Primal Simplex Algorithm for the Pure Minimal Cost Flow Problem

This appendix describes the adaptation of the primal simplex algorithm for solving a pure network flow program; that is, the problem of finding the optimal flow distribution for the minimum cost flow network model given by Eqs. (4a) - (4c) in the text. We begin with a statement of the algorithm and then provide an example to demonstrate how the computations are performed.

Algorithm

Step 1. Start with a primal feasible basis tree consisting of the arcs in the set $\mathbf{n}_B$ and the sets $\mathbf{n}_0$ and $\mathbf{n}_1$ corresponding to nonbasic arcs at their lower and upper bounds, respectively. Compute the primal and dual solutions. If necessary, use the phase 1 procedure described below to find an initial feasible basis.

Step 2. Compute the reduced costs, $d_k = c_k + \pi_i - \pi_j$, for all nonbasic arcs $k(i,j)$.

Step 3. If one of the following conditions holds for each nonbasic arc k, *stop* with the optimal solution.

Optimality conditions: if $x_k = 0$, then $d_k \geq 0$

if $x_k = u_k$, then $d_k \leq 0$

Otherwise, select some nonbasic arc that violates this condition and call it the entering arc.

Step 4. Find the increase or decrease in the entering arc that will either drive it to its opposite bound or drive some basic arc to one of its bounds. If the entering arc is driven to its bound, go to Step 3. If a basic arc is driven to one of its bounds, let that be the leaving arc.

Step 5. Change the basis by removing the leaving arc and adding the entering arc. Compute the primal and dual solutions associated with the new basis and go to Step 2.

Initial Solution

As in general linear programming, we have the problem of finding an initial basic feasible solution for the primal simplex method. We use a two-phase approach where an artificial arc with unit cost is introduced for every node but the slack node *m*. The artificial arcs and their initial flows are constructed in the following manner.

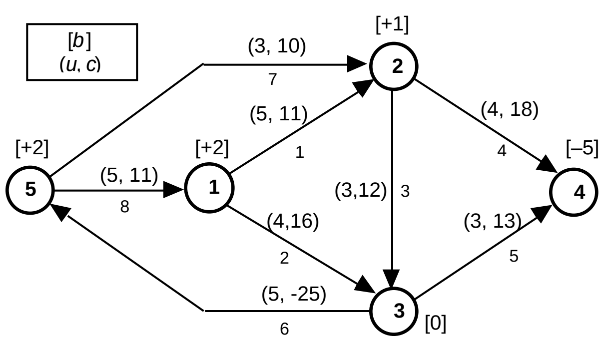
For each node $i = 1$ to $m - 1$.

a. If node *i* has $b_i > 0$, introduce an artificial arc from node *i* to the slack node *m* and assign it the flow b_i. The upper bound on the artificial arc is also b_i.

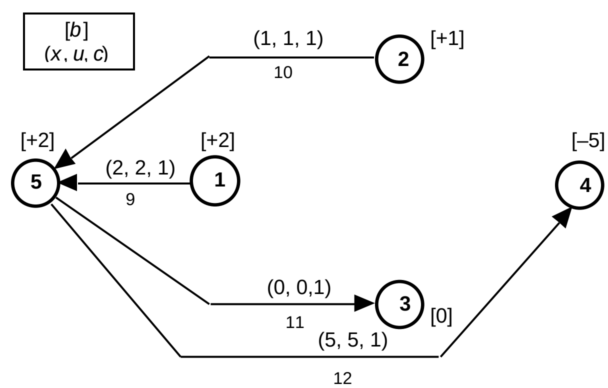
b. If node *i* has $b_i \le 0$, introduce an artificial arc from the slack node *m* to node *i* and assign it the flow $-b_i$. The upper bound on the artificial arc is also $-b_i$.

The costs on the artificial arcs depend on the phase of the solution algorithm.

Figure 35a shows the example network; Fig. 35b shows the spanning tree created by introducing artificial arcs. The tree is rooted at slack node 5. During phase 1 the two networks in the figure are combined. The resultant network consists of 12 arcs but only the unit arc costs on the artificial arcs are used at this stage in the computations. The arcs in the original network carry a unit cost of zero in phase 1.



a. Example network



b. Artificial arcs

Figure 35. Construction of artificial arcs for phase 1

As in general linear programming, we will apply the simplex algorithm in two phases as follows.

Phase 1

1. Assign an arc cost of +1 to each of the artificial arcs and a cost of 0 to each of the original arcs.
2. Using the artificial arcs as the initial basis, solve the minimum cost flow problem with the primal simplex algorithm. If the total cost at optimality is greater than zero, an artificial arc has nonzero flow; stop, there is no feasible solution to the original problem. If the total cost at optimality is zero, all the artificial arcs have zero flow and a feasible solution has been found. Proceed with phase 2.

Phase 2

3. Assign the original arc costs to the original arcs. Delete nonbasic artificial arcs and assign 0 cost and 0 capacity to each artificial arc remaining in the basis. Starting with the basic solution found in phase 1, solve the network problem with the primal simplex algorithm.

The two-phase procedure applies the primal simplex algorithm twice. In phase 1, a basic feasible solution is found if one exists. Starting with this solution, phase 2 works towards optimality. In most cases all the artificial variables are driven from the basis during phase 1; however, it is conceivable that some artificial arcs remain. Setting their capacity to zero assures that the flows in these arcs remain at zero during phase 2.

Finding the Entering Arc

The reduced cost, d_k, for a nonbasic arc $k(i, j)$ is the cost of increasing the flow on that arc by one unit, and can be expressed as:

$$d_k = c_k + \pi_i - \pi_j \tag{7}$$

The three terms on the right-hand side of Eq. (7) include the unit cost of flow on arc k, c_k, the cost, π_i, of bringing a unit of flow to node i from the slack node through a path defined by the basis tree, and the decrease in cost, $-\pi_j$, achieved by reducing the flow through the basic path to node j. For a nonbasic arc with flow at its upper bound, d_k is the savings in cost (marginal benefit) associated with reducing the flow on arc k by one unit.

