way that it can be solved as an alternating sequence of linear programs and pure integer programs. Fig.12 shows a schematic representation of the theoretical development that leads to the decomposition. Starting from P , the first step to the right assumes that the integer vector \mathbf{y} is fixed to some specific value to obtain model PX . The terms representing \mathbf{y} now contribute constants to the objective function and constraints. We have moved the constants to the right side of the constraints and leave the constant term in the objective function. Model PX is a linear programming model. For the moment assume that it has a feasible solution. The value of the objective function at optimality, $v^0(\mathbf{y})$, is shown as a function of \mathbf{y} because it will assume different values for different \mathbf{y} . It must be a lower bound on the optimal objective function of P since it represents a feasible solution for some given value of \mathbf{y} . Thus

$$v^0(\mathbf{y}) \leq Z^* \quad \text{for any given } \mathbf{y}. \quad (1)$$

Since PX is a linear program, we can take its dual to obtain model DI . The constant term of the objective of PX has been temporarily dropped to obtain the dual. It is important to note that the objective coefficients of DI depend on \mathbf{y} while the constraints do not. Thus, the feasible region of DI is independent of \mathbf{y} . The optimal solution to the dual has the same value as the optimal solution to the primal, so

$$w^0(\mathbf{y}) + \mathbf{c}_2\mathbf{y} = v^0(\mathbf{y}). \quad (2)$$

A different way to write DI is shown as $D2$. The latter statement follows from the fact that an optimum solution to DI must be at one of the extreme points of its feasible region. Although $D2$ suggests a search over all the feasible extreme points in the set U to solve the linear program, this obviously inefficient procedure will not be adopted. Model $D2$ is an intermediate formulation in the development.



A schematic showing the theoretical development of Benders' Method.

Replacing $w^0(\mathbf{y})$ by its equivalent in Eq. (2), results in model PI .

$$v^0(\mathbf{y}) = \mathbf{c}_2\mathbf{y} + \underset{u_k \in U}{\text{Min}} \{u_k (\mathbf{b} - \mathbf{A}_2\mathbf{y})\} \quad (3)$$

In P^* we again let \mathbf{y} be variable. The optimum \mathbf{y} is found by minimizing $v^0(\mathbf{y})$ over all alternative \mathbf{y} . P^* is equivalent to the original P . To remove the minimization from the objective function, we use a standard trick of mathematical programming to obtain the equivalent model P^K . In this model there are all integer variables \mathbf{y} with one real variable Z . Here K is the number of extreme points of $D2$, and there is a constraint for each extreme point. The goal is to maximize Z . Model P^K completes the development because it is equivalent to the original model P . Although the new model cannot be practically solved because its formulation requires that all extreme points of DI be identified, it does provide the basis for the iterative algorithm that follows.

Model DI is an important component of the algorithm so it is necessary to discuss conditions under which this model might have no feasible solution or might have an unbounded solution. From duality theory, we know that if DI has no feasible solution, its dual, model PX , must either have no feasible solution or must also be unbounded. The feasible region of DI does not depend on \mathbf{y} . When no feasible solution exists for DI , then PX must be either infeasible or unbounded for all \mathbf{y} . This case is not interesting for practical problems, so we assume that DI has feasible solutions.

When model DI has an unbounded solution for some \mathbf{y} , model PX is infeasible for that \mathbf{y} . This possibility occurs frequently in practical problems and must be accommodated in the algorithm. Here we adopt a simple solution that may result in numerical difficulties when solving large problems. We add the following constraint to DI that assures a bounded solution.

$$u_1 + u_2 + \dots + u_m \leq M, \quad (4)$$

where M is a large positive number. This is equivalent to adding an artificial variable to PX that assures feasibility.

Bounds on the Optimal Objective

As will be described in the next section, we will solve a relaxation of P^K that includes only a subset of its constraints. Let a subset of the extreme points U be the set

$$U^r = \{u_1, u_2, \dots, u_r\}.$$

Define the model P^r as

Model P^r :

$$\begin{aligned}
 Z^r = & \text{Max } Z \\
 \text{Subject to:} \\
 Z \leq & \mathbf{c}_2\mathbf{y} + \mathbf{u}_1(\mathbf{b} - \mathbf{A}_2\mathbf{y}) \\
 Z \leq & \mathbf{c}_2\mathbf{y} + \mathbf{u}_2(\mathbf{b} - \mathbf{A}_2\mathbf{y}) \\
 & \vdots \\
 Z \leq & \mathbf{c}_2\mathbf{y} + \mathbf{u}_r(\mathbf{b} - \mathbf{A}_2\mathbf{y}) \\
 & \mathbf{y} \geq \mathbf{0} \text{ and integer.}
 \end{aligned}$$

