6.8 Spherical Harmonics

Spherical harmonics occur in a large variety of physical problems, for example, whenever a wave equation, or Laplace's equation, is solved by separation of variables in spherical coordinates. The spherical harmonic $Y_{lm}(\theta,\phi)$, $-l \le m \le l$, is a function of the two coordinates θ, ϕ on the surface of a sphere.

The spherical harmonics are orthogonal for different l and m, and they are normalized so that their integrated square over the sphere is unity:

$$\int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta) Y_{l'm'}{}^*(\theta,\phi) Y_{lm}(\theta,\phi) = \delta_{l'l}\delta_{m'm} \tag{6.8.1}$$

Here asterisk denotes complex conjugation.

Mathematically, the spherical harmonics are related to *associated Legendre polynomials* by the equation

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi} \tag{6.8.2}$$

By using the relation

$$Y_{l,-m}(\theta,\phi) = (-1)^m Y_{lm}{}^*(\theta,\phi) \tag{6.8.3}$$

we can always relate a spherical harmonic to an associated Legendre polynomial with $m \ge 0$. With $x \equiv \cos\theta$, these are defined in terms of the ordinary Legendre polynomials (cf. §4.5 and §5.5) by

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x) \tag{6.8.4}$$

The first few associated Legendre polynomials, and their corresponding normalized spherical harmonics, are

$P_0^0(x) = 1$	$Y_{00} = \sqrt{\frac{1}{4\pi}}$
$P_1^1(x) = -(1-x^2)^{1/2}$	$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi}$
$P_1^0(x) = x$	$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$
$P_2^2(x) = 3\,(1-x^2)$	$Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta e^{2i\phi}$
$P_2^1(x) = -3\,(1-x^2)^{1/2}x$	$Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta e^{i\phi}$
$P_2^0(x) = \frac{1}{2}\,(3x^2-1)$	$Y_{20} = \sqrt{\frac{5}{4\pi}}(\frac{3}{2}\cos^2\theta - \frac{1}{2})$

(6.8.5)

There are many bad ways to evaluate associated Legendre polynomials numerically. For example, there are explicit expressions, such as

$$\begin{aligned} P_l^m(x) = \frac{(-1)^m(l+m)!}{2^m m!(l-m)!}(1-x^2)^{m/2} \Bigg[1 &- \frac{(l-m)(m+l+1)}{1!(m+1)} \left(\frac{1-x}{2}\right) \\ &+ \frac{(l-m)(l-m-1)(m+l+1)(m+l+2)}{2!(m+1)(m+2)} \left(\frac{1-x}{2}\right)^2 - \cdots \Bigg] \end{aligned} \tag{6.8.6}$$

where the polynomial continues up through the term in $(1-x)^{l-m}$. (See [1] for this and related formulas.) This is not a satisfactory method because evaluation of the polynomial involves delicate cancellations between successive terms, which alternate in sign. For large l, the individual terms in the polynomial become very much larger than their sum, and all accuracy is lost.

In practice, (6.8.6) can be used only in single precision (32-bit) for l up to 6 or 8, and in double precision (64-bit) for l up to 15 or 18, depending on the precision required for the answer. A more robust computational procedure is therefore desirable, as follows:

The associated Legendre functions satisfy numerous recurrence relations, tabulated in [1-2]. These are recurrences on l alone, on m alone, and on both l and m simultaneously. Most of the recurrences involving m are unstable, and so dangerous for numerical work. The following recurrence on l is, however, stable (compare 5.5.1):

$$(l-m)P_l^m = x(2l-1)P_{l-1}^m - (l+m-1)P_{l-2}^m \tag{6.8.7}$$

It is useful because there is a closed-form expression for the starting value,

$$P_m^m = (-1)^m(2m-1)!!(1-x^2)^{m/2} \tag{6.8.8}$$

(The notation $n!!$ denotes the product of all *odd* integers less than or equal to n.) Using (6.8.7) with $l = m+1$, and setting $P_{m-1}^m = 0$, we find

$$P_{m+1}^m = x(2m+1)P_m^m \tag{6.8.9}$$

Equations (6.8.8) and (6.8.9) provide the two starting values required for (6.8.7) for general l.