If the solution is optimal, one of the following conditions must hold for each nonbasic arc $k(i,j)$:

$$\text{if } x_k = 0, \text{ then } d_k \geq 0 \tag{8a}$$

$$\text{if } x_k = u_k, \text{ then } d_k \leq 0 \tag{8b}$$

Condition (8a) indicates that if the nonbasic flow is at zero, the cost of increasing the flow on the arc must be nonnegative for the optimal solution; otherwise increasing the flow would improve the solution. Alternatively, if the flow on the nonbasic arc is at its upper bound as in

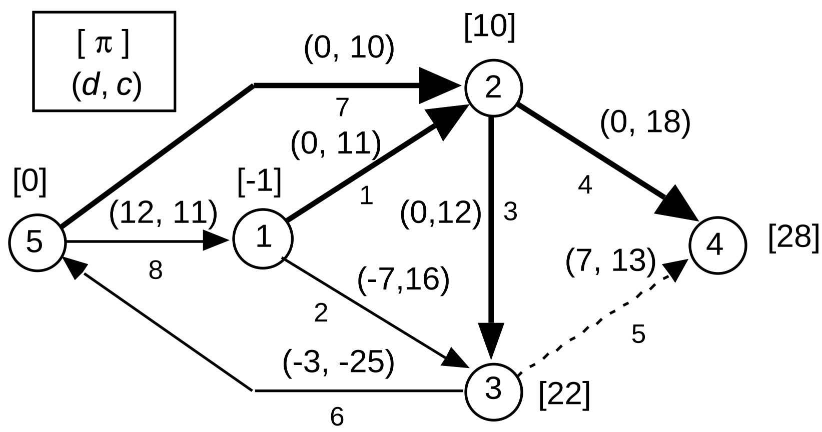
(8b), the cost of decreasing the flow on the arc must be nonnegative (the cost of decreasing flow is $-d_k$) for an optimal solution; otherwise decreasing the flow would improve the solution. Values of $d_k = 0$ imply that changing the flow on the nonbasic arc will not change the objective function. In the case of an optimal solution, this indicates the existence of alternative optima.

When the simplex method is applied to the network flow problem, a nonbasic arc with flow at zero or its upper bound may enter the basis when it fails to satisfy the optimality condition. In our example, we select the entering arc with the largest reduced cost d_E. Thus we choose the entering arc k_E according to the rule

$$d_E = \max \begin{cases} \max\{-d_k : d_k < 0 \text{ for } k \in \mathbf{n}_0\} \\ \max\{d_k : d_k > 0 \text{ for } k \in \mathbf{n}_1\} \end{cases} \qquad (9)$$

For large networks, this rule is not efficient because it requires a search over all nonbasic arcs at every iteration. Alternatives include choosing the first nonbasic nonoptimal arc encountered or selecting an arc from some list of candidates.

To illustrate this concept, consider the example network with the basis $\mathbf{n}_B = \{1, 3, 4, 7\}$ in Fig. 36. The arcs drawn with heavy lines depict the tree representation of this basis. We call this the *dual network* because, together with the original network structure, it shows the variables and parameters primarily associated with the dual of the min-cost flow problem; i.e., the π variables for the nodes, the reduced arc costs, and the original arc costs. Arcs in $\mathbf{n}_0$ are shown with narrow solid lines while arcs in $\mathbf{n}_1$ are shown with dashed lines. The structure and basis information on the figure is sufficient to compute the π vector and the reduced costs $\mathbf{d}$. These can then be used to test the solution for optimality and, if necessary, select an arc to enter the basis. From the figure we note that arcs 2, 5 and 6 fail the optimality test and are candidates to enter the basis. Using the rule stated in Eq. (9) either arc 2 or arc 5 would be selected to enter the basis. The choice is arbitrary and in the following we will see the effects of both selections.



Candidates to enter basis: {2, 5, 6}

Figure 36. Dual network with $\mathbf{n}_B$ = {1, 3, 4, 7}

Finding the Leaving Arc

After an arc has been selected to enter the basis, we move to an adjacent BFS by increasing or decreasing the flow on the entering arc. In this operation, the flows in the basic arcs in the cycle formed by the entering arc must also change in order to maintain conservation of flow at the nodes. The flow changes in the entering arc by an amount that just drives the flow on one of the basic arcs to zero or to its upper bound, or drives the flow on the entering arc to zero or to its upper bound.

Constructing the Cycle

Call the entering arc k_E and assume that it originates at node i_E and terminates at node j_E. We must now construct a signed list of arcs that describes the directed cycle formed by arc k_E and the basic arcs. There are two distinct cases.

a. *Flow is increasing on the entering arc*

   Find the directed path in the basis tree from node j_E to node i_E with arc signs indicated by the direction of arc traversal. Let $\boldsymbol{C}$ be the set of arcs formed by adding arc k_E to the path to create a cycle.

b. *Flow is decreasing on the entering arc*

   Find the directed path in the basis tree from node i_E to node j_E with arc signs indicated by the direction of arc traversal. The addition of arc $-k_E$ to the path forms the set $\boldsymbol{C}$.

Figure 37 depicts the *primal network* for the example. Included in the diagram is information related to arc flows, particularly the external flow at the nodes, the current arc flows, and the upper bounds on flow. The oval in the center of the figure shows the cycle formed when arc 2 is selected to enter the basis. The information contained in the primal and dual networks could be combined but the use of two separate diagrams simplifies the presentation.

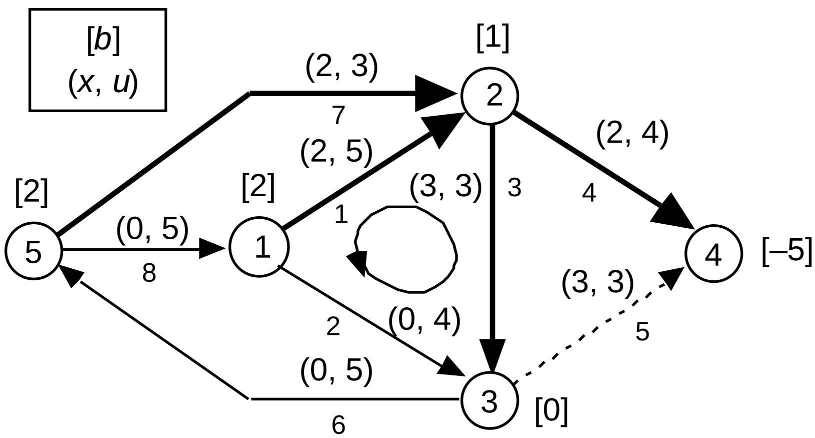


Figure 37. Primal network with $\mathbf{n}_B$ = {1, 3, 4, 7} with arc 2 selected to enter the basis

Because the current flow on arc 2 is 0, it must enter the basis with an increased flow. Thus k_E = 2(1,3) and the directed path from node 3 to node 1 is (–3, –1). The directed cycle identified as $\vec{C}$ = (2, –3, –1). The signs on the arc indices show the direction in which flow is to be changed on the arcs as arc 2 enters the basis.