When $r < K$, this is a relaxation of P^K , so the value of Z^r is an upper bound on the optimal solution.

$$Z^r \geq Z^*. \quad (5)$$

Combining Eq. (1) and Eq. (2), the solution to DI for any given \mathbf{y} will provide a lower bound to the optimal objective value

$$v^0(\mathbf{y}) = w^0(\mathbf{y}) + \mathbf{c}_2\mathbf{y} \leq Z^* \quad \text{for any given } \mathbf{y}. \quad (6)$$

Thus we have procedures for finding both lower and upper bounds on the optimal objective value.

Benders' Algorithm

Benders' algorithm involves sequentially solving the linear program DI for specific values of \mathbf{y} , and solving the integer program P^r for increasing values of r . At each iteration, lower and upper bounds are computed. When the bounds become equal, the algorithm terminates with the optimal solution. The hope is that termination will come when r is considerably less than K . Finite termination is assured because model DI must have a finite number of extreme points.

Step 0 Initialization : Set the objective coefficients of DI to zero and solve. If no feasible solution exists, stop, because either P has no feasible solution or is unbounded for all \mathbf{y} . If DI has an optimum, let \mathbf{u}_1 be the optimal solution. Using \mathbf{u}_1 form the constraint

$$Z \leq \mathbf{c}_2\mathbf{y} + \mathbf{u}_1(\mathbf{b} - \mathbf{A}_2\mathbf{y}). \quad (7)$$

$$\text{Let } Z_L = -\infty.$$

$$\text{Let } r = 1.$$

Step 1 Solve the Integer Program : Solve problem P^r to find the optimum solution \mathbf{y}^r with objective value Z^r .

Compute the upper bound to the optimum solution of P .

$$Z_U = Z^r. \quad (8)$$

Step 2 Solve the Linear Program : Let $\mathbf{y} = \mathbf{y}^r$, compute the new objective function for the linear program **DI**,

$$w^0(\mathbf{y}^r) = \text{Min } \mathbf{u} (\mathbf{b} - \mathbf{A}_2 \mathbf{y}^r), \quad (9)$$

and solve. Let \mathbf{u}_{r+1} be the optimal solution with objective value

$$w^0(\mathbf{y}^r) = \mathbf{u}_{r+1} (\mathbf{b} - \mathbf{A}_2 \mathbf{y}^r). \quad (10)$$

Compute

$$v^0(\mathbf{y}^r) = \mathbf{c}_2 \mathbf{y}^r + w^0(\mathbf{y}^r). \quad (11)$$

$$\text{If } v^0(\mathbf{y}^r) > Z_L ,$$

then replace the lower bound.

$$Z_L = v^0(\mathbf{y}^r).$$

$$\text{If } v^0(\mathbf{y}^r) \leq Z_L ,$$

do not replace the lower bound.

Step 3 Test for Optimality

$$\text{If } Z_L < Z_U$$

the solution is not optimal, so form a new constraint for the integer program.

$$Z \leq \mathbf{c}_2 \mathbf{y} + \mathbf{u}_{r+1} (\mathbf{b} - \mathbf{A}_2 \mathbf{y}). \quad (12)$$

Replace r by $r+1$ and return to step 1.

$$\text{If } Z_L = Z_U ,$$

the solution is optimum, so proceed to step 4.

Step 4 Termination : The optimum solution is \mathbf{y}^r . The optimum \mathbf{x} is the dual of the optimum solution of problem **DI** solved with \mathbf{y}^r .

Example Problem

Consider the following mixed integer program with three 0–1 integer and three real variables.

$$\begin{aligned}
 Z^* = \quad & \text{Max} \quad 8x_1 + 6x_2 - 2x_3 - 42y_1 - 18y_2 - 33y_3 \\
 & \text{Subject to:} \\
 & \quad 2x_1 + x_2 - x_3 - 10y_1 - 8y_2 \leq -4 \\
 & \quad x_1 + x_2 + x_3 - 5y_1 - 8y_3 \leq -3 \\
 & \quad x_j \geq 0 \text{ for } j = 1, 2, 3 \\
 & \quad 0 \leq y_j \leq 1 \text{ and integer for } j = 1, 2, 3.
 \end{aligned}$$