The function that implements this is

```
FUNCTION plgndr(l,m,x)
INTEGER l,m
REAL plgndr,x
   Computes the associated Legendre polynomial P_l^m(x). Here m and l are integers satisfying
   0 <= m <= l, while x lies in the range -1 <= x <= 1.
INTEGER i,ll
REAL fact,pll,pmm,pmmp1,somx2
if(m.lt.0.or.m.gt.l.or.abs(x).gt.1.)pause 'bad arguments in plgndr'
pmm=1.                              Compute P_m^m.
if(m.gt.0) then
    somx2=sqrt((1.-x)*(1.+x))
    fact=1.
    do 11 i=1,m
        pmm=-pmm*fact*somx2
        fact=fact+2.
    enddo 11
endif
if(l.eq.m) then
    plgndr=pmm
else
    pmmp1=x*(2*m+1)*pmm             Compute P_{m+1}^m.
    if(l.eq.m+1) then
        plgndr=pmmp1
    else                            Compute P_l^m, l > m+1.
        do 12 ll=m+2,l
```

```
            pll=(x*(2*ll-1)*pmmp1-(ll+m-1)*pmm)/(ll-m)
            pmm=pmmp1
            pmmp1=pll
        enddo 12
        plgndr=pll
    endif
endif
return
END
```

CITED REFERENCES AND FURTHER READING:

Magnus, W., and Oberhettinger, F. 1949, *Formulas and Theorems for the Functions of Mathematical Physics* (New York: Chelsea), pp. 54ff. [1]

Abramowitz, M., and Stegun, I.A. 1964, *Handbook of Mathematical Functions*, Applied Mathematics Series, Volume 55 (Washington: National Bureau of Standards; reprinted 1968 by Dover Publications, New York), Chapter 8. [2]

6.9 Fresnel Integrals, Cosine and Sine Integrals

Fresnel Integrals

The two Fresnel integrals are defined by

$$C(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right) dt, \qquad S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right) dt \tag{6.9.1}$$

The most convenient way of evaluating these functions to arbitrary precision is to use power series for small x and a continued fraction for large x. The series are

$$\begin{aligned} C(x) &= x - \left(\frac{\pi}{2}\right)^2 \frac{x^5}{5\cdot 2!} + \left(\frac{\pi}{2}\right)^4 \frac{x^9}{9\cdot 4!} - \cdots \\ S(x) &= \left(\frac{\pi}{2}\right) \frac{x^3}{3\cdot 1!} - \left(\frac{\pi}{2}\right)^3 \frac{x^7}{7\cdot 3!} + \left(\frac{\pi}{2}\right)^5 \frac{x^{11}}{11\cdot 5!} - \cdots \end{aligned} \tag{6.9.2}$$

There is a complex continued fraction that yields both $S(x)$ and $C(x)$ simultaneously:

$$C(x) + iS(x) = \frac{1+i}{2}\,\mathrm{erf}\, z, \qquad z = \frac{\sqrt{\pi}}{2}(1-i)x \tag{6.9.3}$$

where

$$\begin{aligned} e^{z^2}\,\mathrm{erfc}\, z &= \frac{1}{\sqrt{\pi}}\left(\frac{1}{z+}\;\frac{1/2}{z+}\;\frac{1}{z+}\;\frac{3/2}{z+}\;\frac{2}{z+}\cdots\right) \\ &= \frac{2z}{\sqrt{\pi}}\left(\frac{1}{2z^2+1-}\;\frac{1\cdot 2}{2z^2+5-}\;\frac{3\cdot 4}{2z^2+9-}\cdots\right) \end{aligned} \tag{6.9.4}$$