Computing the Flow Change and Selecting the Leaving Arc

In general terms, let the flow change in the entering arc by an amount Δ. Arcs with positive signs in the set ***C*** experience a flow increase of Δ , while arcs with negative signs in ***C*** experience a flow decrease of Δ. The largest flow change that results in a feasible solution is the smallest value that will drive the flow on one of the cycle arcs to zero or to its upper bound.

$$\Delta = \min \begin{cases} \min\{u_k - x_k : k > 0 \text{ for } k \quad \boldsymbol{C}\} \\ \min\{x_{-k} : k < 0 \text{ for } k \quad \boldsymbol{C}\} \end{cases} \tag{10}$$

The leaving arc k_L is the arc that provides the minimum value for Eq. (10). In the event of ties, the selection is arbitrary. When k_L ***C***, the arc will leave the basis with its flow at its upper bound. When $-k_L$ $\boldsymbol{C}$, the arc will leave the basis with its flow at zero.

In the example, when arc 2 enters the basis we have $\boldsymbol{C} = \{2, -3, -1\}$ so the flow change is computed as follows.

$$\Delta = \min\left\{\begin{array}{l}\min\{u_2 - x_2\}\\ \min\{x_1, x_3\}\end{array}\right\} = \min\left\{\begin{array}{l}\min\{4-0\}\\ \min\{2,3\}\end{array}\right\} = 2$$

Because the minimum is obtained for arc 1, it must leave the basis. This results in a flow change around the cycle of 2.

Changing the Basis

Changing the Basis Tree and Computing the Dual Values

The new basis tree is constructed by deleting k_L from the previous tree and adding k_E. The dual values can be computed directly from the tree representation using the condition that $\pi_j = c_k + \pi_i$ for all arcs in the basis. A more efficient computational procedure is recommended, however, based on the observation that prior to adding k_E, the deletion of k_L from the basis breaks the tree into two parts, one of which necessarily containing the slack node. This is illustrated in Fig. 38. Call $\boldsymbol{I}_L$ the set of nodes separated from the slack node by deleting arc k_L. In the figure, $\boldsymbol{I}_L$ consists of node 1.

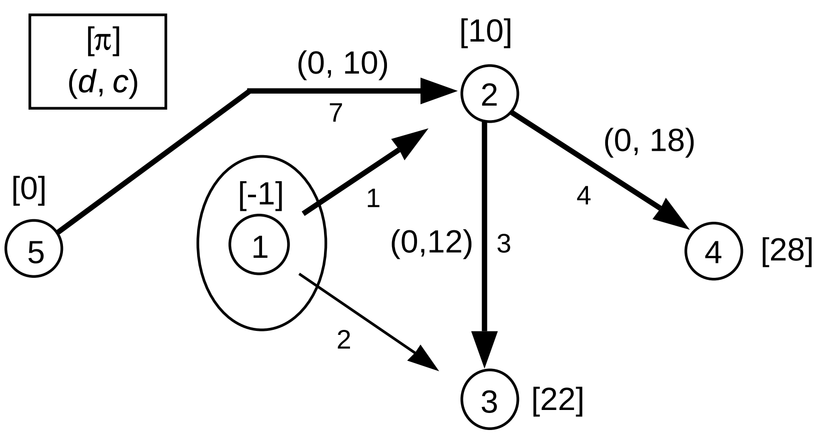


Figure 38. Subtree formed by deleting the leaving arc 1

The values of π will change for the nodes in $\boldsymbol{I}_L$ by the reduced cost of the entering arc. When the entering arc is $k_E(i_E, j_E)$, its reduced cost is

$$d_E = c_{k_E} + \pi_{i_E} - \pi_{j_E}.$$

The revised dual costs, π', for the nodes $i \in \boldsymbol{I}_L$ depend on whether the origin or the terminal node of k_E is a member of $\boldsymbol{I}_L$.

$$\pi_i' = \begin{cases} \pi_i + d_E : \text{for } j_E \in \boldsymbol{I}_L \\ \pi_i - d_E : \text{for } i_E \in \boldsymbol{I}_L \end{cases} \tag{11}$$

For the example, when $k_E = 2$ and $k_L = 1$ the basis tree with dual values assigned is shown in Fig. 39. Because the origin node of arc 2 is in the set $\boldsymbol{I}_L = \{1\}$, we subtract –7 from the dual variables in $\boldsymbol{I}_L$. Thus $\pi_1' = \pi_1 - d_E = -1 - (-7) = 6$ in the new basis tree.

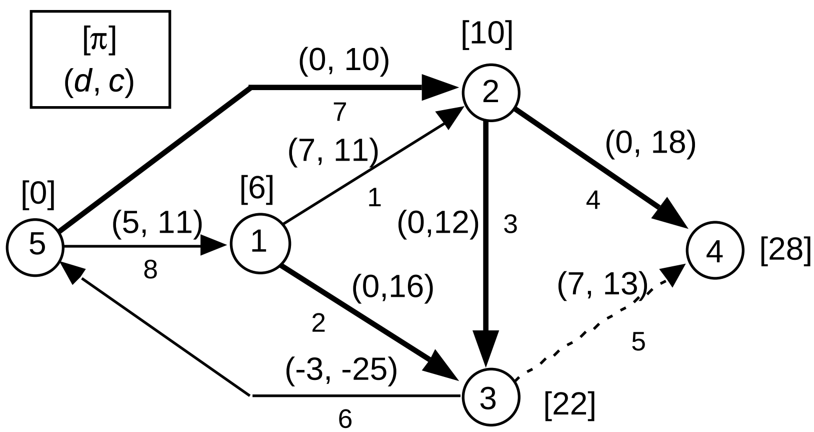


Figure 39. Dual network with $\mathbf{n}_B = \{2, 3, 4, 7\}$

Flow Change

Flow for the new basic solution is easily determined by adjusting the flows on the cycle arcs — increasing the flow on the arcs that have positive indices in ***C*** and decreasing the flow on the arcs with negative indices in ***C***. The new flows on these arcs are computed with Eq. (12).

$$x_k' = \begin{cases} x_k + \Delta : k \in \boldsymbol{C} \text{ and } k > 0 \\ x_{-k} - \Delta : k \in \boldsymbol{C} \text{ and } k < 0 \end{cases} \tag{12}$$

The values of x_k' become the basic flows at the next iteration. Figure 40 illustrates the new flows after the basis change.