Defining the matrices used in model **P**,

$$c_1 = [8 \quad 6 \quad -2]$$

$$c_2 = [-42 \quad -18 \quad -33]$$

$$A_1 = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -10 & -8 & 0 \\ -5 & 0 & -8 \end{bmatrix}$$

$$b = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Model **DI** has two variables

$$u = [u_1, u_2].$$

The objective coefficients of **DI** are

$$b + A_2 y = \begin{bmatrix} -4 + 10y_1 + 8y_2 + 0y_3 \\ -3 + 5y_1 + 0y_2 + 8y_3 \end{bmatrix}$$

The objective function of **DI** is then

$$w^0(y) = \text{Min } u_1[-4 + 10y_1 + 8y_2] + u_2[-3 + 5y_1 + 8y_3].$$

The constraints of model **DI** are

$$(1) \quad 2u_1 + u_2 \geq 8$$

$$\begin{aligned}
(2) \quad & u_1 + u_2 \geq 6 \\
(3) \quad & -u_1 + u_2 \geq -2 \\
(4) \quad & u_1 + u_2 \leq 9999 \\
& u_1 \geq 0, u_2 \geq 0.
\end{aligned}$$

Constraint (4) is added to assure that the solution is bounded.

The constraints of the integer program are formed from solutions of DI . The general form of the constraint is

$$\begin{aligned}
Z &\leq \mathbf{c}_2 \mathbf{y} + \mathbf{u}_k (\mathbf{b} - \mathbf{A}_2 \mathbf{y}), \\
Z &\leq (\mathbf{c}_2 - \mathbf{u}_k \mathbf{A}_2) \mathbf{y} + \mathbf{u}_k \mathbf{b} \\
(-\mathbf{c}_2 + \mathbf{u}_k \mathbf{A}_2) \mathbf{y} + Z &\leq \mathbf{u}_k \mathbf{b}
\end{aligned}$$

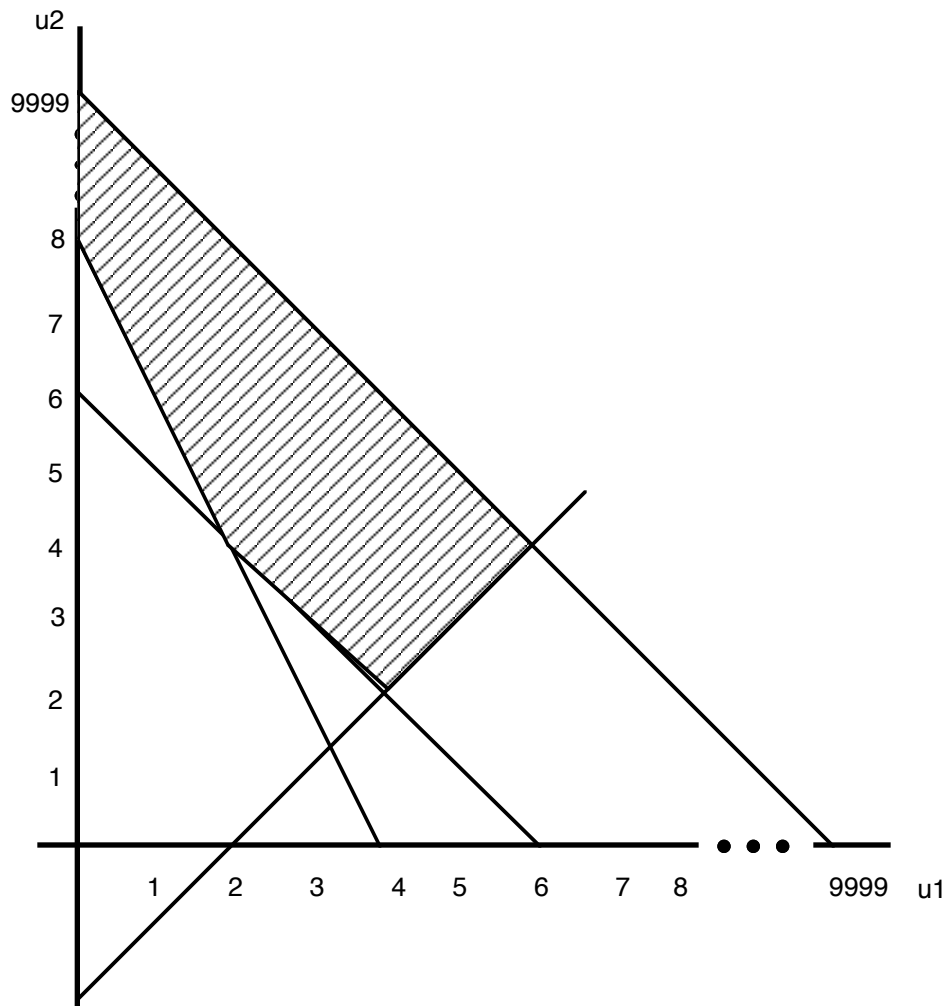
where \mathbf{u}_k is a solution to DI . Substituting the appropriate vectors and matrices we obtain

$$Z \leq -42y_1 - 18y_2 - 33y_3 + u_1[-4 + 10y_1 + 8y_2] + u_2[-3 + 5y_1 + 8y_3].$$