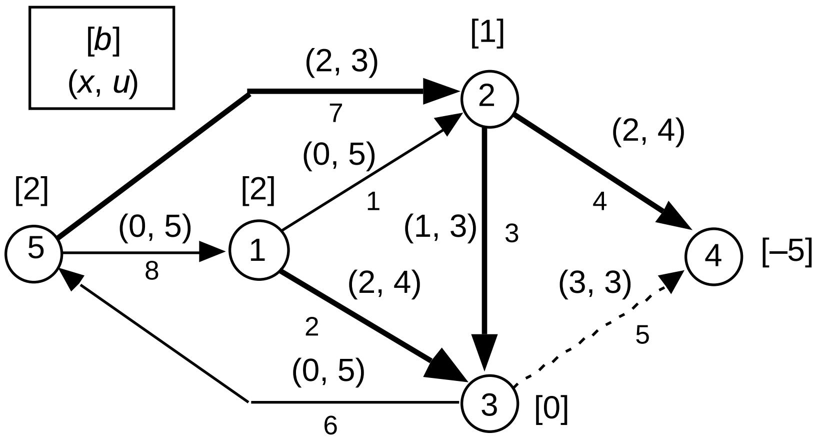


Figure 40. Primal network with $\mathbf{n}_B$ = {2, 3, 4, 7}

Continuing the Algorithm

After changing the dual values, arc flows and basis tree, the algorithm continues by returning to Step 2 and computing the reduced costs for the nonbasic arcs. From Fig. 39, we see that arcs 5 and 6 are candidates to enter the basis, with arc 5 providing the largest violation of the optimality condition. Since arc 5 has flow at its upper bound, it must enter the basis with flow decreasing. The cycle formed by arc 5 and the basis arcs is $\vec{C}$ = (–5, –3, 4). The maximum flow change on the cycle is 1 with arc 3 leaving the basis. Deleting arc 3 from the basis forms the subtree consisting of nodes $\boldsymbol{I}_L$ = {1, 3}. According to Eq. 11, the dual values for these nodes will decrease by 7. The resulting dual and primal networks appear in Figs. 41 and 42.

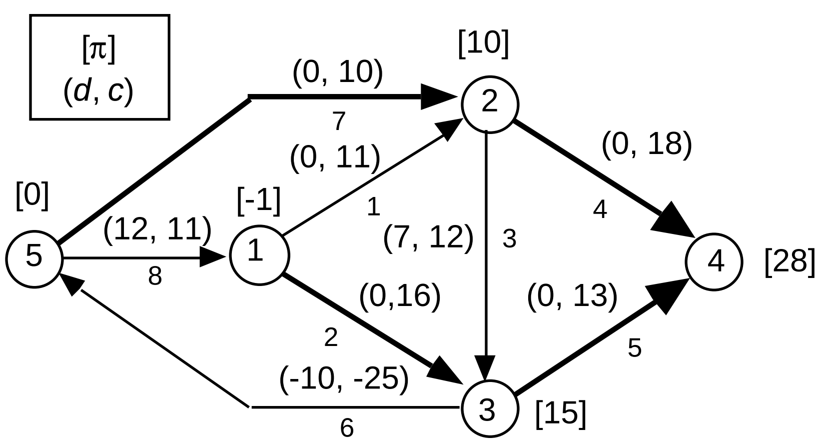


Figure 41. Dual network with $\mathbf{n}_B$ = {2, 4, 5, 7}

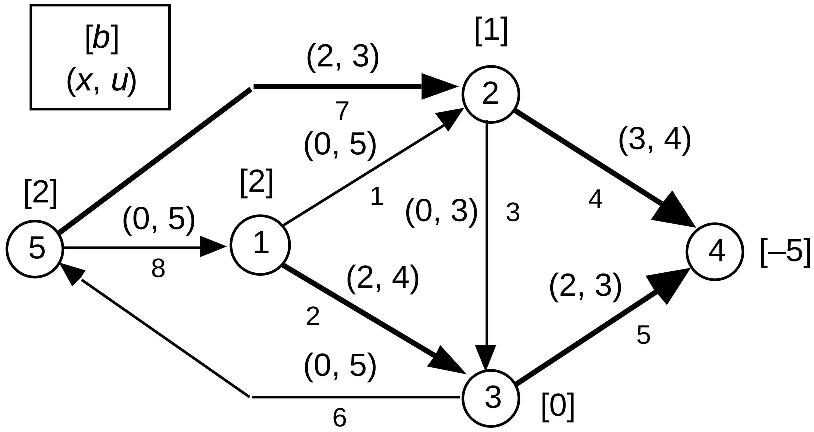


Figure 42. Primal network with $\mathbf{n}_B$ = {2, 4, 5, 7}

The solution is still not optimal because arc 6 violates the optimality condition, implying that more work is required. At each iteration, the basis changes with the deletion of one arc and the addition of another. Except when degeneracy causes a 0 flow change, the objective function decreases by the product of the reduced cost and the flow change Δ. When an iteration begins with a primal feasible solution, the rules for changing the flow assure that the solution remains primal feasible.

Complete Example

We now provide a complete solution to the problem introduced in Fig. 23 starting with phase 1 and a basis consisting of artificial arcs. Figure 43 repeats the example for easy reference. Each iteration of the primal simplex algorithm is illustrated in Fig. 44. We have combined the primal and dual networks for a more compact representation.

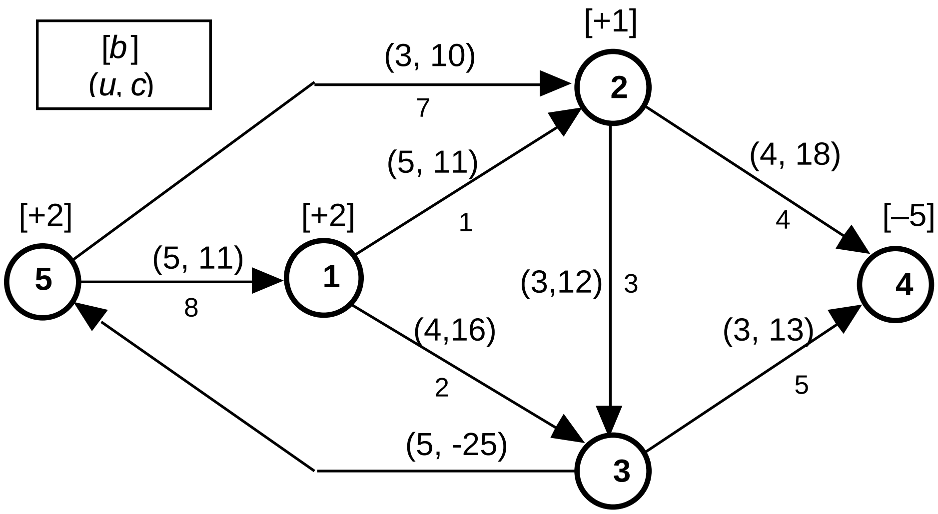
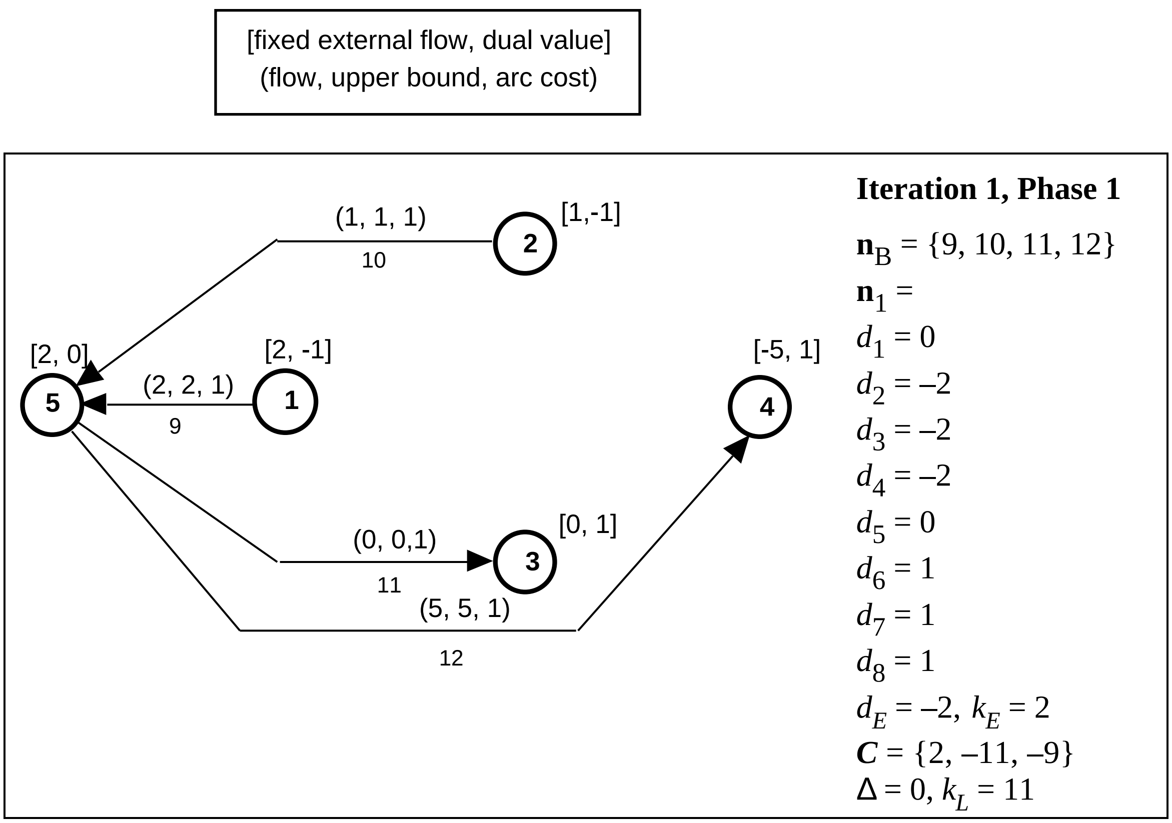
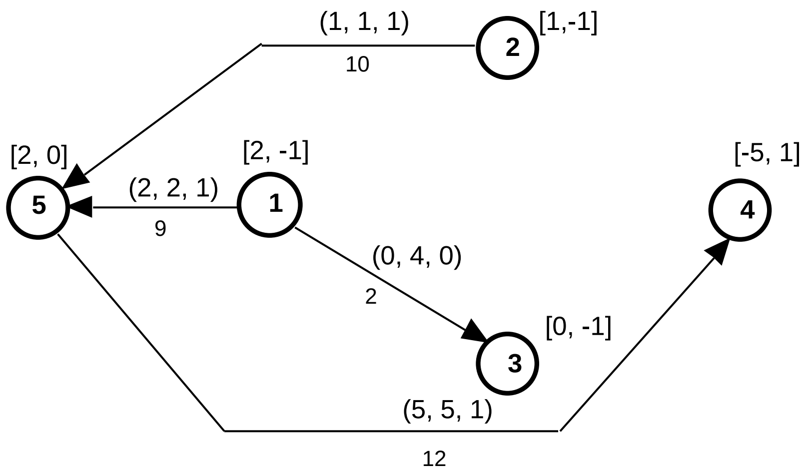