Combining terms and moving the variable terms to the left we have

$$y_1(42 - 10u_1 - 5u_2) + y_2(18 - 8u_1) + y_3(33 - 8u_2) + Z \leq -4u_1 - 3u_2.$$

Note that u_1 and u_2 are known for specific solutions of DI . The figure shows the feasible region of DI .



The feasible region for DI in the example problem.

Solution of Example Problem

Description

Step 0: Solve DI with 0 objective coefficients.

Form an integer constraint with $\mathbf{u} = (4, 2)$.

Step 1: Solve P^1 with one constraint.

Step 2: Find new objective for DI with $\mathbf{y} = (1, 1, 0)$.

Solve DI .

Step 3: Test for Optimality

Form new integer constraint with $\mathbf{u} = (0, 8)$.

Step 1: Solve P^2 with two constraints.

Step 2: Find new objective for DI with $\mathbf{y} = (1, 1, 1)$.

Computation

Linear Programming Solution

Variable	w^0	u_1	u_2
Value	0	4	2

New Integer Constraint

$$-8y_1 - 14y_2 + 17y_3 + Z \leq -22$$

Integer Programming Solution

Variable	Z	y_1	y_2	y_3
Value	0	1	1	0

Objective Function for DI

$$w^0 = +14u_1 + 2u_2$$

Linear Programming Solution

Variable	w^0	u_1	u_2
Value	16	0	8

Current Bounds

Lower: -44
Upper: -0

New Integer Constraint

$$2y_1 + 18y_2 - 31y_3 + Z \leq -24$$

Integer Programming Solution

Variable	Z	y_1	y_2	y_3
Value	-17	1	1	1

Objective Function for DI

$$w^0 = +14u_1 + 10u_2$$

Solve **DI** .

Linear Programming Solution

Variable	w^0	u_1	u_2
Value	68	2	4

Step 3: Test for Optimality

Current Bounds

Lower: -25
Upper: -17

Form new integer constraint with $\mathbf{u} = (2, 4)$.

New Integer Constraint

$$2y_1 + 2y_2 + y_3 + Z \leq -20$$

Step 1: Solve **P³** with three constraints.

Integer Programming Solution

Variable	Z	y_1	y_2	y_3
Value	-24	0	0	0

Step 2: Find new objective for **DI** with $\mathbf{y} = (0, 0, 0)$.

Objective Function for DI

$$w^0 = -4u_1 - 3u_2$$

Solve **DI** . This solution hits the constraint limiting an unbounded solution. Model **PX** has no feasible solution with $\mathbf{y} = (0, 0, 0)$

Linear Programming Solution

Variable	w^0	u_1	u_2
Value	-34997.5	5000.5	4998.5

Step 3: Test for Optimality

Current Bounds

Lower: -25
Upper: -24

Form new integer constraint with $\mathbf{u} = (5000.5, 4998.5)$.

New Integer Constraint

$$-74955.5y_1 - 39986y_2 - 3995517y_3 + Z \leq -34997.5$$

Step 1: Solve **P⁴** with four constraints.

Integer Programming Solution

Variable	Z	y_1	y_2	y_3
Value	-25	1	1	1

Step 2: Find new objective for **DI**
with $\mathbf{y} = (1, 1, 1)$.

Objective Function for DI
 $w^0 = 14u_1 + 10u_2$

Solve **DI**.

Linear Programming Solution

Variable	w^0	u_1	u_2
Value	68	2	4

Step 3: Test for Optimality
The optimal solution has been found.

Current Bounds

Lower:	-25
Upper:	-25

Optimal integer solution.

Integer Solution

Variable	Z	y_1	y_2	y_3
Value	-25	1	1	1

The optimum real solution comes
from the dual variables of **DI**.

Real Solution

Variable	x_1	x_2	x_3
Value	4	6	0

References

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Salkin, H. M., *Integer Programming*, Chapter 8, Addison – Wesley (1975)