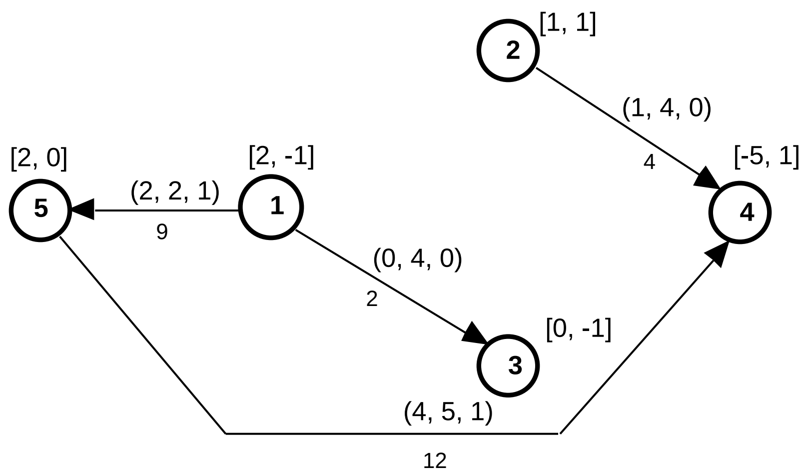
Figure 43. Network for example





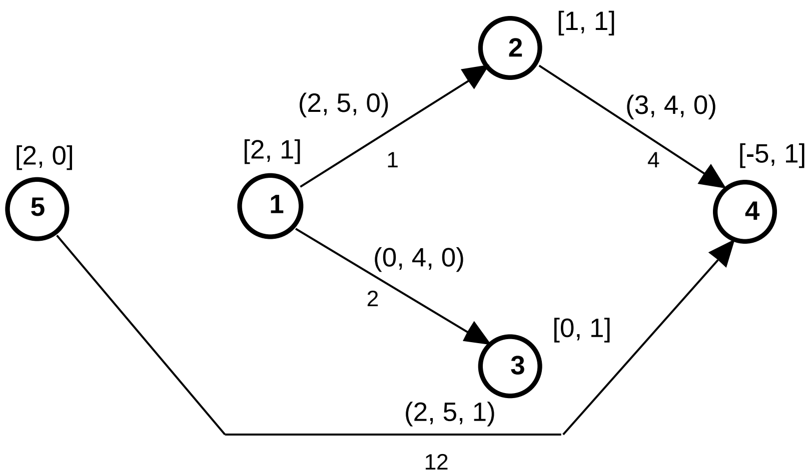
Iteration 2, Phase 1

$\mathbf{n}_B$ = {2, 9, 11, 12}
$\mathbf{n}_1$ =
$d_1 = 0$
$d_3 = 0$
$d_4 = -2$
$d_5 = -2$
$d_6 = -1$
$d_7 = 1$
$d_8 = 1$
$d_E = 4$, $k_E = 2$
$\boldsymbol{C}$ = {4, –10, –12}
$\Delta = 1$, $k_L = 10$



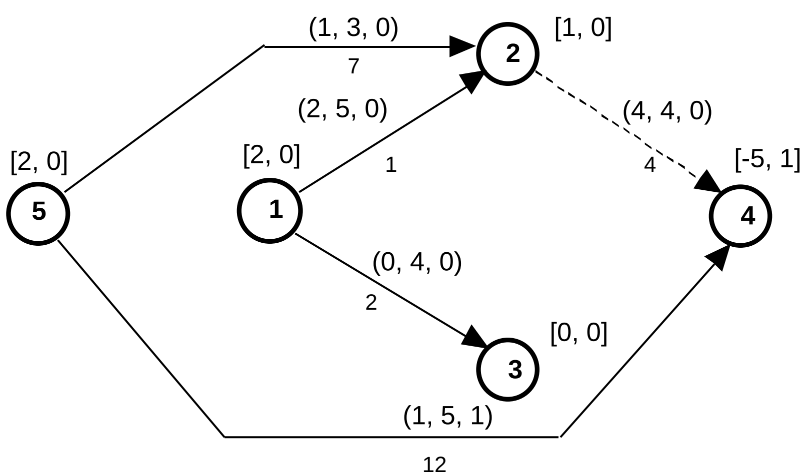
Iteration 3, Phase 1

$\mathbf{n}_B$ = {2, 4, 9, 12}
$\mathbf{n}_1$ =
$d_1 = -2$
$d_3 = 2$
$d_5 = -2$
$d_6 = -1$
$d_7 = -1$
$d_8 = 1$
$d_E = -2$, $k_E = 1$
$\boldsymbol{C}$ = {1, 4, –12, –9}
$\Delta = 2$, $k_L = 9$



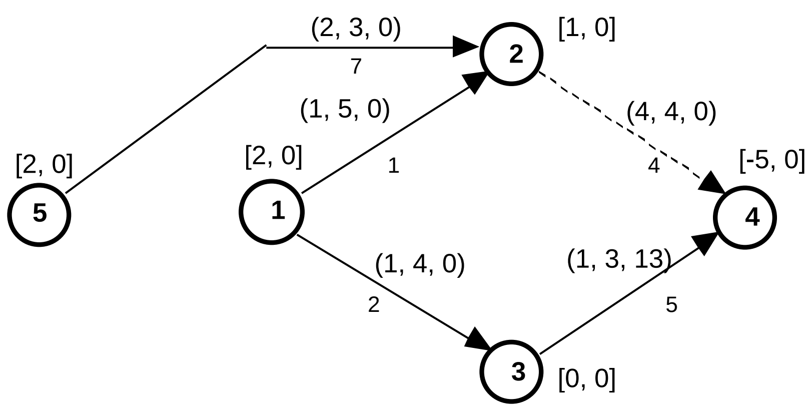
Iteration 4, Phase 1

$\mathbf{n}_B = \{1, 2, 4, 12\}$
$\mathbf{n}_1 =$
$d_3 = 0$
$d_5 = 0$
$d_6 = 1$
$d_7 = -1$
$d_8 = -1$
$d_E = -1,\ k_E = 7$
$\boldsymbol{C} = \{7, 4, -12\}$
$\Delta = 1,\ k_L = 4$



Iteration 5, Phase 1

$\mathbf{n}_B = \{1, 2, 7, 12\}$
$\mathbf{n}_1 = \{4\}$
$d_3 = 0$
$d_4 = -1$
$d_5 = -1$
$d_6 = 0$
$d_8 = 0$
$d_E = 5,\ k_E = 7$
$\boldsymbol{C} = \{5, -12, 7, -1, 2\}$
$\Delta = 1,\ k_L = 12$

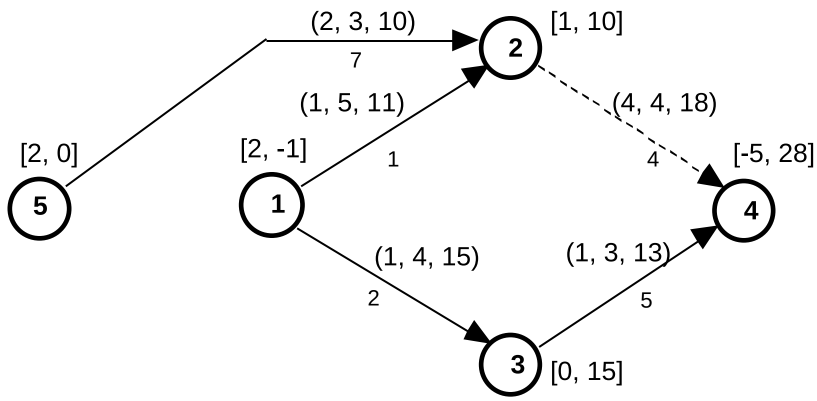


Iteration 6, Phase 1

$\mathbf{n}_B = \{1, 2, 5, 7\}$
$\mathbf{n}_1 = [4]$
$d_3 = 0$
$d_4 = 0$
$d_6 = 0$
$d_8 = 0$

Optimal for Phase 1

Switch to Phase 2



Iteration 7, Phase 2

$\mathbf{n}_B = \{1, 2, 5, 7\}$

$\mathbf{n}_1 = \{4\}$

$d_3 = 7$

$d_4 = 0$

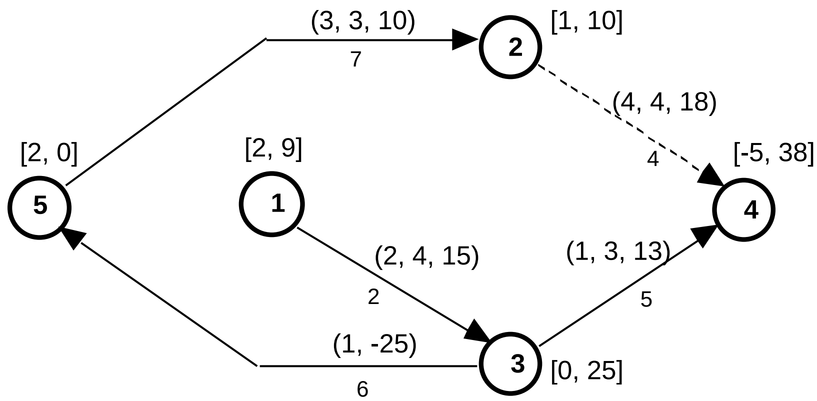
$d_6 = -10$

$d_8 = 12$

$d_E = -10,\ k_E = 6$

$\boldsymbol{C} = \{6, 7, -1, 2\}$

$\Delta = 1,\ k_L = 1$



Iteration 8, Phase 2

$\mathbf{n}_B = \{2, 5, 6, 7\}$

$\mathbf{n}_1 = \{4\}$

$d_1 = 10$

$d_3 = -3$

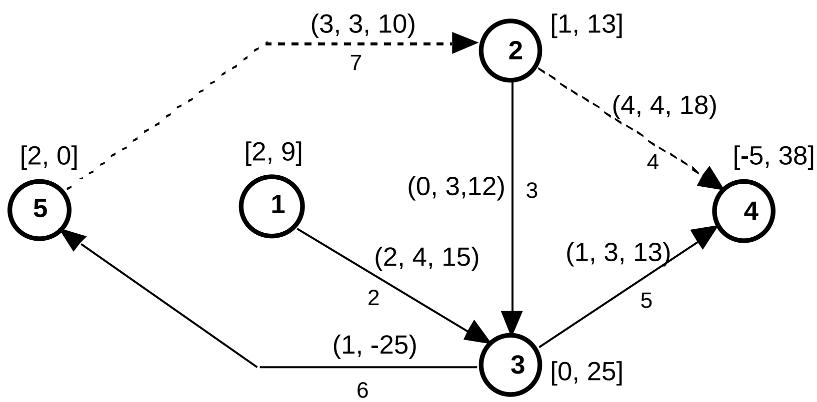
$d_4 = -10$

$d_8 = 2$

$d_E = -3,\ k_E = 3$

$\boldsymbol{C} = \{3, 6, 7\}$

$\Delta = 0,\ k_L = 7$



Iteration 9, Phase 2

$\mathbf{n}_B = \{2, 3, 5, 6\}$

$\mathbf{n}_1 = \{4, 7\}$

$d_1 = 7$

$d_4 = -7$

$d_7 = -3$

$d_8 = 2$

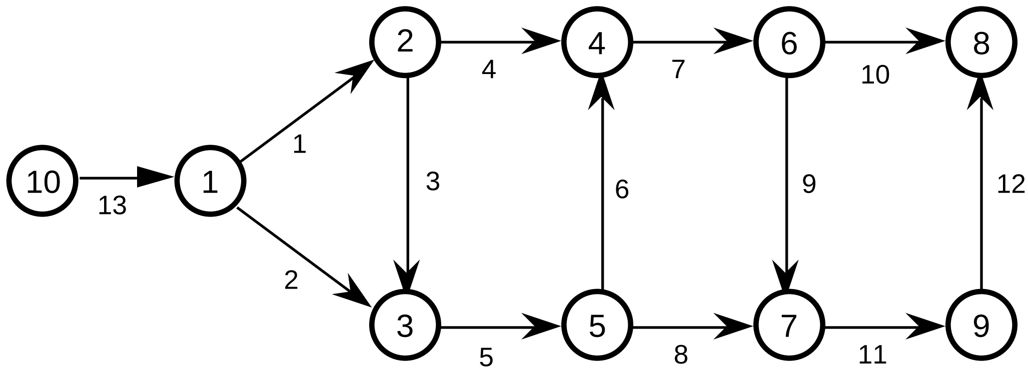
Optimal for Phase 2

Stop

Figure 44. Simplex iterations for example

Exercises

1. Consider the network below with slack node 10.



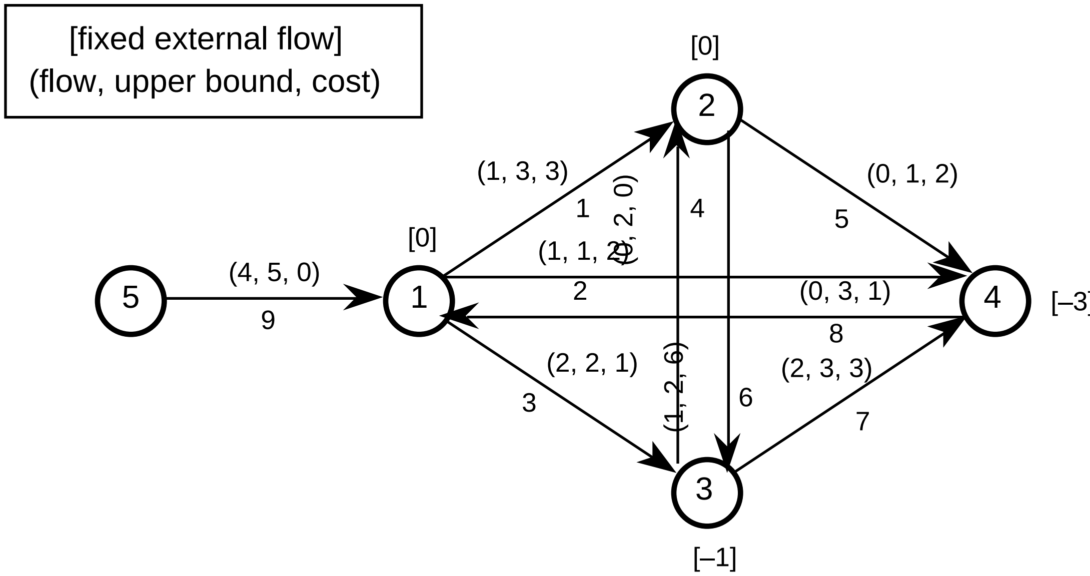
   a. Draw the basis tree for $\mathbf{n}_B$ = {13, 3, 2, 4, 5, 10, 8, 12, 11}.

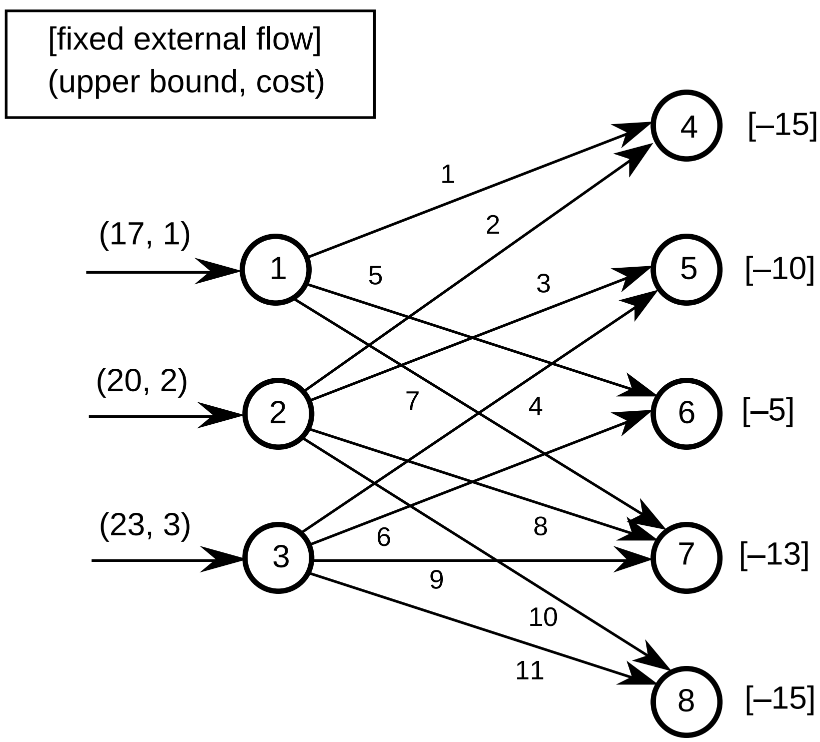
   b. What cycle is formed if arc 7 is added to the tree of part *a*? Express the result as a cycle starting and ending at node 4.

2. Refer to problem 22 in the text. Starting from $\mathbf{n}_B$ = {8, 7, 2, 5}, let each of the nonbasic variables enter the basis. Each basis change should start from the original basis $\mathbf{n}_B$ = {8, 7, 2, 5}. Select the arc with the smallest index in the cycle to leave the basis. Note that the leaving arc will not necessarily be chosen by Eq. (10). Show $\mathbf{n}_B$ and the basis tree obtained in each case.

3. Consider the following network diagram and the cost parameters for the arcs in the table below.

4. For the network shown below, determine if the flows given are optimal. The basic arcs for this solution are 1, 6, 7 and 9. If it is not optimal which arcs violate the optimality conditions.





Arc	1	2	3	4	5	6	7	8	9	10	11
Cost	5	7	10	8	12	13	15	11	9	7	9

a. Show the slack arcs that must be added to represent the slack external flows for phase 1 of the primal simplex algorithm.

b. Determine the basic flows when the basic arcs are

$\mathbf{n}_B$ = {2, 3, 4, 5, 7, 9, 11, slack arc entering node 3}.

The slack arcs that enter nodes 1 and 2 are at their upper bounds. Show the arcs with nonzero values together with their flows.

c. For the basis given in part *b*, show the basis tree.

d. For the basis given in part *b*, compute the dual values for the nodes.

e. For the basis given in part *b*, compute the reduced costs for all nonbasic arcs. Identify the set of nonbasic arcs that are candidates to enter the basis.

f. If arc 8 enter the basis, list the arcs that will experience a flow change and indicate the direction of the change. Identify the arc that will leave the basis.